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Decay for travelling waves in the Gross–Pitaevskii equation

Décroissance asymptotique des ondes progressives dans l'équation de Gross–Pitaevskii

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Abstract

We study the limit at infinity of the travelling waves of finite energy in the Gross–Pitaevskii equation in dimension larger than two: their uniform convergence to a constant of modulus one and their asymptotic decay.

Résumé

Nous étudions la limite à l'infini des ondes progressives d'énergie finie pour les équations de Gross–Pitaevskii en dimension supérieure ou égale à deux : leur convergence uniforme vers une constante de module un et leur comportement asymptotique.

Keywords: Nonlinear Schrödinger equation; Travelling waves; Decay at infinity

Introduction

In this article, we focus on the travelling waves in the Gross–Pitaevskii equation

$$i\partial_t u = \Delta u + u\big(1 - |u|^2\big) \tag{1}$$

of the form $u(t,x) = v(x_1 - ct, \ldots, x_N)$: the parameter $c \geqslant 0$ is the speed of the travelling wave. The profile v then satisfies the equation

$$ic\partial_1 v + \Delta v + v\big(1 - |v|^2\big) = 0. \tag{2}$$

The Gross–Pitaevskii equation is a physical model for superconductivity and superfluidity associated to the energy

$$E(v) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v|^2 + \frac{1}{4}\int_{\mathbb{R}^N} \big(1 - |v|^2\big)^2 = \int_{\mathbb{R}^N} e(v). \tag{3}$$

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The non-constant travelling waves of finite energy play an important role in the long time dynamics of general solutions and were first considered by C.A. Jones and P.H. Roberts [11]: they conjectured that they only exist when $c < \sqrt{2}$ and that they are axisymmetric around axis x_1. They also proposed an asymptotic development at infinity for the travelling waves up to a multiplicative constant of modulus one. In particular, in dimension two, they conjectured that

$$v(x) - 1 \underset{|x|\to+\infty}{\sim} \frac{i\alpha x_1}{x_1^2 + (1 - \frac{c^2}{2})x_2^2} \tag{4}$$

and in dimension three, that

$$v(x) - 1 \underset{|x|\to+\infty}{\sim} \frac{i\alpha x_1}{(x_1^2 + (1 - \frac{c^2}{2})(x_2^2 + x_3^2))^{3/2}}, \tag{5}$$

where the real number α is the so-called stretched dipole coefficient.

The non-existence of non-constant travelling waves of finite energy for the case $c > \sqrt{2}$ was recently established in [10]. Therefore, we will suppose throughout that $0 \leqslant c < \sqrt{2}$. Concerning existence, F. Béthuel and J.C. Saut [1, 2] first showed the existence of travelling waves in dimension two when c is small, and also gave a mathematical evidence for their limit at infinity.

Theorem. *In dimension two, a travelling wave for the Gross–Pitaevskii equation of finite energy and of speed $0 \leqslant c < \sqrt{2}$ satisfies up to a multiplicative constant of modulus one*

$$v(x) \underset{|x|\to+\infty}{\to} 1.$$

In dimension $N \geqslant 3$, F. Béthuel, G. Orlandi and D. Smets [3] showed their existence when c is small, and in every dimension, A. Farina [8] proved a universal bound for their modulus.

In this paper, we complement the previous analysis by proving the convergence of the travelling waves at infinity in dimension $N \geqslant 3$ (see also [9]) and by giving a first estimate of their asymptotic decay, which is consistent with the conjectures (4) and (5) of C.A. Jones and P.H. Roberts [11].

More precisely, we are going to prove the following theorem.

Theorem 1. *In dimension $N \geqslant 3$, a travelling wave v for the Gross–Pitaevskii equation of finite energy and of speed $0 \leqslant c < \sqrt{2}$ satisfies up to a multiplicative constant of modulus one*

$$v(x) \underset{|x|\to+\infty}{\to} 1.$$

Moreover, in dimension $N \geqslant 2$, the function $x \mapsto |x|^{N-1}(v(x) - 1)$ is bounded on $\mathbb{R}^N$.

Remark. In view of conjectures (4) and (5) of C.A. Jones and P.H. Roberts [11], it is likely that Theorem 1 yields the optimal decay rate for $v - 1$.

However, we do not know if there is some argument which prevents the solutions to decay faster as it is the case for constant solutions. Actually, it is commonly conjectured that Theorem 1 gives the optimal decay rate of the travelling waves which are non-constant and axisymmetric around axis x_1.

We deduce immediately from Theorem 1 some integrability properties for $v - 1$.

Corollary 2. *The function $v - 1$ belongs to all the spaces $L^p(\mathbb{R}^N)$ for*

$$\frac{N}{N-1} < p \leqslant +\infty.$$

Remark. We conjecture that the function $v-1$ does not belong to $L^{\frac{N}{N-1}}(\mathbb{R}^N)$ unless it is constant.

Corollary 2 has interesting consequences in dimension $N \geqslant 3$ because, in this case, the function $v-1$ belongs to the space $L^2(\mathbb{R}^N)$, and therefore, in view of the energy bound, to the space $H^1(\mathbb{R}^N)$: thus, the function

$$(x,t) \mapsto v(x_1 - ct, x_2, \dots, x_N)$$

is solution in $C^0(\mathbb{R}, 1 + H^1(\mathbb{R}^N))$ of the Cauchy problem associated to equation (1) with the initial data

$$u(0,x) = v(x).$$

The next theorem due to F. Béthuel and J.C. Saut [1] asserts that Eq. (1) is well-posed in this space.

Theorem. *Let $v_0 \in 1 + H^1(\mathbb{R}^N)$. There is a unique solution $v \in C^0(\mathbb{R}, 1 + H^1(\mathbb{R}^N))$ of Eq.* (1).
Moreover, the energy E is conserved and the solution v depends continuously on the initial data v_0.

Therefore, we are now able to study the stability of a travelling wave in the space $1 + H^1(\mathbb{R}^N)$, and to understand better the long time dynamics of the time-dependent Gross–Pitaevskii equation.

The proof of Corollary 2 being an immediate consequence of Theorem 1, the paper is organized around the proof of Theorem 1.

In the first part, we study the local smoothness and the Sobolev regularity of a travelling wave v.

Theorem 3. *If v is a solution of finite energy of Eq.* (2) *in $L^1_{\mathrm{loc}}(\mathbb{R}^N)$, then, v is C^∞, bounded, and the functions $\eta := 1 - |v|^2$ and ∇v belong to all the spaces $W^{k,p}(\mathbb{R}^N)$ for $k \in \mathbb{N}$ and $1 < p \leqslant +\infty$.*

Remark. We do not know if the functions η and ∇v belong to some spaces $W^{k,1}(\mathbb{R}^N)$: we will only show that all the derivatives of η are in $L^1(\mathbb{R}^N)$. In fact, it is commonly conjectured that η and ∇v do not belong to $L^1(\mathbb{R}^N)$ except for the constant case, but that all their derivatives are in $L^1(\mathbb{R}^N)$ (see for example the article of C.A. Jones and P.H. Roberts [11] for more details).

By a bootstrap argument adapted from the articles of F. Béthuel and J.C. Saut [1,2], we first prove that v is C^∞ on $\mathbb{R}^N$ and that η and ∇v belong to all the L^p-spaces for $2 \leqslant p \leqslant +\infty$: it follows that the modulus ρ of v converges to 1 at infinity (see Lemma 14 in Section 1.2). In particular, there is some real number R_0 such that

$$\rho \geqslant \frac{1}{2} \quad \text{on } {}^c B_o(0, R_0).$$

We then construct a lifting θ of v on ${}^c B_o(0, R_0)$, i.e. a function in $C^\infty({}^c B_o(0, R_0), \mathbb{R})$ such that

$$v = \rho e^{i\theta}.$$

The construction is actually different in dimension $N = 2$, where it involves to determine the topological degree of the function $\frac{v}{\rho}$ at infinity, and in dimension $N \geqslant 3$ (see Lemma 15 in Section 1.2).

We next compute new equations for the new functions η and $\nabla\theta$: those functions are more suitable to study the asymptotic decay of v. In order to do so, since θ is not defined on $\mathbb{R}^N$, we introduce a cut-off function $\psi \in C^\infty(\mathbb{R}^N, [0,1])$ such that

$$\begin{cases} \psi = 0 & \text{on } B_o(0, 2R_0), \\ \psi = 1 & \text{on } {}^c B_o(0, 3R_0). \end{cases}$$

All the asymptotic estimates obtained subsequently will be independent of the choice of ψ. The functions η and $\psi\theta$ then satisfy the equations

$$\Delta^2\eta - 2\Delta\eta + c^2\partial^2_{1,1}\eta = -\Delta F - 2c\partial_1 \mathrm{div}(\mathrm{G}) \tag{6}$$

and

$$\Delta(\psi\theta) = \frac{c}{2}\partial_1\eta + \mathrm{div}(G), \tag{7}$$

where

$$F = 2|\nabla v|^2 + 2\eta^2 - 2ci\partial_1 v.v - 2c\partial_1(\psi\theta) \tag{8}$$

and

$$G = i\nabla v.v + \nabla(\psi\theta). \tag{9}$$

An important aspect of Eqs. (6) and (7) is the fact that F and G behave like quadratic functions of η and ∇v at infinity: it allows to apply the bootstrap argument in Lemma 6.

Remark. In this paragraph, we try to motivate the introduction of the lifting θ. Without lifting, Eqs. (6) and (7) may be written as

$$\begin{cases} \Delta^2\eta - 2\Delta\eta + c^2\partial_{1,1}^2\eta = -\Delta\widetilde{F} - 2c\partial_1\,\mathrm{div}(\widetilde{G}), \\ \frac{c}{2}\partial_1\eta + \mathrm{div}(\widetilde{G}) = 0, \end{cases}$$

where

$$\begin{cases} \widetilde{F} = 2|\nabla v|^2 + 2\eta^2 - 2ci\partial_1 v.v, \\ \widetilde{G} = i\nabla v.v. \end{cases}$$

However, $\widetilde{F}$ and $\widetilde{G}$ do not behave like quadratic functions of η and ∇v at infinity: for instance, at infinity, the function $\widetilde{G}$ is given by

$$\widetilde{G} = -\rho^2\nabla\theta$$

and behaves like $-\nabla\theta$. It seems rather difficult to determine the asymptotic decay of v with such an equation.

Starting with Eqs. (6) and (7), we can develop an argument due to J.L. Bona and Yi A. Li [4], and A. de Bouard and J.C. Saut [6] (see also the articles of M. Maris [13,14] for many more details): it relies on the transformation of a partial differential equation in a convolution equation. Actually, Eqs. (6) and (7) can be written as

$$\eta = K_0 * F + 2c\sum_{j=1}^{N} K_j * G_j, \tag{10}$$

where K_0 and K_j are the kernels of Fourier transformation,

$$\widehat{K_0}(\xi) = \frac{|\xi|^2}{|\xi|^4 + 2|\xi|^2 - c^2\xi_1^2}, \tag{11}$$

respectively,

$$\widehat{K_j}(\xi) = \frac{\xi_1\xi_j}{|\xi|^4 + 2|\xi|^2 - c^2\xi_1^2}, \tag{12}$$

and for every $j \in \{1,\dots,N\}$,

$$\partial_j(\psi\theta) = \frac{c}{2}K_j * F + c^2\sum_{k=1}^{N} L_{j,k} * G_k + \sum_{k=1}^{N} R_{j,k} * G_k, \tag{13}$$

where $L_{j,k}$ and $R_{j,k}$ are the kernels of Fourier transformation,

$$\widehat{L_{j,k}}(\xi) = \frac{\xi_1^2 \xi_j \xi_k}{|\xi|^2(|\xi|^4 + 2|\xi|^2 - c^2\xi_1^2)}, \tag{14}$$

respectively,

$$\widehat{R_{j,k}}(\xi) = \frac{\xi_j \xi_k}{|\xi|^2}. \tag{15}$$

Eqs. (10) and (13) seem more involved than Eq. (2), but are presumably more adapted to study the Sobolev regularity of the functions η and ∇v, as well as their decay properties. Indeed, concerning regularity, we complete the proof of Theorem 3 by showing that the kernels K_0, K_j, $L_{j,k}$ and $R_{j,k}$ are L^p-multipliers for $1 < p < +\infty$: it follows from Lizorkin's theorem [12] and standard arguments on Riesz operators (see for instance the books of J. Duoandikoetxea [7], and E.M. Stein and G. Weiss [17]). We can then deduce from Eqs. (10) and (13) that the functions η and ∇v belong to all the spaces $W^{k,p}(\mathbb{R}^N)$ for $k \in \mathbb{N}$ and $1 < p < 2$ (see Proposition 18 in Section 1.3).

Finally, we infer from Theorem 3 the convergence of the travelling waves towards a constant of modulus one at infinity (see also [9]).

Corollary 4. *In dimension $N \geqslant 3$, a travelling wave v for the Gross–Pitaevskii equation of finite energy and of speed $0 \leqslant c < \sqrt{2}$ satisfies up to a multiplicative constant of modulus one*

$$v(x) \underset{|x| \to +\infty}{\to} 1.$$

As mentioned, Eqs. (10) and (13) are also presumably more adapted to study the asymptotic decay of the functions η and ∇v. In order to clarify this claim, let us study a simple example: consider a convolution equation of the form

$$g = K * f,$$

where we suppose that the functions K and f are smooth functions. We want to estimate the algebraic decay of the function g, i.e. to determine all the indices α for which it belongs to the space

$$M_\alpha^\infty(\mathbb{R}^N) = \left\{u : \mathbb{R}^N \mapsto \mathbb{C} \mid \|u\|_{M_\alpha^\infty(\mathbb{R}^N)} = \sup\left\{|x|^\alpha \left|u(x)\right|,\ x \in \mathbb{R}^N\right\} < +\infty\right\},$$

in function of the algebraic decay of K and f. We have the following lemma.

Lemma 5. *Assume K and f are continuous functions on $\mathbb{R}^N$ which are in the space $M_{\alpha_1}^\infty(\mathbb{R}^N)$, respectively $M_{\alpha_2}^\infty(\mathbb{R}^N)$, where $\alpha_1 > N$ and $\alpha_2 > N$. Then, the function g belongs to the space $M_\alpha^\infty(\mathbb{R}^N)$ for $\alpha \leqslant \min\{\alpha_1, \alpha_2\}$.*

Proof. The proof of Lemma 5 relies on Young's inequalities

$$\begin{aligned}\forall x \in \mathbb{R}^N, \quad |x|^\alpha \left|g(x)\right| &\leqslant |x|^\alpha \int_{\mathbb{R}^N} \left|K(x-y)\right|\left|f(y)\right| dy \\ &\leqslant A \int_{\mathbb{R}^N} \left(|x-y|^\alpha \left|K(x-y)\right|\left|f(y)\right| + \left|K(x-y)\right||y|^\alpha \left|f(y)\right|\right) dy \\ &\leqslant A\left(\|K\|_{M_\alpha^\infty(\mathbb{R}^N)}\|f\|_{L^1(\mathbb{R}^N)} + \|K\|_{L^1(\mathbb{R}^N)}\|f\|_{M_\alpha^\infty(\mathbb{R}^N)}\right).\end{aligned}$$

Since $\alpha_1 > N$ and $\alpha_2 > N$, K and f belong to $L^1(\mathbb{R}^N)$: thus, if $\alpha \leqslant \min\{\alpha_1, \alpha_2\}$, the last term is finite and the function g belongs to the space $M_\alpha^\infty(\mathbb{R}^N)$. □

The assumptions $\alpha_1 > N$ and $\alpha_2 > N$ are quite restrictive, but we can generalize this method by using Young's inequalities involving not only the L^1–L^∞ estimate, but the L^p–$L^{p'}$ estimate, and determine the algebraic decay of functions which satisfy such a convolution equation.

Our situation is close to the previous example. Indeed, Eqs. (10) and (13) are of the form

$$\big(\eta, \nabla(\psi\theta)\big) = K * F\big(\eta, \nabla(\psi\theta)\big),$$

where F behaves like a quadratic function in terms of the variables η and $\nabla(\psi\theta)$.

In order to understand what happens in this case, we consider the non-linear model

$$f = K * f^2,$$

where f and K are both smooth functions. We get

Lemma 6. *Assume K and f are continuous functions on $\mathbb{R}^N$ which are in the space $M^\infty_{\alpha_1}(\mathbb{R}^N)$, respectively $M^\infty_{\alpha_2}(\mathbb{R}^N)$, where $\alpha_1 > N$, $\alpha_2 > N/2$ and $\alpha_1 > \alpha_2$. Then, the function f belongs to the space $M^\infty_\alpha(\mathbb{R}^N)$ for $\alpha \leqslant \alpha_1$.*

Proof. The proof of Lemma 6 also relies on Young's inequalities

$$\begin{aligned}
\forall x \in \mathbb{R}^N, \quad |x|^\alpha \big|f(x)\big| &\leqslant |x|^\alpha \int_{\mathbb{R}^N} \big|K(x-y)\big|\big|f(y)\big|^2\,dy \\
&\leqslant A \int_{\mathbb{R}^N} \big(|x-y|^\alpha \big|K(x-y)\big|\big|f(y)\big|^2 + \big|K(x-y)\big||y|^\alpha\big|f(y)\big|^2\big)\,dy \\
&\leqslant A\big(\|K\|_{M^\infty_\alpha(\mathbb{R}^N)}\|f\|^2_{L^2(\mathbb{R}^N)} + \|K\|_{L^1(\mathbb{R}^N)}\|f\|^2_{M^\infty_{\frac{\alpha}{2}}(\mathbb{R}^N)}\big).
\end{aligned}$$

Since $\alpha_1 > N$ and $\alpha_2 > N/2$, K and f belong to $L^1(\mathbb{R}^N)$ and $L^2(\mathbb{R}^N)$: thus, if $\alpha \leqslant \min\{\alpha_1, 2\alpha_2\}$, the last term is finite and the function f belongs to the space $M^\infty_\alpha(\mathbb{R}^N)$. By iterating this step, the function f belongs to the space $M^\infty_\alpha(\mathbb{R}^N)$ if $\alpha \leqslant \min\{\alpha_1, 2^k\alpha_2\}$ for every $k \in \mathbb{N}$, i.e. for $\alpha \leqslant \alpha_1$. □

Lemma 6 provides a striking optimal decay property for super linear equations. Indeed, assuming f possesses *some* algebraic decay, then, if f is moreover solution of such a convolution equation, it decays as fast as the kernel. However, some decay of f must be established *first*, in order to initiate the inductive argument.

Turning back to the functions η and $\nabla(\psi\theta)$ and convolution Eqs. (10) and (13), the situation is a little more involved, since we have a system of equations and since the kernels are singular at the origin. However, the conclusion is similar: **the decay of the solution is determined by the decay of the kernel**.

Thus, in our case, we will determine the decay at infinity of the kernels K_0, K_j, $L_{j,k}$ and $R_{j,k}$, *some* decay at infinity for the functions η and $\nabla(\psi\theta)$, before getting their optimal decay by the previous inductive argument.

In view of the previous discussion, the second part of the paper will be devoted to the analysis of the kernels K_0, K_j, $L_{j,k}$ and $R_{j,k}$: we will estimate their algebraic decay at the origin, where they are singular, and at infinity. It relies on three different arguments.

• We first use an L^1–L^∞ inequality, which generalizes the classical one between a function and its Fourier transformation: it follows from the next lemma which is presumably well-known to the experts.

Lemma 7. *Let $0 < s < 1$ and $\hat{f} \in S(\mathbb{R}^N)$. Then, the function $x \mapsto |x|^s f(x)$ is in $C^0_0(\mathbb{R}^N) := \{g \in C^0(\mathbb{R}^N) \mid g(x) \to_{|x|\to+\infty} 0\}$, and satisfies for every $x \in \mathbb{R}^N$,*

$$|x|^s f(x) = I_N \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\hat{f}(y) - \hat{f}(z)}{|y-z|^{N+s}} e^{ix.y}\,dy\,dz, \tag{16}$$

where we denote

$$I_N = -\left((2\pi)^{N+1} \int_0^{+\infty} \left(J_{\frac{N}{2}-1}(2\pi u) - \frac{\pi^{\frac{N}{2}-1}}{\Gamma(\frac{N}{2})} u^{\frac{N}{2}-1}\right) u^{-\frac{N}{2}-s}\, du\right)^{-1} > 0,$$

and where $J_{\frac{N}{2}-1}$ is the Bessel function defined by

$$\forall u \in \mathbb{R}, \quad J_{\frac{N}{2}-1}(u) = \left(\frac{u}{2}\right)^{\frac{N}{2}-1} \sum_{n=0}^{+\infty} \frac{(-1)^n u^{2n}}{4^n n! \Gamma(n + \frac{N}{2})}.$$

We deduce from Lemma 7 the following theorem.

Theorem 8. *Let $N-2 < \alpha < N$, $n \in \mathbb{N}$ and $(j,k) \in \{1,\ldots,N\}^2$. The functions $d^n K_0$, $d^n K_j$ and $d^n L_{j,k}$ belong to $M^\infty_{\alpha+n}(\mathbb{R}^N)$.*

• We then prove independently that all those functions are bounded even in the critical case, i.e. when $\alpha = N$. This is done by another duality argument in $S'(\mathbb{R}^N)$, and by a standard integration by parts.

Theorem 9. *Let $n \in \mathbb{N}$ and $(j,k) \in \{1,\ldots,N\}^2$. The functions $d^n K_0$, $d^n K_j$ and $d^n L_{j,k}$ belong to $M^\infty_{N+n}(\mathbb{R}^N)$.*

Remark. We conjecture Theorem 9 is optimal, i.e. the functions $|.|^{\alpha+n} d^n K_0$, $|.|^{\alpha+n} d^n K_j$ and $|.|^{\alpha+n} d^n L_{j,k}$ are not bounded on $\mathbb{R}^N$ for $\alpha > N$.

• Finally, we study what we shall call the composed Riesz kernels, i.e. the kernels $R_{j,k}$. We exactly know their form by standard Riesz operator theory (see for example the books of J. Duoandikoetxea [7], and E.M. Stein and G. Weiss [17]). If f is a smooth function and if we denote $g_{j,k} = R_{j,k} * f$ for every $(j,k) \in \{1,\ldots,N\}^2$, we have the formula

$$\forall x \in \mathbb{R}^N, \quad g_{j,k}(x) = A_N \int_{|y|>1} \frac{\delta_{j,k}|y|^2 - N y_j y_k}{|y|^{N+2}} f(x-y)\, dy + A_N \int_{|y|\leqslant 1} \frac{\delta_{j,k}|y|^2 - N y_j y_k}{|y|^{N+2}} \big(f(x-y) - f(x)\big)\, dy. \tag{17}$$

Therefore, in this section, we do not study the decay of the kernels $R_{j,k}$ at infinity, but directly, the decay of the functions $g_{j,k}$, when the function f belongs to $L^1(\mathbb{R}^N)$ and the functions $|.|^\alpha f$ and $|.|^\alpha \nabla f$ are bounded for some positive number α.

In the third part, we turn to the decay of the functions η and ∇v at infinity: we first give a refined energy estimate due to F. Béthuel, G. Orlandi and D. Smets [3].

Lemma 10. *Let v, a solution of finite energy of Eq.* (2) *in $L^1_{\mathrm{loc}}(\mathbb{R}^N)$. For every $0 \leqslant c < \sqrt{2}$, there is a strictly positive constant α_c such that the function*

$$R \to R^{\alpha_c} \int_{B(0,R)^c} e(v)$$

is bounded on $\mathbb{R}_+$.

It is the starting point of the whole study of the decay of v at infinity. Indeed, it enables to prove *some* algebraic decay for the functions η and ∇v, which leads to the following theorem by the inductive method yet mentioned.

Theorem 11. *Let $\alpha \in \mathbb{N}^N$. Then, the functions η, $\nabla(\psi\theta)$ and ∇v satisfy*

$$\begin{cases} (\eta, \partial^\alpha \nabla(\psi\theta), \partial^\alpha \nabla v) \in M_N^\infty(\mathbb{R}^N)^3, \\ \partial^\alpha \nabla \eta \in M_{N+1}^\infty(\mathbb{R}^N). \end{cases}$$

Remark. The key result of Theorem 11 is that the algebraic decay of the functions η, $\nabla\eta$ and $\nabla(\psi\theta)$ is imposed by the kernels of the equations they satisfy: we believe that Theorem 11 is optimal for $\alpha = 0$, but not for higher derivatives. The functions $\partial^\alpha\eta$, $\partial^\alpha\nabla(\psi\theta)$ and $\partial^\alpha\nabla v$ are commonly supposed to belong to $M_{N+|\alpha|}^\infty(\mathbb{R}^N)$.

As mentioned, we can deduce from Theorem 11 some integrability for the derivatives of the function η.

Corollary 12. *Let $\alpha \in \mathbb{N}^N$. Then, $\partial^\alpha \nabla\eta \in L^1(\mathbb{R}^N)$.*

The proof of Corollary 12 being an immediate consequence of Theorems 3 and 11, we will omit it, and instead, we will conclude the paper by proving the asymptotic estimate of Theorem 1 for $v - 1$.

1. Regularity and convergence at infinity of travelling waves for the Gross–Pitaevskii equation

The first part is devoted to the proofs of Theorem 3 and Corollary 4, i.e. to determine the Sobolev regularity and the convergence at infinity of a travelling wave v of finite energy and of speed $0 \leqslant c < \sqrt{2}$ in dimension $N \geqslant 2$ (see also [9]).

The proofs essentially stem from the articles of F. Béthuel and J.C. Saut [1,2], and are based on Eqs. (10) and (13): we first determine the Sobolev regularity of η and ∇v for Sobolev exponents $p \in [2, +\infty]$. We then derive properly Eqs. (10) and (13) by introducing some lifting θ of v. This yields the Sobolev regularity of η and ∇v for Sobolev exponents $p \in]1, 2[$ by using some Fourier multiplier theory. At last, Corollary 4 follows from a general argument connecting the existence of a limit at infinity for some function with its Sobolev regularity (see Proposition 19 in Section 1.4).

1.1. L^p-integrability for $2 \leqslant p \leqslant +\infty$

We first prove the Sobolev regularity of η and ∇v for Sobolev exponents $p \in [2, +\infty]$. The following proposition holds even if $c \geqslant \sqrt{2}$.

Proposition 13. *If v is a solution of finite energy of Eq.* (2) *in $L^1_{\mathrm{loc}}(\mathbb{R}^N)$, then the function v is C^∞, bounded, and the functions η and ∇v belong to all the spaces $W^{k,p}(\mathbb{R}^N)$ for $k \in \mathbb{N}$ and $2 \leqslant p \leqslant +\infty$.*

Proof. We only prove Proposition 13 in dimension three because the general proof is identical with small changes of Sobolev indices. The proof is adapted from the article of F. Béthuel and J.C. Saut [1], where it is written in dimension two. It is based on a bootstrap argument.

We first consider a point z_0 in $\mathbb{R}^3$ and we denote Ω, the unit ball with center z_0. Then, we consider the solutions v_1 and v_2 of the equations

$$\begin{cases} \Delta v_1 = 0 & \text{on } \Omega, \\ v_1 = v & \text{on } \partial\Omega, \end{cases}$$

and

$$\begin{cases} -\Delta v_2 = v(1-|v|^2) + ic\partial_1 v := g(v) & \text{on } \Omega, \\ v_2 = 0 & \text{on } \partial\Omega. \end{cases}$$

Since the energy $E(v)$ of v is finite, v is uniformly bounded in $L^4(\Omega)$, which means that the norm of v in $L^4(\Omega)$ is finite and bounded by a constant which only depends on c and $E(v)$, but not on z_0. Thus, $v(1-|v|^2)$ is uniformly bounded in $L^{\frac{4}{3}}(\Omega)$, and likewise, $\partial_1 v$ is also uniformly bounded in $L^{\frac{4}{3}}(\Omega)$, such as $g(v)$. By standard elliptic theory, v_2 is then uniformly bounded in $W^{2,\frac{4}{3}}(\Omega)$, and by Sobolev embeddings, v_1 is uniformly bounded in $L^4(\Omega)$.

If we denote ω, the ball with center z_0 and with radius $\frac{1}{2}$, then, by Caccioppoli inequalities, v_1 is uniformly bounded in $W^{3,\frac{4}{3}}(\omega)$: thus, v is uniformly bounded in $W^{2,\frac{4}{3}}(\omega)$, and, by Sobolev embeddings, in $L^{12}(\omega)$.

Furthermore, we compute

$$\forall j \in \{1,2,3\}, \quad \partial_j g(v) = \partial_j v\big(1-|v|^2\big) - 2(v.\partial_j v)v + ic\partial_{1,j}^2 v.$$

So, $\partial_j g(v)$ is uniformly bounded in $L^{\frac{4}{3}}(\omega)$, and by standard elliptic theory, v_2 and v are uniformly bounded in $W^{3,\frac{4}{3}}(\omega)$. Finally, by Sobolev embeddings once more, v is uniformly bounded in $C^{0,\frac{3}{4}}(\omega)$: therefore, v is continuous and bounded on $\mathbb{R}^3$.

However, its gradient $w = \nabla v$ satisfies

$$-\Delta w - ic\partial_1 w + \left(\frac{c^2}{2} + 2\right) w = w\big(1-|v|^2\big) - 2(v.w)v + \left(\frac{c^2}{2} + 2\right) w := h(w),$$

and $h(w)$ belongs to $L^2(\mathbb{R}^3)$, which proves that w belongs to $H^2(\mathbb{R}^3)$. So, w is continuous and bounded, and by iterating, we conclude that v is C^∞, bounded and that all its derivatives belong to the spaces $L^2(\mathbb{R}^3)$ and $L^\infty(\mathbb{R}^3)$. Proposition 13 then follows from a standard interpolation between L^p-spaces. □

Remark. Proposition 13 shows that every weak solution of finite energy of Eq. (2) is a classical solution.

1.2. Convolution equations

In this section, we establish the convolution equations, i.e. Eqs. (10) and (13): we will use them to complete the study of the Sobolev regularity of the travelling waves, and to determine their decay at infinity.

We first construct a lifting θ of v: in order to do so, we first prove that v does not vanish at infinity. It follows from Proposition 13.

Lemma 14. *The modulus ρ of v and all its derivatives $\partial^\alpha v$ satisfy*

$$\begin{cases} \rho(x) \underset{|x|\to+\infty}{\to} 1, \\ \partial^\alpha v(x) \underset{|x|\to+\infty}{\to} 0. \end{cases}$$

Remark. Lemma 14 holds even if $c \geqslant \sqrt{2}$.

Proof. Indeed, on one hand, v is bounded and lipschitzian by Proposition 13, so, η^2 is uniformly continuous on $\mathbb{R}^N$: as $\int_{\mathbb{R}^N} \eta^2$ is finite, we get

$$\eta(x) \underset{|x|\to+\infty}{\to} 0,$$

which gives

$$\rho(x) \underset{|x|\to+\infty}{\to} 1.$$

On the other hand, ∇v belongs to all the spaces $W^{k,p}(\mathbb{R}^N)$ for every $k \in \mathbb{N}$ and $p \in [2, +\infty]$, so, $\partial^\alpha v$ is uniformly continuous and satisfies

$$\int_{\mathbb{R}^N} |\partial^\alpha v|^2 < +\infty,$$

and we get likewise

$$\partial^\alpha v(x) \underset{|x|\to+\infty}{\to} 0. \qquad \square$$

Therefore, v does not vanish at the neighbourhood of infinity, and we can construct a smooth lifting of v there.

Lemma 15. *There is some real number $R_0 \geqslant 0$ and a function $\theta \in C^\infty({}^c B_o(0, R_0), \mathbb{R})$ such that*

$$v = \rho e^{i\theta} \quad \text{on } {}^c B_o(0, R_0).$$

Remark. Lemma 15 holds even if $c \geqslant \sqrt{2}$.

Proof. By Lemma 14, there is some real number $R_0 \geqslant 0$ such that ρ satisfies

$$\rho \geqslant \frac{1}{2} \quad \text{on } {}^c B_o(0, R_0).$$

Thus, the map $v/|v|$ is a C^∞ function from ${}^c B_o(0, R_0)$ to the circle $\mathbb{S}^1$.

In dimension $N \geqslant 3$, the fundamental group $\pi_1(\mathbb{S}^{N-1})$ of the sphere $\mathbb{S}^{N-1}$ is reduced to $\{0\}$, and therefore, there is a function $\theta \in C^\infty({}^c B_o(0, R_0), \mathbb{R})$ such that

$$v = |v| e^{i\theta} = \rho e^{i\theta}.$$

In dimension $N = 2$, the fundamental group $\pi_1(\mathbb{S}^1)$ of the circle $\mathbb{S}^1$ is $\mathbb{Z}$: so, there is a function $\theta \in C^\infty({}^c B_o(0, R_0), \mathbb{R})$ such that v is equal to $|v| e^{i\theta}$ on ${}^c B_o(0, R_0)$, if and only if the topological degree of v on the circle $S(0, R_0)$ is 0.

Let us denote $d \in \mathbb{Z}$, the topological degree of v on this circle. Since v does not vanish on ${}^c B_o(0, R_0)$, d is the degree of v on each circle $S(0, R)$ for every $R \geqslant R_0$, and we get

$$2\pi d R = -\int_{S(0,R)} i\partial_\tau\left(\frac{v}{|v|}\right)(\xi).\frac{v(\xi)}{|v(\xi)|}\,d\xi = -\int_{S(0,R)} \frac{i\partial_\tau v(\xi).v(\xi)}{|v(\xi)|^2}\,d\xi,$$

whence

$$|d| \leqslant \frac{1}{2\pi R}\int_{S(0,R)} \frac{|\partial_\tau v(\xi)|}{|v(\xi)|}\,d\xi \leqslant \frac{1}{\pi R}\int_{S(0,R)} \left|\nabla v(\xi)\right| d\xi \leqslant \sqrt{\frac{2}{\pi R}}\left(\int_{S(0,R)} \left|\nabla v(\xi)\right|^2 d\xi\right)^{1/2}.$$

Since ∇v belongs to $L^2(\mathbb{R}^N)$, there is some real number $R > \max\{1, R_0\}$ such that

$$\int_{S(0,R)} \left|\nabla v(\xi)\right|^2 d\xi \leqslant 1,$$

which gives

$$|d| \leqslant \sqrt{\frac{2}{\pi}} < 1.$$

As $d \in \mathbb{Z}$, it yields

$$d = 0,$$

and there is a function $\theta \in C^\infty({}^cB_o(0, R_0), \mathbb{R})$ such that

$$v = \rho e^{i\theta}. \qquad \square$$

Now, we can compute Eqs. (6) and (7) on $\mathbb{R}^N$: thus, we introduce a cut-off function $\psi \in C^\infty(\mathbb{R}^N, [0, 1])$ such that

$$\begin{cases} \psi = 0 & \text{on } B_o(0, 2R_0), \\ \psi = 1 & \text{on } {}^cB_o(0, 3R_0), \end{cases}$$

and we then prove

Proposition 16. *If $v := v_1 + iv_2$ is a solution of finite energy of Eq.* (2) *in $L^1_{\mathrm{loc}}(\mathbb{R}^N)$, the functions η and $\psi\theta$ satisfy the equations*

$$\Delta^2\eta - 2\Delta\eta + c^2\partial^2_{1,1}\eta = -\Delta F - 2c\partial_1 \operatorname{div}(G), \tag{6}$$

and

$$\Delta(\psi\theta) = \frac{c}{2}\partial_1\eta + \operatorname{div}(G), \tag{7}$$

where

$$F = 2|\nabla v|^2 + 2\eta^2 + 2c(v_1\partial_1 v_2 - v_2\partial_1 v_1) - 2c\partial_1(\psi\theta) \tag{8}$$

and

$$G = -v_1\nabla v_2 + v_2\nabla v_1 + \nabla(\psi\theta). \tag{9}$$

Remark. Proposition 16 holds even if $c \geqslant \sqrt{2}$.

Proof. Denoting $v = v_1 + iv_2$, we have by Eq. (2)

$$\Delta v_1 - c\partial_1 v_2 + v_1\big(1 - |v|^2\big) = 0, \tag{18}$$

$$\Delta v_2 + c\partial_1 v_1 + v_2\big(1 - |v|^2\big) = 0. \tag{19}$$

We then compute

$$\Delta^2\eta - 2\Delta\eta + c^2\partial^2_{1,1}\eta = -2\Delta|\nabla v|^2 - 2\Delta(v.\Delta v) - 2\Delta\eta + c^2\partial^2_{1,1}\eta.$$

By Eqs. (18) and (19), we have on one hand

$$v.\Delta v = v_1\Delta v_1 + v_2\Delta v_2 = c(v_1\partial_1 v_2 - v_2\partial_1 v_1) - |v|^2\eta,$$

and on the other hand,

$$c\partial_1\eta = -2c(v_1\partial_1 v_1 + v_2\partial_1 v_2) = 2(\Delta v_2 v_1 - \Delta v_1 v_2) = 2\operatorname{div}(\nabla v_2 v_1 - \nabla v_1 v_2). \tag{20}$$

Therefore, we get

$$\begin{aligned} \Delta^2\eta - 2\Delta\eta + c^2\partial^2_{1,1}\eta &= -2\Delta|\nabla v|^2 - 2\Delta\eta^2 - 2c\Delta(v_1\partial_1 v_2 - v_2\partial_1 v_1) + 2c\partial_1\operatorname{div}(v_1\nabla v_2 - v_2\nabla v_1) \\ &= -\Delta\big(2|\nabla v|^2 + 2\eta^2 + 2c(v_1\partial_1 v_2 - v_2\partial_1 v_1) - 2c\partial_1(\psi\theta)\big) \\ &\quad + 2c\partial_1 \operatorname{div}\big(v_1\nabla v_2 - v_2\nabla v_1 - \nabla(\psi\theta)\big) \\ &= -\Delta F - 2c\partial_1 \operatorname{div}(G), \end{aligned}$$

which gives Eq. (6).

For Eq. (7), we introduce the function $\psi\theta$ in Eq. (20) and we get

$$\Delta(\psi\theta) = \frac{c}{2}\partial_1\eta + \operatorname{div}\big(\nabla v_1 v_2 - \nabla v_2 v_1 + \nabla(\psi\theta)\big) = \frac{c}{2}\partial_1\eta + \operatorname{div}(G). \qquad \square$$

Finally, so as to study Eqs. (6) and (7), we transform them in convolution equations.

Proposition 17. *The functions η and $\nabla(\psi\theta)$ satisfy the equations*

$$\eta = K_0 * F + 2c\sum_{j=1}^{N} K_j * G_j, \tag{10}$$

$$\partial_j(\psi\theta) = \frac{c}{2}K_j * F + c^2\sum_{k=1}^{N} L_{j,k} * G_k + \sum_{k=1}^{N} R_{j,k} * G_k, \tag{13}$$

where K_0, K_j, $L_{j,k}$ and $R_{j,k}$ are the kernels of Fourier transformation,

$$\widehat{K_0}(\xi) = \frac{|\xi|^2}{|\xi|^4 + 2|\xi|^2 - c^2\xi_1^2}, \tag{11}$$

$$\widehat{K_j}(\xi) = \frac{\xi_1\xi_j}{|\xi|^4 + 2|\xi|^2 - c^2\xi_1^2}, \tag{12}$$

$$\widehat{L_{j,k}}(\xi) = \frac{\xi_1^2\xi_j\xi_k}{|\xi|^2(|\xi|^4 + 2|\xi|^2 - c^2\xi_1^2)}, \tag{14}$$

$$\widehat{R_{j,k}}(\xi) = \frac{\xi_j\xi_k}{|\xi|^2}. \tag{15}$$

Though Eqs. (10) and (13) look rather involved than Eq. (2), they simplify a lot the study of the regularity and of the decay of v in the next sections.

1.3. L^p-integrability for $1 < p < 2$

In this section, we achieve the proof of Theorem 3 by proving the following proposition in the case $c < \sqrt{2}$.

Proposition 18. *If v is a solution of finite energy of Eq.* (2) *in $L^1_{\mathrm{loc}}(\mathbb{R}^N)$, then the functions η and ∇v belong to all the spaces $W^{k,p}(\mathbb{R}^N)$ for $k \in \mathbb{N}$ and $1 < p < 2$.*

Proof. The proof is adapted from an article of F. Béthuel and J.C. Saut [2] and based on Eqs. (10) and (13). We first study the Sobolev regularity of the functions F and G for Sobolev exponents $p \in [1, +\infty]$.

Step 1. *F and G belong to all the spaces $W^{k,p}(\mathbb{R}^N)$ for $k \in \mathbb{N}$ and $1 \leqslant p \leqslant +\infty$.*

By formulae (8) and (9), F and G are equal to

$$F = 2|\nabla v|^2 + 2\eta^2 + 2c(v_1\partial_1 v_2 - v_2\partial_1 v_1) - 2c\partial_1(\psi\theta)$$

and

$$G = -v_1\nabla v_2 + v_2\nabla v_1 + \nabla(\psi\theta).$$

So, by Proposition 13, they are C^∞ on $\mathbb{R}^N$, and it is sufficient to prove that they belong to all the spaces $W^{k,p}({}^cB_o(0,3R_0))$ for $k \in \mathbb{N}$ and $1 \leqslant p \leqslant +\infty$.

On the set ${}^cB_o(0,3R_0)$, F is equal to

$$F = 2|\nabla v|^2 + 2\eta^2 - 2c\eta\partial_1\theta.$$

On one hand, by Proposition 13, η and ∇v belong to all the spaces $W^{k,p}(\mathbb{R}^N)$ for $k \in \mathbb{N}$ and $2 \leqslant p \leqslant +\infty$.

On the other hand, ρ is higher than $\frac{1}{2}$ on the set ${}^cB_o(0,3R_0)$ by definition of R_0 (see the proof of Lemma 15), and v belongs to all the spaces $W^{k,\infty}(\mathbb{R}^N)$ for $k \in \mathbb{N}$: therefore, the map $\nabla(\psi\theta)$, given by

$$\nabla(\psi\theta) = \frac{iv.\nabla v}{|v|^2}$$

at infinity, also belongs to all the spaces $W^{k,p}({}^cB_o(0,3R_0))$ for $k \in \mathbb{N}$ and $2 \leqslant p \leqslant +\infty$.
As F is a quadratic function of η, $\nabla(\psi\theta)$ and ∇v, it is in all the spaces $W^{k,p}({}^cB_o(0,3R_0))$ for $k \in \mathbb{N}$ and $1 \leqslant p \leqslant +\infty$.

Likewise, the function G is given by

$$G = \eta\nabla(\psi\theta)$$

on the set ${}^cB_o(0,3R_0)$, and it is also a quadratic function of η and $\nabla(\psi\theta)$: thus, G belongs to all the spaces $W^{k,p}({}^cB_o(0,3R_0))$ for $k \in \mathbb{N}$ and $1 \leqslant p \leqslant +\infty$.

We then establish a first property of the Gross–Pitaevskii kernels K_0, K_j, $L_{j,k}$ and $R_{j,k}$.

Step 2. *The functions $\widehat{K_0}$, $\widehat{K_j}$, $\widehat{L_{j,k}}$ and $\widehat{R_{j,k}}$ are L^p-multipliers for $1 < p < +\infty$.*

Step 2 follows from Lizorkin's theorem [12].

Lizorkin's theorem. *Let $\widehat{K}$ a bounded function in $C^N(\mathbb{R}^N \setminus \{0\})$ and assume*

$$\prod_{j=1}^N (\xi_j^{k_j})\partial_1^{k_1}\dots\partial_N^{k_N}\widehat{K}(\xi) \in L^\infty(\mathbb{R}^N)$$

as soon as $(k_1,\dots,k_N) \in \{0,1\}^N$ satisfies

$$0 \leqslant \sum_{j=1}^N k_j \leqslant N.$$

Then, $\widehat{K}$ is a L^p-multiplier for $1 < p < +\infty$.

By a straightforward computation, $\widehat{K_0}$, $\widehat{K_j}$ and $\widehat{L_{j,k}}$ satisfy all the hypothesis of Lizorkin's theorem, and so, they are L^p-multipliers for $1 < p < +\infty$.

By standard Riesz operator theory, the functions $\widehat{R_{j,k}}$ are L^p-multipliers too (see for example the books of J. Duoandikoetxea [7] and E.M. Stein and G. Weiss [17]).

Step 3. *η and $\nabla(\psi\theta)$ belong to all the spaces $W^{k,p}(\mathbb{R}^N)$ for $k \in \mathbb{N}$ and $1 < p < 2$.*

By Steps 1 and 2, and Eqs. (10) and (13), η and $\nabla(\psi\theta)$ belong to $L^p(\mathbb{R}^N)$ for $1 < p < 2$. We then iterate the proof for all the derivatives of η and $\nabla(\psi\theta)$ using the equations

$$\partial^\alpha \eta = K_0 * \partial^\alpha F + 2c \sum_{j=1}^N K_j * \partial^\alpha G_j, \tag{21}$$

$$\partial^\alpha \partial_j (\psi\theta) = \frac{c}{2} K_j * \partial^\alpha F + c^2 \sum_{k=1}^N L_{j,k} * \partial^\alpha G_k + \sum_{k=1}^N R_{j,k} * \partial^\alpha G_k, \tag{22}$$

for every $\alpha \in \mathbb{N}^N$. By Step 1, $\partial^\alpha F$ and $\partial^\alpha G$ belong to all the spaces $L^p(\mathbb{R}^N)$ for $1 \leqslant p \leqslant +\infty$: Step 3 then follows from Step 2 and Eqs. (21) and (22).

Step 4. *∇v belongs to all the spaces $W^{k,p}(\mathbb{R}^N)$ for $k \in \mathbb{N}$ and $1 < p < 2$.*

The function v being C^∞ on $\mathbb{R}^N$ by Proposition 13, it is sufficient to prove that ∇v belongs to all the spaces $W^{k,p}({}^c B_o(0, 3R_0))$ for $k \in \mathbb{N}$ and $1 < p < 2$.

In order to do so, we first claim that $\nabla\rho$ belongs to the spaces $W^{k,p}({}^c B_o(0, 3R_0))$ for $k \in \mathbb{N}$ and $1 < p < 2$: indeed, ρ is given by

$$\rho = \sqrt{1-\eta}.$$

By Lemma 14, η is higher than $\frac{3}{4}$ on the set ${}^c B_o(0, 3R_0)$, so, by Step 3 and by the L^p-chain rule theorem, $\nabla\rho$ belongs to all the spaces $W^{k,p}({}^c B_o(0, 3R_0))$ for $k \in \mathbb{N}$ and $1 < p < 2$.

Thus, ρ and $\nabla(\psi\theta)$ belong to all the spaces $W^{k,\infty}({}^c B_o(0, 3R_0))$ for $k \in \mathbb{N}$, and $\nabla\rho$ and $\nabla(\psi\theta)$ belong to all the spaces $W^{k,p}({}^c B_o(0, 3R_0))$ for $k \in \mathbb{N}$ and $1 < p < 2$: since ∇v is given by

$$\nabla v = \nabla\rho e^{i\psi\theta} + i\rho\nabla(\psi\theta)e^{i\psi\theta}$$

at infinity, by Leibnitz's formula and by the L^p-chain rule theorem, ∇v belongs to all the spaces $W^{k,p}({}^c B_o(0, 3R_0))$ for $k \in \mathbb{N}$ and $1 < p < 2$. □

1.4. Convergence at infinity in dimension $N \geqslant 3$

We now deduce Corollary 4 from Theorem 3: indeed, by the following proposition, the convergence at infinity of a travelling wave v follows from its regularity.

Proposition 19. *Let $v \in C^2(\mathbb{R}^N)$, and suppose that $N \geqslant 3$ and that the gradient of v belongs to the spaces $W^{1,p_0}(\mathbb{R}^N)$ and $W^{1,p_1}(\mathbb{R}^N)$, where*

$$1 < p_0 < N - 1 < p_1 < +\infty.$$

Then, there is a constant $v_\infty \in \mathbb{C}$ such that

$$v(x) \underset{|x| \to +\infty}{\to} v_\infty.$$

Proof. Proposition 19 relies on a radial construction of the limit v_∞: we focus on the functions $(v_r)_{r>0}$ defined by

$$\forall \xi \in \mathbb{S}^{N-1}, \quad v_r(\xi) = v(r\xi).$$

We first prove their convergence almost everywhere towards a measurable function v_∞ on $\mathbb{S}^{N-1}$ when r tends to $+\infty$. Then, we show the uniformity of this convergence by a standard embedding theorem involving Lorentz spaces, and we conclude by showing that v_∞ is a constant function.

At first, we construct the limit v_∞: we compute

$$\int_{\mathbb{S}^{N-1}} \int_1^{+\infty} \left|\partial_r v(r\xi)\right| dr\, d\sigma \leqslant \int_{\mathbb{S}^{N-1}} \left(\int_1^{+\infty} \left|\nabla v(r\xi)\right|^{p_0} r^{N-1}\, dr \right)^{\frac{1}{p_0}} \left(\int_1^{+\infty} r^{-\frac{N-1}{p_0-1}}\, dr \right)^{\frac{1}{p_0'}} d\sigma$$

$$\leqslant A_{N,p_0} \left(\int_{{}^c B_o(0,1)} \left|\nabla v(x)\right|^{p_0} dx \right)^{\frac{1}{p_0}} < +\infty,$$

and therefore,

$$\int_1^{+\infty} \left|\partial_r v(r\xi)\right| dr < +\infty \quad \text{a.e.}$$

Hence, there is a measurable function v_∞ on $\mathbb{S}^{N-1}$ such that

$$v_r(\xi) \underset{r\to+\infty}{\to} v_\infty(\xi) \quad \text{a.e.}$$

We now claim

Lemma 20. *v_∞ is the limit in $L^\infty(\mathbb{S}^{N-1})$ of the functions $(v_r)_{r>0}$ when r tends to $+\infty$, i.e.*

$$\|v_r - v_\infty\|_{L^\infty(\mathbb{S}^{N-1})} \underset{r\to+\infty}{\to} 0.$$

Indeed, denote

$$\forall p \in [p_0, p_1], \quad \forall r > 0,\, I_p(r) = r^{N-1} \int_{\mathbb{S}^{N-1}} \left|\nabla v(r\xi)\right|^p d\sigma.$$

The function I_p is C^1 on $\mathbb{R}_+^*$ and its derivative satisfies

$$\forall r > 0, \quad \left|I_p'(r)\right| \leqslant (N-1) r^{N-2} \int_{\mathbb{S}^{N-1}} \left|\nabla v(r\xi)\right|^p d\sigma + p r^{N-1} \int_{\mathbb{S}^{N-1}} \left|\nabla v(r\xi)\right|^{p-1} \left|\partial_r \nabla v(r\xi)\right| d\sigma,$$

so,

$$\int_0^{+\infty} \left|I_p'(r)\right| dr \leqslant A\left(\|\nabla v\|_{L^p(\mathbb{R}^N)}^p + \|\nabla v\|_{L^p(\mathbb{R}^N)}^{p-1} \|\nabla v\|_{W^{1,p}(\mathbb{R}^N)}\right) < +\infty.$$

Hence, I_p has a limit at $+\infty$, and since

$$\int_0^{+\infty} I_p(r)\, dr = \|\nabla v\|_{L^p(\mathbb{R}^N)}^p < +\infty,$$

this limit is zero.

Furthermore, we notice that

$$\left|\nabla v(r\xi)\right|^2 = \left|\partial_r v(r\xi)\right|^2 + r^{-2} \left|\nabla^{\mathbb{S}^{N-1}} v_r(\xi)\right|^2,$$

where $\nabla^{\mathbb{S}^{N-1}} v_r$ denotes the gradient of the function v_r on the sphere $\mathbb{S}^{N-1}$. It yields

$$r^{N-1-p} \int\limits_{\mathbb{S}^{N-1}} \left|\nabla^{\mathbb{S}^{N-1}} v_r(\xi)\right|^p d\sigma \underset{r\to+\infty}{\to} 0. \tag{23}$$

So, we know at least partly the L^p-convergence of the gradients of the functions v_r: we now estimate the L^q-convergence of the functions v_r to prove their uniform convergence by using embedding theorems.

Thus, if $p_0 \leqslant q < \min\{p_1, N\}$, we get for every $r > 0$,

$$\begin{aligned}\int\limits_{\mathbb{S}^{N-1}} \left|v_r(\xi) - v_\infty(\xi)\right|^q d\sigma &\leqslant \int\limits_{\mathbb{S}^{N-1}} \left(\int\limits_r^{+\infty} \left|\partial_r v(s\xi)\right| ds\right)^q d\sigma \\ &\leqslant \left(\frac{q-1}{N-q}\right)^{q-1} \int\limits_{\mathbb{S}^{N-1}} r^{q-N} \int\limits_r^{+\infty} \left|\nabla v(s\xi)\right|^q s^{N-1}\, ds\, d\sigma \\ &\leqslant A_{N,q} \|\nabla v\|^q_{L^q(\mathbb{R}^N)} r^{q-N}. \end{aligned} \tag{24}$$

By assertions (23) and (24), the functions v_r converge to v_∞ in $L^q(\mathbb{S}^{N-1})$ for every $q \in [p_0, \min\{p_1, N\}[$, and their gradient converge to 0 in $L^q(\mathbb{S}^{N-1})$ for every $q \in [p_0, N-1]$: hence, the functions v_r converge to v_∞ in $W^{1,q}(\mathbb{S}^{N-1})$ for every $q \in [p_0, N-1]$, and since their gradient converge to 0, the gradient of v_∞ in $\mathcal{D}'(\mathbb{S}^{N-1})$ is 0, i.e. the function v_∞ is constant.

Actually, by standard Sobolev embedding theorem, the spaces $W^{1,q}(\mathbb{S}^{N-1})$ do not embed in $L^\infty(\mathbb{S}^{N-1})$ for any $q \in [p_0, N-1]$: that is the reason why we introduce the Lorentz space $L^{N-1,1}(\mathbb{S}^{N-1})$.

At first, let us recall briefly the definition of this space: we consider a measurable function f on $\mathbb{S}^{N-1}$ and we define its distribution function λ_f by

$$\forall t > 0, \quad \lambda_f(t) := \mu\left(\left\{\xi \in \mathbb{S}^{N-1}, \left|f(\xi)\right| > t\right\}\right),$$

where μ is the standard measure of $\mathbb{S}^{N-1}$, and its decreasing rearrangement f^* by

$$\forall t > 0, \quad f^*(t) := \inf\left\{s > 0,\ \lambda_f(s) \leqslant t\right\}.$$

The Lorentz space $L^{N-1,1}(\mathbb{S}^{N-1})$ is the set of all measurable functions f such that

$$\|f\|_{L^{N-1,1}(\mathbb{S}^{N-1})} := \int\limits_0^{+\infty} t^{\frac{1}{N-1}-1} f^*(t)\, dt < +\infty.$$

The interest of this space relies on the theorem of A. Cianchi and L. Pick [5].

Cianchi and Pick's theorem. *Denote*

$$W\left(L^{N-1,1}(\mathbb{S}^{N-1})\right) := \left\{u \in L^{N-1,1}(\mathbb{S}^{N-1}), \nabla^{\mathbb{S}^{N-1}} u \in L^{N-1,1}(\mathbb{S}^{N-1})\right\}.$$

Then,

$$W(\mathbb{S}^{N-1}) \hookrightarrow L^\infty(\mathbb{S}^{N-1}),$$

i.e. there is some constant $C > 0$ such that for every function $f \in W(L^{N-1,1}(\mathbb{S}^{N-1}))$,

$$\|f\|_{L^\infty(\mathbb{S}^{N-1})} \leqslant C\left(\|f\|_{L^{N-1,1}(\mathbb{S}^{N-1})} + \left\|\nabla^{\mathbb{S}^{N-1}} f\right\|_{L^{N-1,1}(\mathbb{S}^{N-1})}\right).$$

Remark. In fact, A. Cianchi and L. Pick [5] proved a stronger result (Theorem 3.5 and Remark 3.7 there), which is not used here, but which explains why we introduce the Lorentz space $L^{N-1,1}(\mathbb{S}^{N-1})$.

Let X, a rearrangement-invariant Banach function space on the sphere $\mathbb{S}^{N-1}$, and denote

$$W(X) = \left\{u \in X, \nabla^{\mathbb{S}^{N-1}} u \in X\right\}.$$

Then, $W(X)$ embeds in $L^\infty(\mathbb{S}^{N-1})$ if and only if

$$X \hookrightarrow L^{N-1,1}(\mathbb{S}^{N-1}).$$

Thus, in some sense, $W(L^{N-1,1}(\mathbb{S}^{N-1}))$ is the largest space (among the admissible $W(X)$) which embeds in $L^\infty(\mathbb{S}^{N-1})$: that is the reason why the space $L^{N-1,1}(\mathbb{S}^{N-1})$ appears naturally in our proof.

By Cianchi and Pick's theorem, it only remains to prove that the functions v_r and their gradients converge to v_∞, respectively ∇v_∞ in $L^{N-1,1}(\mathbb{S}^{N-1})$: by assertion (24), we have for every $N-1 < q < \min\{p_1, N\}$

$$\begin{aligned}
\|v_r - v_\infty\|_{L^{N-1,1}(\mathbb{S}^{N-1})} &= \int_0^{|\mathbb{S}^{N-1}|} t^{-\frac{N-2}{N-1}} |v_r - v_\infty|^*(t)\, dt \\
&\leqslant \left(\int_0^{|\mathbb{S}^{N-1}|} |v_r - v_\infty|^{*q}(t)\, dt\right)^{\frac{1}{q}} \left(\int_0^{|\mathbb{S}^{N-1}|} t^{-\frac{q'(N-2)}{N-1}}\, dt\right)^{\frac{1}{q'}} \\
&\leqslant A_{N,q} \|v_r - v_\infty\|_{L^q(\mathbb{S}^{N-1})} \\
&\leqslant A_{N,q} \|\nabla v\|^q_{L^q(\mathbb{R}^N)} r^{q-N} \underset{r\to+\infty}{\to} 0.
\end{aligned}$$

Now, fix $\varepsilon > 0$. By assertion (23), there is some real number $r_\varepsilon > 0$ such that

$$\forall r > r_\varepsilon, \quad \forall q \in \{p_0, p_1\}, r^{N-1-q} \int_{\mathbb{S}^{N-1}} \left|\nabla^{\mathbb{S}^{N-1}} v_r(\xi)\right|^q d\sigma \leqslant \varepsilon^q.$$

Thus, denoting $\lambda_r = \lambda_{\nabla^{\mathbb{S}^{N-1}} v_r}$ and $f_r = |\nabla^{\mathbb{S}^{N-1}} v_r|^*$, we obtain

$$\forall t > 0, \quad \lambda_r(t) \leqslant \min\left\{\frac{\varepsilon^{p_0}}{r^{N-1-p_0} t^{p_0}}, \frac{\varepsilon^{p_1}}{r^{N-1-p_1} t^{p_1}}\right\},$$

and

$$\forall t > 0, \quad f_r(t) \leqslant \min\left\{\frac{\varepsilon}{r^{\frac{N-1}{p_0}-1} t^{\frac{1}{p_0}}}, \frac{\varepsilon}{r^{\frac{N-1}{p_1}-1} t^{\frac{1}{p_1}}}\right\}.$$

Finally, we compute

$$\begin{aligned}
\left\|\nabla^{\mathbb{S}^{N-1}} v_r\right\|_{L^{N-1,1}(\mathbb{S}^{N-1})} &= \int_0^{|\mathbb{S}^{N-1}|} f_r(t) t^{-\frac{N-2}{N-1}}\, dt \\
&\leqslant \varepsilon r^{1-\frac{N-1}{p_1}} \int_0^{r^{1-N}} t^{-\frac{N-2}{N-1}-\frac{1}{p_1}}\, dt + \varepsilon r^{1-\frac{N-1}{p_0}} \int_{r^{1-N}}^{|\mathbb{S}^{N-1}|} t^{-\frac{N-2}{N-1}-\frac{1}{p_0}}\, dt \leqslant A_{N,p_0,p_1}\varepsilon.
\end{aligned}$$

It yields that $\nabla^{\mathbb{S}^{N-1}} v_r$ converges to 0 in $L^{N-1,1}(\mathbb{S}^{N-1})$ when r tends to $+\infty$. By Cianchi and Pick's theorem, we then get

$$\|v_r - v_\infty\|_{L^\infty(\mathbb{S}^{N-1})} \underset{r\to+\infty}{\to} 0,$$

which achieves the proof of Lemma 20.

The proof of Proposition 19 is then complete because the functions v_r converge uniformly to v_∞ by Lemma 20, and because the proof of Lemma 20 yields that v_∞ is a constant function. □

Corollary 4 then follows from Theorem 3 and Proposition 19.

Proof of Corollary 4. If v is a travelling wave of finite energy and of speed $0 \leqslant c < \sqrt{2}$, it satisfies the assumptions of Proposition 19 by Theorem 3. So, there is a constant $v_\infty \in \mathbb{C}$ such that

$$v(x) \underset{|x|\to+\infty}{\to} v_\infty.$$

By Lemma 14, the modulus of v_∞ is one. □

Remark. To simplify the notations, and since the solutions are defined up to a rotation, we will assume from now on that

$$v_\infty = 1.$$

2. Linear estimates for the Gross–Pitaevskii kernels

In the second part, we estimate the algebraic decay of the kernels associated to the Gross–Pitaevskii equation K_0, K_j, $L_{j,k}$ and $R_{j,k}$, i.e. the exponents α for which the functions $|.|^\alpha K_0$, $|.|^\alpha K_j$, $|.|^\alpha L_{j,k}$ and $|.|^\alpha R_{j,k}$ are bounded on $\mathbb{R}^N$. We then deduce some L^p-regularity for those kernels.

2.1. Inequalities L^1–L^∞

In this section, for sake of completeness, we first prove Lemma 7, which is presumably well-known to the experts. We then deduce three generalizations of it for functions which are not necessarily in $S(\mathbb{R}^N)$. The first one concerns the functions in the fractional Sobolev space $W^{s,1}(\mathbb{R}^N)$ defined by

$$W^{s,1}(\mathbb{R}^N) := \left\{u \in L^1(\mathbb{R}^N)\colon \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(z)-u(y)|}{|z-y|^{N+s}}\,dy\,dz < +\infty\right\} \tag{25}$$

for $0 < s < 1$, the second one, the functions in the fractional Deny–Lions space $D^{s,1}(\mathbb{R}^N)$ defined by

$$D^{s,1}(\mathbb{R}^N) := \left\{u \in L^{p_s}(\mathbb{R}^N)\colon \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(z)-u(y)|}{|z-y|^{N+s}}\,dy\,dz < +\infty\right\} \tag{26}$$

for $0 < s < 1$: they are both useful to study the algebraic decay of the kernels K_0, K_j and $L_{j,k}$. The last one concerns the functions in the homogeneous fractional Sobolev space $\dot{W}^{s,1}(\mathbb{R}^N)$, whose definition is more involved: it is likely to be the largest space in which the L^1–L^∞ estimate of Lemma 7 holds.

Proof of Lemma 7. Let $\hat{f}$, a function in $S(\mathbb{R}^N)$. At first, f is also in $S(\mathbb{R}^N)$, so, the function $x \mapsto |x|^s f(x)$ is in $C_0^0(\mathbb{R}^N)$. Now , fix $x \in \mathbb{R}^N$: we get

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\hat{f}(z)-\hat{f}(y)}{|z-y|^{N+s}} e^{ix.y}\,dy\,dz = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\hat{f}(y+t)-\hat{f}(y)}{|t|^{N+s}} e^{ix.y}\,dy\,dt$$
$$= \int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N} f(\sigma)e^{i(x.y-\sigma.y)}\frac{e^{-it.\sigma}-1}{|t|^{N+s}}\,d\sigma\right)dy\right)dt.$$

We then compute

$$\int_{\mathbb{R}^N} \frac{e^{-it.\sigma}-1}{|t|^{N+s}}\,dt$$

by a general formula for the Fourier transformation of radial functions (see for example the book of L. Schwartz [16]):

$$\int_{\mathbb{R}^N} \frac{e^{-it.\sigma}-1}{|t|^{N+s}}\,dt = 2\pi \int_0^{+\infty}\left(J_{\frac{N}{2}-1}\big(2\pi r|\sigma|\big) - \frac{\pi^{\frac{N}{2}-1}}{\Gamma(\frac{N}{2})}\big(r|\sigma|\big)^{\frac{N}{2}-1}\right) r^{-s-\frac{N}{2}}|\sigma|^{1-\frac{N}{2}}\,dr$$
$$= 2\pi|\sigma|^s \int_0^{+\infty}\left(J_{\frac{N}{2}-1}(2\pi u) - \frac{\pi^{\frac{N}{2}-1}}{\Gamma(\frac{N}{2})}u^{\frac{N}{2}-1}\right)u^{-\frac{N}{2}-s}\,du.$$

So, if we denote

$$A_N = 2\pi \int_0^{+\infty}\left(J_{\frac{N}{2}-1}(2\pi u) - \frac{\pi^{\frac{N}{2}-1}}{\Gamma(\frac{N}{2})}u^{\frac{N}{2}-1}\right)u^{-\frac{N}{2}-s}\,du < 0,$$

we get

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\hat{f}(z)-\hat{f}(y)}{|z-y|^{N+s}} e^{ix.y}\,dy\,dz = A_N \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} f(\sigma)|\sigma|^s e^{i(x.y-\sigma.y)}\,d\sigma\,dy$$
$$= A_N \int_{\mathbb{R}^N} \widehat{|.|^s f}(y)e^{iy.x}\,dy = (2\pi)^N A_N f(x)|x|^s,$$

and formula (16) holds for every $\hat{f} \in S(\mathbb{R}^N)$. □

We have assumed in Lemma 7 that $\hat{f}$ is a smooth function in $S(\mathbb{R}^N)$. However, we can extent Lemma 7 in three ways at least by an argument of density.

• Consider first the fractional Sobolev space $W^{s,1}(\mathbb{R}^N)$ defined by (25) for every $0 < s < 1$. $W^{s,1}(\mathbb{R}^N)$ is a Banach space for the norm

$$\|u\|_{W^{s,1}(\mathbb{R}^N)} := \|u\|_{L^1(\mathbb{R}^N)} + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(z)-u(y)|}{|z-y|^{N+s}}\,dy\,dz,$$

in which the space $S(\mathbb{R}^N)$ is dense (see the books of J. Peetre [15] and H. Triebel [18] for many more details: $W^{s,1}(\mathbb{R}^N)$ is equal to the Besov space $B^s_{1,1}(\mathbb{R}^N)$).

We deduce from the property of density of $S(\mathbb{R}^N)$ and from Lemma 7 the next corollary.

Corollary 21. *Let* $0 < s < 1$ *and* $\hat{f} \in W^{s,1}(\mathbb{R}^N)$. *Then, the function* $x \mapsto |x|^s f(x)$ *is in* $C^0_0(\mathbb{R}^N)$ *and satisfies*

$$\big\||.|^s f\big\|_{L^\infty(\mathbb{R}^N)} \leqslant I_N \|\hat{f}\|_{W^{s,1}(\mathbb{R}^N)}, \tag{27}$$

where I_N *is the constant given by Lemma* 7.

Proof. Let $\hat{f} \in W^{s,1}(\mathbb{R}^N)$. Since $S(\mathbb{R}^N)$ is dense in $W^{s,1}(\mathbb{R}^N)$, there is a sequence $(\widehat{f_n})_{n\in\mathbb{N}}$ of functions of $S(\mathbb{R}^N)$ such that

$$\|\hat{f} - \widehat{f_n}\|_{W^{s,1}(\mathbb{R}^N)} \underset{n\to+\infty}{\to} 0.$$

Thus, by Lemma 7, the sequence of functions

$$g_n : x \mapsto g_n(x) = |x|^s f_n(x)$$

is a Cauchy sequence in the space $C^0_0(\mathbb{R}^N)$: therefore, there is a function $g \in C^0_0(\mathbb{R}^N)$ such that

$$\|g_n - g\|_{L^\infty(\mathbb{R}^N)} \underset{n\to+\infty}{\to} 0.$$

By assumption, the functions $\widehat{f_n}$ converge to $\hat{f}$ in $L^1(\mathbb{R}^N)$, so, the functions f_n converge to f in $L^\infty(\mathbb{R}^N)$, and up to an extraction, almost everywhere. It follows that

$$g = |.|^s f.$$

By Lemma 7, we have for every $n \in \mathbb{N}$,

$$\big\||.|^s f_n\big\|_{L^\infty(\mathbb{R}^N)} \leqslant I_N \|\widehat{f_n}\|_{W^{s,1}(\mathbb{R}^N)},$$

which yields inequality (27) by taking the limit $n \to +\infty$. □

• Actually, we are going to work on functions which do not belong to the space $W^{s,1}(\mathbb{R}^N)$. That is the reason why we introduce a second space in which Lemma 7 holds: by standard Sobolev embeddings, we know that

$$W^{s,1}(\mathbb{R}^N) \hookrightarrow L^{p_s}(\mathbb{R}^N)$$

for every $0 < s < 1$ and $p_s = \frac{N}{N-s}$. Thus, we are led to consider the fractional Deny-Lions space $D^{s,1}(\mathbb{R}^N)$ defined by (26) for every $0 < s < 1$. $D^{s,1}(\mathbb{R}^N)$ is also a Banach space for the norm

$$\|u\|_{W^{s,1}(\mathbb{R}^N)} := \|u\|_{L^{p_s}(\mathbb{R}^N)} + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(z) - u(y)|}{|z-y|^{N+s}}\,dy\,dz,$$

and the space $S(\mathbb{R}^N)$ is also dense in $D^{s,1}(\mathbb{R}^N)$ (see the books of J. Peetre [15] and H. Triebel [18] for many more details).

We deduce from the property of density of $S(\mathbb{R}^N)$ and from Lemma 7 the next corollary.

Corollary 22. *Let* $0 < s < 1$ *and* $\hat{f} \in D^{s,1}(\mathbb{R}^N)$. *Then, the function* $x \mapsto |x|^s f(x)$ *is in* $C^0_0(\mathbb{R}^N)$ *and satisfies*

$$\big\||.|^s f\big\|_{L^\infty(\mathbb{R}^N)} \leqslant I_N \|\hat{f}\|_{D^{s,1}(\mathbb{R}^N)},$$

where I_N *is the constant given by Lemma* 7.

Proof. The proof being nearly identical to the proof of Corollary 21, we omit it: the main difference is that the functions $\widehat{f_n}$ do not converge to $\hat{f}$ in $L^1(\mathbb{R}^N)$ anymore. However, they converge to $\hat{f}$ in $L^{p_s}(\mathbb{R}^N)$: since $p_s \leqslant 2$, the functions f_n converge to f in $L^{p'_s}(\mathbb{R}^N)$ where $p'_s = \frac{p_s}{p_s-1}$, so, up to an extraction, they also converge to f almost everywhere. Corollary 22 then follows from the same arguments as in the proof of Corollary 21. □

• Finally, we introduce a last space to which the conclusion of Lemma 7 can be extended: the homogeneous fractional Sobolev space $\dot{W}^{s,1}(\mathbb{R}^N)$. Its definition is rather involved. We first consider the space

$$Z(\mathbb{R}^N) = \{u \in S(\mathbb{R}^N) \mid \forall \alpha \in \mathbb{N}^N,\ \partial^\alpha \hat{u}(0) = 0\},$$

and its topological dual space $Z'(\mathbb{R}^N)$. We are going to identify $Z'(\mathbb{R}^N)$ with the factor space $S'(\mathbb{R}^N)/P(\mathbb{R}^N)$, where $P(\mathbb{R}^N)$ denotes the set of all polynomial functions on $\mathbb{R}^N$. In this case, an element of $Z'(\mathbb{R}^N)$ is a class of tempered distributions defined modulo a polynomial function: we will denote $\dot{u}$, a representative of the class u in $S'(\mathbb{R}^N)$. The space $\dot{W}^{s,1}(\mathbb{R}^N)$ is then given by

$$\dot{W}^{s,1}(\mathbb{R}^N) = \left\{u \in Z'(\mathbb{R}^N)\colon \inf_{P \in P(\mathbb{R}^N)} \left(\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|\dot{u}(z) + P(z) - \dot{u}(y) - P(y)|}{|z-y|^{N+s}}\, dy\, dz \right) < +\infty \right\}$$

for every $0 < s < 1$. $\dot{W}^{s,1}(\mathbb{R}^N)$ is a Banach space for the norm

$$\|u\|_{\dot{W}^{s,1}(\mathbb{R}^N)} := \inf_{P \in P(\mathbb{R}^N)} \left(\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|\dot{u}(z) + P(z) - \dot{u}(y) - P(y)|}{|z-y|^{N+s}}\, dy\, dz \right).$$

The space $Z(\mathbb{R}^N)$ is dense in $\dot{W}^{s,1}(\mathbb{R}^N)$ and $\dot{W}^{s,1}(\mathbb{R}^N)$ is continuously embedded in $Z'(\mathbb{R}^N)$ (see the book of J. Peetre [15] and H. Triebel [18] for many more details: $\dot{W}^{s,1}(\mathbb{R}^N)$ is equal to the homogeneous Besov space $\dot{B}^s_{1,1}(\mathbb{R}^N)$).

We deduce from the property of density of $Z(\mathbb{R}^N)$ and from Lemma 7 the following corollary.

Corollary 23. *Let $0 < s < 1$ and $\hat{f} \in \dot{W}^{s,1}(\mathbb{R}^N)$. Then, there is a distribution $\tilde{f}$ in the class f such that the function $x \mapsto |x|^s \tilde{f}(x)$ is in $C^0_0(\mathbb{R}^N)$ and satisfies*

$$\big\| |.|^s \tilde{f} \big\|_{L^\infty(\mathbb{R}^N)} \leqslant I_N \|\hat{f}\|_{\dot{W}^{s,1}(\mathbb{R}^N)}, \tag{28}$$

where I_N is the constant given by Lemma 7.

Remark. We must clarify some points: $\hat{f}$ is a class of distributions modulo a polynomial function. Thus, f is also a class of tempered distributions, but modulo a finite linear combination of the Dirac mass δ_0 in 0 and of some of its derivatives: we will denote $\tilde{f}$, a representative of the class f in $S'(\mathbb{R}^N)$.

Proof. Let $\hat{f} \in \dot{W}^{s,1}(\mathbb{R}^N)$. $Z(\mathbb{R}^N)$ is dense in $\dot{W}^{s,1}(\mathbb{R}^N)$, so, there is a sequence $(\widehat{f_n})_{n\in\mathbb{N}}$ of functions of $Z(\mathbb{R}^N)$ such that

$$\|\hat{f} - \widehat{f_n}\|_{\dot{W}^{s,1}(\mathbb{R}^N)} \underset{n\to+\infty}{\to} 0. \tag{29}$$

Thus, $(\widehat{f_n})_{n\in\mathbb{N}}$ is a Cauchy sequence in $\dot{W}^{s,1}(\mathbb{R}^N)$, so, by Lemma 7, the sequence of functions

$$g_n : x \mapsto g_n(x) = |x|^s f_n(x)$$

is a Cauchy sequence in the space $C^0_0(\mathbb{R}^N)$. Therefore, there is a function $g \in C^0_0(\mathbb{R}^N)$ such that

$$\|g_n - g\|_{L^\infty(\mathbb{R}^N)} \underset{n\to+\infty}{\to} 0.$$

On the other hand, since $\dot{W}^{s,1}(\mathbb{R}^N)$ is continuously embedded in $Z'(\mathbb{R}^N)$, assertion (29) yields that

$$\widehat{f_n} \underset{n\to+\infty}{\to} \hat{f} \quad \text{in } Z'(\mathbb{R}^N).$$

So, if we consider a function $\phi \in S(\mathbb{R}^N)$ such that

$$\widehat{|.|^s\phi} \in Z(\mathbb{R}^N),$$

i.e. a function $\phi \in S(\mathbb{R}^N)$ such that $|.|^s\phi$ is in $C^\infty(\mathbb{R}^N)$ and

$$\forall \alpha \in \mathbb{N}^N, \quad \partial^\alpha\big(|.|^s\phi\big)(0) = 0,$$

we get

$$\langle g, \phi\rangle = \lim_{n\to+\infty}\big\langle |.|^s f_n, \phi\big\rangle = (2\pi)^{-N}\lim_{n\to+\infty}\big\langle \widehat{f_n}, \widehat{|.|^s\phi}\big\rangle$$
$$= (2\pi)^{-N}\big\langle \hat{f}, \widehat{|.|^s\phi}\big\rangle = \big\langle |.|^s f, \phi\big\rangle.$$

We deduce that there is some representative $\tilde{f}$ in the class of f which is in $C_0^0(\mathbb{R}^N \setminus \{0\})$ and which satisfies

$$|.|^s\tilde{f} = g \quad \text{on } \mathbb{R}^N \setminus \{0\}.$$

Since g is in $C_0^0(\mathbb{R}^N)$, $\frac{g}{|.|^s}$ is in $L^1_{\mathrm{loc}}(\mathbb{R}^N)$, and so, is a tempered distribution. Consequently, $\tilde{f} - \frac{g}{|.|^s}$ is also a tempered distribution whose support is included in the set $\{0\}$.

By Schwartz lemma, it is a finite linear combination of δ_0 and of some of its derivatives, i.e. the classes of $\tilde{f}$ and $\frac{g}{|.|^s}$ modulo a finite linear combination of δ_0 and of some of its derivatives are the same: up to the choice of a new representative $\tilde{f}$ in the class f, we will assume that we have exactly

$$\tilde{f} = \frac{g}{|.|^s} \quad \text{in } S'(\mathbb{R}^N).$$

Then, $\tilde{f}$ is in $L^1_{\mathrm{loc}}(\mathbb{R}^N)$, and $|.|^s\tilde{f}$ is a tempered distribution in $L^1_{\mathrm{loc}}(\mathbb{R}^N)$ which satisfies

$$g = |.|^s\tilde{f} \quad \text{on } \mathbb{R}^N.$$

Finally, $|.|^s\tilde{f}$ is in $C_0^0(\mathbb{R}^N)$, and since for every $n \in \mathbb{N}$,

$$\big\| |.|^s f_n\big\|_{L^\infty(\mathbb{R}^N)} \leqslant I_N \|\widehat{f_n}\|_{\dot{W}^{s,1}(\mathbb{R}^N)},$$

estimate (28) holds by taking the limit $n \to +\infty$. □

2.2. First estimates for the Gross–Pitaevskii kernels

In this section, we deduce from Lemma 7 and Corollaries 21, 22 and 23 some L^∞-estimates for the Gross–Pitaevskii kernels, i.e. Theorem 8.

Proof of Theorem 8. We first report some properties of the functions K_0, K_j, $L_{j,k}$ and of their derivatives.

Step 1. *Let $(n, p) \in \mathbb{N}^2$ and f, either the function $d^p\widehat{d^nK_0}$, $d^p\widehat{d^nK_j}$ or $d^p\widehat{d^nL_{j,k}}$. f is a rational fraction on $\mathbb{R}^N$, whose denominator only vanishes at* 0 *and such that*

$$|.|^{p-n} f \in L^\infty\big(B(0,1)\big) \quad \text{and} \quad |.|^{p-n+2} f \in L^\infty\big(B(0,1)^c\big).$$

Step 1 follows from a straightforward inductive argument based on formulae (11), (12) and (14): we only give its sketch. For instance, for $n = 0$, by formula (11), the function $\widehat{K_0}$ is a rational fraction equal to

$$\widehat{K_0}(\xi) = \frac{|\xi|^2}{|\xi|^4 + 2|\xi|^2 - c^2\xi_1^2},$$

so, it satisfies the estimates of Step 1. Moreover, its derivative $\partial_j\widehat{K_0}$ is

$$\partial_j\widehat{K_0}(\xi) = \frac{2\xi_j}{|\xi|^4 + 2|\xi|^2 - c^2\xi_1^2} - \frac{4\xi_j|\xi|^4 + 4\xi_j|\xi|^2 - 2c^2\delta_{j,1}\xi_1|\xi|^2}{(|\xi|^4 + 2|\xi|^2 - c^2\xi_1^2)^2}.$$

It is also a rational fraction which satisfies the conclusion of Step 1: the proof then follows from a straightforward induction on p.

Remark. We infer from Step 1 that the behaviour of all those kernels is identical, and in order to simplify the proof, we focus on the function $d^n K_0$.

We notice that $d^{N-1+n}\widehat{d^n K_0}$ belongs to $L^1(\mathbb{R}^N)$, so, by the standard L^1–L^∞ inequality, $|.|^{N-1+n} d^n K_0$ is bounded on $\mathbb{R}^N$.

To prove the other estimates, we then derive

Step 2. *Let $s \in]0,1[$ and $n \in \mathbb{N}$. The functions*

$$|.|^{N-2+s+n} d^n K_0$$

are bounded on $\mathbb{R}^N$.

Indeed, we apply Corollary 22 to the function

$$\hat{f} = d^{N-2+n}\widehat{d^n K_0}.$$

We first notice by Step 1 that $\hat{f}$ is in $L^p(\mathbb{R}^N)$ for $1 < p < \frac{N}{N-2}$: since $1 < p_s < \frac{N}{N-2}$ for every $0 < s < 1$, $\hat{f}$ is in $L^{p_s}(\mathbb{R}^N)$ for every $0 < s < 1$ and it only remains to compute

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|\hat{f}(z) - \hat{f}(y)|}{|z-y|^{N+s}}\,dy\,dz$$

$$= \int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N} \frac{|\hat{f}(y+t) - \hat{f}(y)|}{|t|^{N+s}}\,dy\right)dt$$

$$= \int_{\mathbb{R}^N}\left(\int_{|t|\leqslant 1} \frac{|\hat{f}(y+t) - \hat{f}(y)|}{|t|^{N+s}}\,dt\right)dy + \int_{|t|>1}\left(\int_{|y|>2|t|} \frac{|\hat{f}(y+t) - \hat{f}(y)|}{|t|^{N+s}}\,dy\right)dt$$

$$+ \int_{|t|>1}\left(\int_{|y|\leqslant 2|t|} \frac{|\hat{f}(y+t) - \hat{f}(y)|}{|t|^{N+s}}\,dy\right)dt.$$

For the first integral, we have

$$\int_{\mathbb{R}^N}\left(\int_{|t|\leqslant 1} \frac{|\hat{f}(y+t) - \hat{f}(y)|}{|t|^{N+s}}\,dt\right)dy \leqslant \int_0^1\left(\int_{\mathbb{R}^N}\left(\int_{|t|\leqslant 1} \frac{|\nabla\hat{f}(y+\sigma t)|}{|t|^{N+s-1}}\,dt\right)dy\right)d\sigma$$

$$\leqslant \left(\int_{\mathbb{R}^N}\left|\nabla\hat{f}(z)\right|dz\right)\left(\int_{|t|\leqslant 1}\frac{dt}{|t|^{N+s-1}}\right)$$

$$\leqslant A\int_{\mathbb{R}^N}\left|d^{N-1+n}\widehat{d^n K_0}(\xi)\right|d\xi < +\infty$$

for the second one,

$$\int_{|t|>1}\left(\int_{|y|>2|t|}\frac{|\hat f(y+t)-\hat f(y)|}{|t|^{N+s}}\,dy\right)dt \leqslant \int_0^1\left(\int_{|t|>1}\left(\int_{|y|>2|t|}\frac{|\nabla\hat f(y+\sigma t)|}{|t|^{N+s-1}}\,dy\right)dt\right)d\sigma$$

$$\leqslant A\int_0^1\left(\int_{|t|>1}\left(\int_{|y|>2|t|}\frac{dy}{|y+\sigma t|^{N+1}}\right)\frac{dt}{|t|^{N+s-1}}\right)d\sigma$$

$$\leqslant A\int_{|t|>1}\left(\int_{|y|>2|t|}\frac{dy}{(|y|-|t|)^{N+1}}\right)\frac{dt}{|t|^{N+s-1}}$$

$$\leqslant A\left(\int_{|t|>1}\frac{dt}{|t|^{N+s}}\right)\left(\int_{|u|>2}\frac{du}{(|u|-1)^{N+1}}\right)<+\infty$$

and for the last one,

$$\int_{|t|>1}\left(\int_{|y|\leqslant 2|t|}\frac{|\hat f(y+t)-\hat f(y)|}{|t|^{N+s}}\,dy\right)dt \leqslant 2\int_{|t|>1}\left(\int_{|y|\leqslant 3|t|}\left|\hat f(y)\right|dy\right)\frac{dt}{|t|^{N+s}}$$

$$\leqslant A\int_{|t|>1}\left(\int_{|y|\leqslant 1}\frac{dy}{|y|^{N-2}}+\int_{1<|y|\leqslant 3|t|}\frac{dy}{|y|^{N}}\right)\frac{dt}{|t|^{N+s}}$$

$$\leqslant A\left(\int_{|t|>1}\frac{dt}{|t|^{N+s}}\right)\left(\int_{|y|\leqslant 1}\frac{dy}{|y|^{N-2}}\right)+A\int_{|t|>1}\frac{\ln(3|t|)}{|t|^{N+s}}\,dt<+\infty.$$

Thus, we get

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|\hat f(z)-\hat f(y)|}{|z-y|^{N+s}}\,dy\,dz<+\infty,$$

and $\hat f$ is in $D^{s,1}(\mathbb{R}^N)$: by Corollary 22, $|.|^{N-2+s+n}d^nK_0$ is then bounded on $\mathbb{R}^N$ for every $0<s<1$.

We achieve the proof by the next similar step

Step 3. *Let* $s\in\,]0,1[$ *and* $n\in\mathbb{N}$. *The functions*

$$|.|^{N-1+s+n}d^nK_0$$

are bounded on $\mathbb{R}^N$.

The proof relies on Corollary 21 for the function

$$\hat f = d^{N-1+n}\widehat{d^nK_0}.$$

By Step 1, $\hat f$ is in $L^1(\mathbb{R}^N)$ and we compute likewise

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|\hat f(z)-\hat f(y)|}{|z-y|^{N+s}}\,dy\,dz=\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|\hat f(y+t)-\hat f(y)|}{|t|^{N+s}}\,dy\,dt$$

$$=\int_{\mathbb{R}^N}\left(\int_{|t|\geqslant 1}\frac{|\hat f(y+t)-\hat f(y)|}{|t|^{N+s}}\,dt\right)dy$$

$$+\int_{|t|<1}\left(\int_{|y|\leqslant 2|t|}\frac{|\hat{f}(y+t)-\hat{f}(y)|}{|t|^{N+s}}\,dy\right)dt$$
$$+\int_{|t|<1}\left(\int_{|y|>2|t|}\frac{|\hat{f}(y+t)-\hat{f}(y)|}{|t|^{N+s}}\,dy\right)dt.$$

For the first integral, we have

$$\int_{\mathbb{R}^N}\left(\int_{|t|\geqslant 1}\frac{|\hat{f}(y+t)-\hat{f}(y)|}{|t|^{N+s}}\,dt\right)dy\leqslant 2\left(\int_{\mathbb{R}^N}\left|\hat{f}(z)\right|dz\right)\left(\int_{|t|\geqslant 1}\frac{dt}{|t|^{N+s}}\right)$$
$$\leqslant\left(\int_{\mathbb{R}^N}\left|d^{N-1+n}\widehat{d^nK_0}(z)\right|dz\right)\left(\int_{|t|\geqslant 1}\frac{dt}{|t|^{N+s}}\right)<+\infty,$$

for the second one,

$$\int_{|t|<1}\left(\int_{|y|\leqslant 2|t|}\frac{|\hat{f}(y+t)-\hat{f}(y)|}{|t|^{N+s}}\,dy\right)dt\leqslant 2\int_{|t|<1}\left(\int_{|y|\leqslant 3|t|}\left|\hat{f}(y)\right|dy\right)\frac{dt}{|t|^{N+s}}$$
$$\leqslant A\int_{|t|<1}\left(\int_{|y|\leqslant 3|t|}\frac{dy}{|y|^{N-1}}\right)\frac{dt}{|t|^{N+s}}$$
$$\leqslant A\left(\int_{|t|<1}\frac{dt}{|t|^{N+s-1}}\right)\left(\int_{|u|\leqslant 3}\frac{du}{|u|^{N-1}}\right)<+\infty,$$

and for the last one,

$$\int_{|t|<1}\left(\int_{|y|>2|t|}\frac{|\hat{f}(y+t)-\hat{f}(y)|}{|t|^{N+s}}\,dy\right)dt$$
$$\leqslant\int_0^1\left(\int_{|t|<1}\left(\int_{|y|>2|t|}\left|\nabla\hat{f}(y+\sigma t)\right|dy\right)\frac{dt}{|t|^{N+s-1}}\right)d\sigma$$
$$\leqslant A\int_{|t|<1}\left(\int_{2>|y|>2|t|}\frac{dy}{(|y|-|t|)^N}+\int_{|y|>2}\frac{dy}{(|y|-|t|)^{N+2}}\right)\frac{dt}{|t|^{N+s-1}}$$
$$\leqslant A\int_{|t|<1}\left(\int_2^{\frac{2}{|t|}}\frac{u^{N-1}}{(u-1)^N}\,du\right)\frac{dt}{|t|^{N+s-1}}+A\left(\int_{|t|<1}\frac{dt}{|t|^{N+s-1}}\right)\left(\int_{|y|>2}\frac{dy}{(|y|-1)^{N+2}}\right)$$
$$\leqslant A\int_{|t|<1}\frac{|\ln(t)|}{|t|^{N+s-1}}\,dt+A\left(\int_{|t|<1}\frac{dt}{|t|^{N+s-1}}\right)\left(\int_{|y|>2}\frac{dy}{(|y|-1)^{N+2}}\right)<+\infty.$$

Thus, we also get

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|\hat{f}(z)-\hat{f}(y)|}{|z-y|^{N+s}}\,dy\,dz<+\infty,$$

and $\hat{f}$ is in $W^{s,1}(\mathbb{R}^N)$: by Corollary 21, $|.|^{N+s-1+n} d^n K_0$ is bounded on $\mathbb{R}^N$ for every $0 < s < 1$, which achieves the proofs of Step 3 and Theorem 8. □

Remark. Here, the key ingredient is the form of the Fourier transformation $\widehat{K}$ of the kernels.

- $\widehat{K}$ is a rational fraction;
- $\widehat{K}$ is only singular at the origin, where the singularity is of the form $\underset{\xi \to 0}{O}(1/|\xi|^\alpha)$;
- at infinity, $\widehat{K}$ is of the form $\underset{|\xi| \to +\infty}{O}(1/|\xi|^\beta)$, where $\beta > \alpha$.

We can obtain the algebraic decay of all the kernels whose Fourier transformation satisfies similar assumptions by the same argument.

Before improving those estimates, we deduce some L^p-integrability for the Gross–Pitaevskii kernels.

Corollary 24. *Let $(j,k) \in \{1,\ldots,N\}^2$. The functions K_0, K_j and $L_{j,k}$ belong to all the spaces $L^p(\mathbb{R}^N)$ for*

$$1 < p < \frac{N}{N-2},$$

and their gradients, for

$$1 \leqslant p < \frac{N}{N-1}.$$

Proof. It follows from the estimates of Theorem 8. □

Remark. We conjecture Corollary 24 is optimal, i.e.

- the functions K_0, K_j and $L_{j,k}$ do not belong either to $L^1(\mathbb{R}^N)$, nor to $L^{\frac{N}{N-2}}(\mathbb{R}^N)$;
- their gradients do not belong to $L^{\frac{N}{N-1}}(\mathbb{R}^N)$.

2.3. *Critical estimates for the Gross–Pitaevskii kernels*

In this section, we improve the linear estimates given by Theorem 8 by proving Theorem 9. It seems very similar to Theorem 8, but its proof is quite different: we conjecture that the functions $|.|^{N+n} d^n K_0$, $|.|^{N+n} d^n K_j$ and $|.|^{N+n} d^n L_{j,k}$ do not tend to 0 at infinity. Thus, we cannot prove Theorem 9 from a general inequality deduced from the density of $S(\mathbb{R}^N)$: it would mean that $|.|^{N+n} d^n K_0$, $|.|^{N+n} d^n K_j$ and $|.|^{N+n} d^n L_{j,k}$ tend to 0 at infinity. Actually, its proof relies on the following lemma.

Lemma 25. *Let $1 \leqslant j \leqslant N$. The function*

$$x \mapsto x_j f(x)$$

is bounded on $B(0,1)^c$ for every $f \in S'(\mathbb{R}^N)$ such that

(i) *$\hat{f}$ is a function C^2 on $\mathbb{R}^N \setminus \{0\}$,*
(ii) *$(|.|^{N+1} + |.|^{N-1})\hat{f}$ is bounded on $\mathbb{R}^N$,*
(iii) *$(|.|^{N+2} + |.|^{N})\partial_j \hat{f}$ is bounded on $\mathbb{R}^N$,*
(iv) *$(|.|^{N+3} + |.|^{N+1})\partial_j \partial_k \hat{f}$ are bounded on $\mathbb{R}^N$ for $1 \leqslant k \leqslant N$.*

Proof. Indeed, we establish the formula

Step 1. *Let* $\lambda > 0$. *The following equality holds almost everywhere*

$$x_j f(x) = \frac{i}{(2\pi)^N}\left(\int\limits_{B(0,\lambda)^c} \partial_j \hat{f}(\xi) e^{ix.\xi}\, d\xi + \int\limits_{B(0,\lambda)} \partial_j \hat{f}(\xi)(e^{ix.\xi} - 1)\, d\xi + \frac{1}{\lambda} \int\limits_{S(0,\lambda)} \xi_j \hat{f}(\xi) e^{ix.\xi}\, d\xi \right). \tag{30}$$

Let $g \in S(\mathbb{R}^N)$. We have

$$\langle x_j f, \hat{g}\rangle = \langle f, x_j \hat{g}\rangle = -i\langle f, \widehat{\partial_j g}\rangle = -i\langle \hat{f}, \partial_j g\rangle.$$

By assumption (ii), $\hat{f}$ is in $L^1(\mathbb{R}^N)$, so, we can write

$$\langle x_j f, \hat{g}\rangle = -i \int\limits_{\mathbb{R}^N} \hat{f}(\xi)\partial_j g(\xi)\, d\xi,$$

and by integrating by parts, we deduce

$$\langle x_j f, \hat{g}\rangle = -i\langle \hat{f}, \partial_j g\rangle = i \int\limits_{B(0,\lambda)^c} \partial_j \hat{f}(\xi) g(\xi)\, d\xi + i \int\limits_{B(0,\lambda)} \partial_j \hat{f}(\xi)\big(g(\xi) - g(0)\big)\, d\xi + \frac{ig(0)}{\lambda} \int\limits_{S(0,\lambda)} \xi_j \hat{f}(\xi)\, d\xi.$$

Since g is in $S(\mathbb{R}^N)$, it satisfies

$$g(\xi) = \frac{1}{(2\pi)^N} \int\limits_{\mathbb{R}^N} \hat{g}(x) e^{ix.\xi}\, dx,$$

which yields

$$\langle x_j f, \hat{g}\rangle = \frac{i}{(2\pi)^N} \int\limits_{\mathbb{R}^N} \hat{g}(x) \left(\int\limits_{B(0,\lambda)^c} \partial_j \hat{f}(\xi) e^{ix.\xi}\, d\xi + \int\limits_{B(0,\lambda)} \partial_j \hat{f}(\xi)(e^{ix.\xi} - 1)\, d\xi + \frac{1}{\lambda} \int\limits_{S(0,\lambda)} \xi_j \hat{f}(\xi)\, d\xi \right) dx.$$

As the function

$$x \mapsto \int\limits_{B(0,\lambda)^c} \partial_j \hat{f}(\xi) e^{ix.\xi}\, d\xi + \int\limits_{B(0,\lambda)} \partial_j \hat{f}(\xi)(e^{ix.\xi} - 1)\, d\xi + \frac{1}{\lambda} \int\limits_{S(0,\lambda)} \xi_j \hat{f}(\xi)\, d\xi$$

belongs to $L^1_{\mathrm{loc}}(\mathbb{R}^N)$, by standard duality, formula (30) is valid almost everywhere.

To proceed further, we estimate each term of formula (30).

Step 2. *The following inequalities hold for every* $x \in \mathbb{R}^N$ *and* $\lambda > 0$

$$\begin{cases} |\int_{B(0,\lambda)} \partial_j \hat{f}(\xi)(e^{ix.\xi} - 1)\, d\xi| \leqslant A\lambda|x|, \\ |\int_{S(0,\lambda)} \xi_j \hat{f}(\xi) e^{ix.\xi}\, d\xi| \leqslant A\lambda, \end{cases}$$

where A *is a real number independent of* x *and* λ.

Indeed, on one hand, we know

$$\forall u \in \mathbb{R}, \quad |e^{iu} - 1| \leqslant A|u|,$$

and therefore,

$$\left|\int_{B(0,\lambda)} \partial_j \hat{f}(\xi)(e^{ix.\xi}-1)\,d\xi\right| \leqslant A|x| \int_{B(0,\lambda)} \left|\partial_j \hat{f}(\xi)\right||\xi|\,d\xi.$$

By assumption (iii), we get

$$\left|\int_{B(0,\lambda)} \partial_j \hat{f}(\xi)(e^{ix.\xi}-1)\,d\xi\right| \leqslant A|x| \int_{B(0,\lambda)} \frac{d\xi}{|\xi|^{N-1}} \leqslant A\lambda|x|.$$

On the other hand, we deduce likewise from assumption (ii)

$$\left|\int_{S(0,\lambda)} \xi_j \hat{f}(\xi)e^{ix.\xi}\,d\xi\right| \leqslant A \int_{S(0,\lambda)} \frac{d\xi}{|\xi|^{N-2}} \leqslant A\lambda,$$

and it only remains a single integral to evaluate.

Step 3. *The following inequality holds for every $x \in B(0,1)^c$ and $0 < \lambda < 1$*

$$\left|\int_{B(0,\lambda)^c} \partial_j \hat{f}(\xi)e^{ix.\xi}\,d\xi\right| \leqslant A\left(1+\frac{1}{\lambda|x|}\right),$$

where A is a real number independent of x and λ.

Indeed, we have

$$\int_{B(0,\lambda)^c} \partial_j \hat{f}(\xi)e^{ix.\xi}\,d\xi = \int_{B(0,1)^c} \partial_j \hat{f}(\xi)e^{ix.\xi}\,d\xi + \int_{B(0,1)\setminus B(0,\lambda)} \partial_j \hat{f}(\xi)e^{ix.\xi}\,d\xi.$$

For the first integral, we deduce from assumption (iii)

$$\left|\int_{B(0,1)^c} \partial_j \hat{f}(\xi)e^{ix.\xi}\,d\xi\right| \leqslant \int_{B(0,1)^c} \left|\partial_j \hat{f}(\xi)\right|\,d\xi \leqslant A.$$

For the second one, by assumption,

$$|x| > 1,$$

so, there is some integer $1 \leqslant k \leqslant N$ such that

$$|x_k| \geqslant \frac{|x|}{N}.$$

By integrating by parts, we then get

$$\begin{aligned}
\int_{B(0,1)\setminus B(0,\lambda)} \partial_j \hat{f}(\xi)e^{ix.\xi}\,d\xi &= \frac{1}{ix_k} \int_{B(0,1)\setminus B(0,\lambda)} \partial_j \hat{f}(\xi)\partial_k(e^{ix.\xi})\,d\xi \\
&= \frac{1}{ix_k}\left(-\int_{B(0,1)\setminus B(0,\lambda)} \partial_j\partial_k \hat{f}(\xi)e^{ix.\xi}\,d\xi + \int_{S(0,1)} \partial_j \hat{f}(\xi)e^{ix.\xi}\xi_k\,d\xi \right.\\
&\quad \left. - \frac{1}{\lambda}\int_{S(0,\lambda)} \partial_j \hat{f}(\xi)e^{ix.\xi}\xi_k\,d\xi\right),
\end{aligned}$$

and by assumptions (iii) and (iv),

$$\left|\int_{B(0,1)\setminus B(0,\lambda)} \partial_j \hat{f}(\xi) e^{ix.\xi}\, d\xi\right| \leqslant \frac{N}{|x|}\left(A \int_{B(0,1)\setminus B(0,\lambda)} \frac{d\xi}{|\xi|^{N+1}} + A + \frac{A}{\lambda}\int_{S(0,\lambda)} \frac{d\xi}{|\xi|^{N-1}}\right) \leqslant \frac{A}{\lambda|x|} + A.$$

Thus, we get

$$\left|\int_{B(0,\lambda)^c} \partial_j \hat{f}(\xi) e^{ix.\xi}\, d\xi\right| \leqslant \frac{A}{\lambda|x|} + A.$$

Finally, by Steps 1, 2 and 3, we get for every $x \in B(0,1)^c$ and $0 < \lambda < 1$,

$$\left|x_j f(x)\right| \leqslant A\lambda|x| + \frac{A}{\lambda|x|} + A.$$

By choosing

$$\lambda = \frac{1}{|x|},$$

we obtain the result of Lemma 25. □

Now, we can deduce the proof of Theorem 9.

Proof of Theorem 9. By Step 1 of the proof of Theorem 8, the functions $d^{N-1+n}\widehat{d^n K_0}$, $d^{N-1+n}\widehat{d^n K_j}$ and $d^{N-1+n}\widehat{d^n L_{j,k}}$ satisfy the four assumptions of Lemma 25, which implies Theorem 9. □

2.4. Estimates for the composed Riesz kernels

We focus next on the kernels $R_{j,k}$, for which we have the explicit expression (17): if f is a smooth function, and if $g_{j,k}$ is the function defined by

$$\forall \xi \in \mathbb{R}^N, \quad \widehat{g_{j,k}}(\xi) = \widehat{R_{j,k}}(\xi)\hat{f}(\xi),$$

we have

$$\forall x \in \mathbb{R}^N, \quad g_{j,k}(x) = A_N \int_{|y|>1} \frac{\delta_{j,k}|y|^2 - N y_j y_k}{|y|^{N+2}} f(x-y)\, dy + A_N \int_{|y|\leqslant 1} \frac{\delta_{j,k}|y|^2 - N y_j y_k}{|y|^{N+2}} \big(f(x-y) - f(x)\big)\, dy.$$

Therefore, we do not need to study the decay of the kernels $R_{j,k}$ directly, and instead, we may restrict ourselves to the decay of the functions $g_{j,k}$ with suitable assumptions on f. In that context, we recall some useful facts, which are presumably well-known to the experts. For sake of completeness, we also mention the proofs.

Proposition 26. *Let f a function C^1 on $\mathbb{R}^N$ which belongs to $L^p(\mathbb{R}^N)$ for $1 < p \leqslant +\infty$, and suppose there is*

$$\delta \in]0, N]$$

such that for every $\beta \in [0, \delta[$,

$$\begin{cases} |.|^\beta f \in L^\infty(\mathbb{R}^N), \\ |.|^\beta \nabla f \in L^\infty(\mathbb{R}^N). \end{cases}$$

Then, the functions

$$|.|^{\beta} g_{j,k} \in L^{\infty}(\mathbb{R}^N)$$

for every $(j,k) \in \{1,\dots,N\}^2$ *and for every* $\beta \in [0,\delta[$.

Proof. Recalling formula (17), we first denote

$$g_{j,k}(x) = A_N \int\limits_{|y|>1} \frac{\delta_{j,k}|y|^2 - N y_j y_k}{|y|^{N+2}} f(x-y)\,dy + A_N \int\limits_{|y|\leqslant 1} \frac{\delta_{j,k}|y|^2 - N y_j y_k}{|y|^{N+2}} \big(f(x-y) - f(x)\big)\,dy$$
$$= I_1(x) + I_2(x).$$

Then, if we fix $\beta \in [0,\delta[$, we get

$$|x|^{\beta}\big|I_1(x)\big| \leqslant A \int\limits_{|y|>1} \frac{|x-y|^{\beta}|f(x-y)|}{|y|^N}\,dy + A \int\limits_{|y|>1} \frac{|f(x-y)|}{|y|^{N-\beta}}\,dy.$$

Hence, if $p > \frac{N}{N-\beta}$, we have

$$\int\limits_{|y|>1} \frac{|f(x-y)|}{|y|^{N-\beta}}\,dy \leqslant \|f\|_{L^{p'}(\mathbb{R}^N)} \bigg(\int\limits_{|y|>1} \frac{dy}{|y|^{p(N-\beta)}} \bigg)^{\frac{1}{p}} < +\infty,$$

and if $\beta < \delta - \varepsilon$ and $|x| > 4$, then,

$$\begin{aligned}
\int\limits_{|y|>1} \frac{|x-y|^{\beta}|f(x-y)|}{|y|^N}\,dy &\leqslant A \int\limits_{|y|>1} \frac{dy}{|y|^N|x-y|^{\varepsilon}} \\
&\leqslant \frac{A}{|x|^{\varepsilon}} \int\limits_{|t|>\frac{1}{|x|}} \frac{dt}{|t|^N|\frac{x}{|x|} - t|^{\varepsilon}} \\
&\leqslant \frac{A}{|x|^{\varepsilon}} \int\limits_{\frac{1}{|x|}<|t|<\frac{1}{2}} \frac{dt}{|t|^N} + \frac{A}{|x|^{\varepsilon}} \int\limits_{\frac{1}{2}<|t|<\frac{3}{2}} \frac{dt}{|\frac{x}{|x|} - t|^{\varepsilon}} + \frac{A}{|x|^{\varepsilon}} \int\limits_{|t|>\frac{3}{2}} \frac{dt}{|t|^N(|t|-1)^{\varepsilon}} \\
&\leqslant \frac{A\ln|x|}{|x|^{\varepsilon}} + A + \frac{A}{|x|^{\varepsilon}} \int\limits_{|t-\frac{x}{|x|}|<\frac{1}{2}} \frac{dt}{|\frac{x}{|x|} - t|^{\varepsilon}} \\
&\leqslant \frac{A\ln|x|}{|x|^{\varepsilon}} + A < +\infty,
\end{aligned}$$

whereas, if $|x| \leqslant 4$, we get

$$\int\limits_{|y|>1} \frac{|x-y|^{\beta}|f(x-y)|}{|y|^N}\,dy \leqslant A \int\limits_{1<|y|<5} \frac{dy}{|y|^N} + A \int\limits_{|y|>5} \frac{dy}{|y|^N(|y|-4)^{\varepsilon}} < +\infty.$$

Thus, $|.|^{\beta} I_1$ is bounded on $\mathbb{R}^N$, and likewise, we have for I_2,

$$|x|^{\beta} I_2(x) \leqslant A \int\limits_{|y|\leqslant 1} \frac{|x-y|^{\beta}|f(x-y) - f(x)|}{|y|^N}\,dy + A \int\limits_{|y|\leqslant 1} \frac{|f(x-y) - f(x)|}{|y|^{N-\beta}}\,dy.$$

On one hand, if $\beta < \delta - \varepsilon$, we compute

$$\int\limits_{|y|\leqslant 1} \frac{|x-y|^\beta |f(x-y)-f(x)|}{|y|^N}\,dy \leqslant \|\nabla f\|_{L^\infty(B(x,1))}\big(|x|+1\big)^\beta \int\limits_{|y|\leqslant 1} \frac{dy}{|y|^{N-1}} \leqslant \frac{A}{(1+|x|)^\varepsilon} < +\infty$$

and on the other hand, we get if $\beta = 0$,

$$\int\limits_{|y|\leqslant 1} \frac{|f(x-y)-f(x)|}{|y|^N}\,dy \leqslant A \int\limits_{|y|\leqslant 1} \frac{dy}{|y|^{N-1}} < +\infty,$$

whereas if $\beta > 0$,

$$\int\limits_{|y|\leqslant 1} \frac{|f(x-y)-f(x)|}{|y|^{N-\beta}}\,dy \leqslant A \int\limits_{|y|\leqslant 1} \frac{dy}{|y|^{N-\beta}}.$$

Therefore, $|.|^\beta I_2$ is also bounded on $\mathbb{R}^N$, such as $|.|^\beta g_{j,k}$. □

Remark. In fact, a similar proposition holds for the Riesz kernels.

Actually, we will make use of the next more precise proposition in the critical case: it is also presumably well-known to the experts, but for sake of completeness, we also mention the proof.

Proposition 27. *Let f a function C^1 on $\mathbb{R}^N$ which belongs to $L^1(\mathbb{R}^N)$, and suppose that*

$$\begin{cases} (1+|.|^N) f \in L^\infty(\mathbb{R}^N), \\ (1+|.|^{N+1})\nabla f \in L^\infty(\mathbb{R}^N). \end{cases}$$

Then, the functions

$$|.|^N g_{j,k} \in L^\infty(\mathbb{R}^N)$$

for every $(j,k) \in \{1,\dots,N\}^2$.

Proof. Recalling formula (17) once more, we notice

$$\begin{aligned} g_{j,k}(x) &= A_N \int\limits_{|y|>\frac{|x|}{4},\,|x-y|>\frac{|x|}{4}} \frac{\delta_{j,k}|y|^2 - N y_j y_k}{|y|^{N+2}} f(x-y)\,dy + A_N \int\limits_{|x-y|\leqslant\frac{|x|}{4}} \frac{\delta_{j,k}|y|^2 - N y_j y_k}{|y|^{N+2}} f(x-y)\,dy \\ &\quad + A_N \int\limits_{|y|\leqslant\frac{|x|}{4}} \frac{\delta_{j,k}|y|^2 - N y_j y_k}{|y|^{N+2}} \big(f(x-y)-f(x)\big)\,dy \\ &= I_1(x) + I_2(x) + I_3(x). \end{aligned}$$

For the first integral, we compute

$$\begin{aligned} \big|I_1(x)\big| &\leqslant A_N \int\limits_{|y|>\frac{|x|}{4},\,|x-y|>\frac{|x|}{4}} \frac{dy}{|y|^N|x-y|^N} \leqslant \frac{A_N}{|x|^N} \int\limits_{|z|>\frac{1}{4},\,|\frac{x}{|x|}-z|>\frac{1}{4}} \frac{dz}{|z|^N|\frac{x}{|x|}-z|^N} \\ &\leqslant \frac{A_N}{|x|^N} \int\limits_{|z|>\frac{1}{4},\,|e_1-z|>\frac{1}{4}} \frac{dz}{|z|^N|e_1-z|^N} \leqslant \frac{A_N}{|x|^N}, \end{aligned}$$

for the second one,

$$\left|I_2(x)\right| \leqslant \frac{A_N}{|x|^N} \int_{|x-y|\leqslant\frac{|x|}{4}} \left|f(x-y)\right| dy \leqslant \frac{A_N}{|x|^N} \int_{|t|<\frac{|x|}{4}} \left|f(t)\right| dt \leqslant \frac{A_N}{|x|^N},$$

and for the last one,

$$\left|I_3(x)\right| \leqslant A_N \int_{|y|\leqslant\frac{|x|}{4}} |y|^{1-N}|x|^{-N-1}\, dy \leqslant \frac{A_N}{|x|^N}.$$

Thus, $|.|^N g_{j,k}$ is bounded on $\mathbb{R}^N$. □

3. Decay at infinity

In the last part, we study the algebraic decay of the functions η, $\nabla(\psi\theta)$, ∇v and of their derivatives, by the inductive argument yet explained in the introduction (see Lemmas 5 and 6), which was introduced by J.L. Bona and Yi A. Li [4], and A. de Bouard and J.C. Saut [6] (see also the articles of M. Maris [13,14] for many more details).

We first prove a refined energy estimate based on Lemma 10, which provides some algebraic decay for the functions η, $\nabla(\psi\theta)$ and ∇v. Then, by convolution equations (10) and (13), we deduce inductively Theorem 11, which gives some decay rate for all those functions.

3.1. A refined energy estimate

We first give an energy estimate for v thanks to arguments from F. Béthuel, G. Orlandi and D. Smets [3]: it will yield in the next section some algebraic decay for the functions η, $\nabla(\psi\theta)$ and ∇v.

Proposition 28. *If v is a solution of finite energy of Eq.* (2) *in $L^1_{\text{loc}}(\mathbb{R}^N)$, there is some real number $\alpha > 0$ such that the integral*

$$\int_{\mathbb{R}^N} |x|^\beta e(v)(x)\, dx$$

is finite for every $0 \leqslant \beta < \alpha$.

The proof relies on Lemma 10 proved by F. Béthuel, G. Orlandi and D. Smets [3] for small c. For sake of completeness, we mention the proof of Lemma 10 for every $0 \leqslant c < \sqrt{2}$.

Proof of Lemma 10. We first invoke Lemma 15 to choose some real number R so large that

$$v = \rho e^{i\theta} \quad \text{on } B(0,R)^c.$$

By Eq. (2), we then compute

$$-\Delta\rho + \rho|\nabla\theta|^2 + c\rho\partial_1\theta = \rho(1-\rho^2), \tag{31}$$

$$\operatorname{div}(\rho^2\nabla\theta) = -\frac{c}{2}\partial_1\rho^2 \tag{32}$$

on the set $B(0,R)^c$.

Then, fix $\lambda > R$ and denote $\Omega = B(0,\lambda) \setminus B(0,R)$, and $\theta_R = \frac{1}{|\mathbb{S}_R|}\int_{\mathbb{S}_R}\theta$. We first multiply Eq. (31) by $\rho^2 - 1$, which gives by integrating by parts,

$$2\int_\Omega \rho|\nabla\rho|^2 - \int_{\mathbb{S}_\lambda} \partial_\nu\rho(\rho^2-1) + \int_{\mathbb{S}_R} \partial_\nu\rho(\rho^2-1) + \int_\Omega \rho(\rho^2-1)|\nabla\theta|^2 + c\int_\Omega \rho(\rho^2-1)\partial_1\theta = -\int_\Omega \rho(\rho^2-1)^2. \tag{33}$$

We already know that $\partial_\nu\rho(\rho^2-1)$ belongs to $L^1(B(0,R)^c)$, so, we can construct an increasing sequence $(\lambda_n)_{n\in\mathbb{N}}$ which diverges to $+\infty$, and such that

$$\int_{\mathbb{S}_{\lambda_n}} \partial_\nu\rho(\rho^2-1) \underset{n\to+\infty}{\to} 0.$$

By taking the limit at infinity in equality (33), we get

$$2\int_{B(0,R)^c} \rho|\nabla\rho|^2 + \int_{\mathbb{S}_R} \partial_\nu\rho(\rho^2-1) + \int_{B(0,R)^c} \rho(\rho^2-1)|\nabla\theta|^2 + c\int_{B(0,R)^c} \rho(\rho^2-1)\partial_1\theta = -\int_{B(0,R)^c} \rho(\rho^2-1)^2. \tag{34}$$

We also get such a result by multiplying Eq. (32) by $\theta - \theta_R$ and by integrating by parts,

$$\int_\Omega \rho^2|\nabla\theta|^2 - \int_{\mathbb{S}_\lambda} \rho^2\partial_\nu\theta(\theta-\theta_R) + \int_{\mathbb{S}_R} \rho^2\partial_\nu\theta(\theta-\theta_R) = -\frac{c}{2}\int_\Omega (\rho^2-1)\partial_1\theta + \frac{c}{2}\int_{\mathbb{S}_\lambda}(\rho^2-1)\nu_1(\theta-\theta_R) - \frac{c}{2}\int_{\mathbb{S}_R}(\rho^2-1)\nu_1(\theta-\theta_R).$$

By Theorem 3, $\nabla\theta$ and $1-\rho^2$ belong to $L^{\frac{N}{N-1}}(B(0,R)^c)$, so, we can construct another increasing sequence $(\lambda_n)_{n\in\mathbb{N}}$ which diverges to $+\infty$, and such that

$$\lambda_n \int_{\mathbb{S}_{\lambda_n}} \left(|\nabla\theta|^{\frac{N}{N-1}} + |1-\rho^2|^{\frac{N}{N-1}}\right) \underset{n\to\infty}{\to} 0.$$

Since

$$\begin{cases} |\int_{\mathbb{S}_\lambda} \rho^2\partial_\nu\theta(\theta-\theta_R)| \leqslant A\int_{\mathbb{S}_\lambda}|\partial_\nu\theta| \leqslant A(\lambda\int_{\mathbb{S}_\lambda}|\partial_\nu\theta|^{\frac{N}{N-1}})^{\frac{N-1}{N}}, \\ |\int_{\mathbb{S}_\lambda}(\rho^2-1)\nu_1(\theta-\theta_R)| \leqslant A\int_{\mathbb{S}_\lambda}|1-\rho^2| \leqslant A(\lambda\int_{\mathbb{S}_\lambda}|1-\rho^2|^{\frac{N}{N-1}})^{\frac{N-1}{N}}, \end{cases}$$

we get

$$\int_{B(0,R)^c} \rho^2|\nabla\theta|^2 + \int_{\mathbb{S}_R} \rho^2\partial_\nu\theta(\theta-\theta_R) = -\frac{c}{2}\left(\int_{B(0,R)^c}(\rho^2-1)\partial_1\theta + \int_{\mathbb{S}_R}(\rho^2-1)\nu_1(\theta-\theta_R)\right). \tag{35}$$

By adding equalities (34) and (35), we infer

$$\int\limits_{B(0,R)^c} e(v) = -\frac{c}{2}\int\limits_{B(0,R)^c} \rho(\rho^2-1)\partial_1\theta - \frac{1}{2}\int\limits_{\mathbb{S}_R} \rho^2\partial_\nu\theta(\theta-\theta_R)$$
$$-\frac{c}{4}\int\limits_{\mathbb{S}_R}(\theta-\theta_R)(\rho^2-1)\nu_1 + \int\limits_{B(0,R)^c}(1-\rho)\left(\frac{|\nabla\rho|^2}{2}+\frac{(1-\rho^2)^2}{4}\right)$$
$$-\frac{c}{4}\int\limits_{B(0,R)^c}(1-\rho)(\rho^2-1)\partial_1\theta - \frac{1}{4}\int\limits_{\mathbb{S}_R}\partial_\nu\rho(\rho^2-1) + \frac{1}{4}\int\limits_{B(0,R)^c}\rho(1-\rho^2)|\nabla\theta|^2. \tag{36}$$

It remains to evaluate each term in the right member of equality (36). For the first one, we can write

$$\left|\frac{c}{2}\int\limits_{B(0,R)^c}\rho(\rho^2-1)\partial_1\theta\right| \leqslant \frac{c}{\sqrt{2}}\int\limits_{B(0,R)^c}\left(\frac{\rho^2\partial_1\theta^2}{2}+\frac{(1-\rho^2)^2}{4}\right) \leqslant \frac{c}{\sqrt{2}}\int\limits_{B(0,R)^c} e(v).$$

For the next one, we get by Sobolev–Poincaré inequality,

$$\left|\frac{1}{2}\int\limits_{\mathbb{S}_R}\rho^2\partial_\nu\theta(\theta-\theta_R)\right| \leqslant A\left(\int\limits_{\mathbb{S}_R}\rho^2\partial\nu\theta^2\right)^{\frac{1}{2}}\left(\int\limits_{\mathbb{S}_R}(\theta-\theta_R)^2\right)^{\frac{1}{2}}$$
$$\leqslant AR\left(\int\limits_{\mathbb{S}_R}\rho^2\partial\nu\theta^2\right)^{\frac{1}{2}}\left(\int\limits_{\mathbb{S}_R}\partial_\nu\theta^2\right)^{\frac{1}{2}} \leqslant AR\int\limits_{\mathbb{S}_R} e(v),$$

and likewise,

$$\begin{cases} |\frac{c}{4}\int_{\mathbb{S}_R}(\theta-\theta_R)(\rho^2-1)| \leqslant AR\int_{\mathbb{S}_R} e(v),\\ |\int_{\mathbb{S}_R}\partial_\nu\rho(\rho^2-1)| \leqslant A\int_{\mathbb{S}_R} e(v). \end{cases}$$

In order to estimate the other terms, we fix $\varepsilon > 0$, and by Lemma 14, we choose R sufficiently large such as $|\rho - 1|$ and $|\nabla\theta|$ are less than ε on the domain $B(0,R)^c$. For such an R, we have

$$\begin{cases} |\int_{B(0,R)^c}(1-\rho)(\frac{|\nabla\rho|^2}{2}+\frac{(1-\rho^2)^2}{4})| \leqslant \varepsilon\int_{B(0,R)^c} e(v),\\ |\frac{c}{4}\int_{B(0,R)^c}(1-\rho)(\rho^2-1)\partial_1\theta| \leqslant A\varepsilon\int_{B(0,R)^c} e(v),\\ |\frac{1}{4}\int_{B(0,R)^c}\rho(1-\rho^2)|\nabla\theta|^2| \leqslant A\varepsilon\int_{B(0,R)^c} e(v), \end{cases}$$

which finally gives,

$$\int\limits_{B(0,R)^c} e(v) \leqslant \left(\frac{c}{\sqrt{2}}+A\varepsilon\right)\int\limits_{B(0,R)^c} e(v) + AR\int\limits_{\mathbb{S}_R} e(v).$$

If ε is sufficiently small such as

$$\frac{c}{\sqrt{2}} + A\varepsilon < 1,$$

it yields

$$\int\limits_{B(0,R)^c} e(v) \leqslant A_c R\int\limits_{\mathbb{S}_R} e(v).$$

Denoting $J(R) = \int_{B(0,R)^c} e(v)$, we get for R sufficiently large

$$J(R) \leqslant -A_c R J'(R)$$

which gives

$$J(R) \leqslant \frac{C}{R^{1/A_c}}.$$

Lemma 10 holds for $\alpha_c = 1/A_c$. □

Finally, we deduce Proposition 28 from Lemma 10.

Proof of Proposition 28. The case $\beta = 0$ being immediate, we choose $\beta \in]0, \alpha_c[$ and compute

$$\int_{\mathbb{R}^N} |x|^\beta e(v)(x)\,dx = \int_0^{+\infty} r^\beta \int_{\mathbb{S}_r} e(v)\,dr = -\left[r^\beta \int_r^{+\infty} \int_{\mathbb{S}_\rho} e(v)\,d\rho \right]_0^{+\infty} + \beta \int_0^{+\infty} r^{\beta-1} \left(\int_r^{+\infty} \int_{\mathbb{S}_\rho} e(v)\,d\rho \right) dr$$

$$= \beta \int_0^{+\infty} r^{\beta-1} \left(\int_r^{+\infty} \int_{\mathbb{S}_\rho} e(v)\,d\rho \right) dr < +\infty. \quad \square$$

Remark. Proposition 28 is crucial to initialize the proof of the next section.

3.2. Decay of the functions η and ∇v

In this section, we prove Theorem 11, i.e. we determine some algebraic decay for the functions η, $\nabla(\psi\theta)$, ∇v and their derivatives.

The proof of Theorem 11 essentially follows from the arguments developed in the introduction in Lemmas 5 and 6, and is of inductive nature. However, as mentioned, it is more involved, since we have to consider a system of convolution equations and to handle the singularities of the convolution kernels at the origin. Thus, we will split the argument in four subsections.

In Section 3.2.1, we show that the functions η and ∇v belong to some spaces $M_\beta^\infty(\mathbb{R}^N)$ for β sufficiently small. It provides an initialization similar to the one needed in Lemma 6.

In Section 3.2.2, we apply the inductive argument of Lemma 6 to Eqs. (10) and (13) to improve the algebraic decay of the functions η, $\nabla\eta$, $\nabla(\psi\theta)$ and ∇v.

In Section 3.2.3, we deduce inductively some algebraic decay for the derivatives of the functions η, $\nabla(\psi\theta)$ and ∇v by the same argument.

Finally, in Section 3.2.4, we improve once more the decay rate of the functions η, $\nabla\eta$, $\nabla(\psi\theta)$ and ∇v by using the critical estimates of Theorem 9 instead of Theorem 8, and Proposition 27 instead of Proposition 26.

3.2.1. Initialization of the proof of Theorem 11

In this first subsection, we deduce some algebraic decay for the functions η, $\nabla\eta$, $\nabla(\psi\theta)$ and ∇v from Proposition 28.

Proposition 29. *There is some real number $\alpha > 0$ such that*

$$\big(\eta, \nabla\eta, \nabla(\psi\theta), \nabla v\big) \in M_\beta^\infty(\mathbb{R}^N)^4$$

for every $0 \leqslant \beta < \alpha$.

Proof. The proof relies on Eqs. (10)

$$\eta = K_0 * F + 2c \sum_{j=1}^N K_j * G_j,$$

and (13)

$$\partial_j(\psi\theta) = \frac{c}{2}K_j * F + c^2\sum_{k=1}^{N} L_{j,k} * G_k + \sum_{k=1}^{N} R_{j,k} * G_k.$$

We estimate each term of those equations beginning by Eq. (10).

Step 1.1. *Let $j \in \{1, \dots, N\}$. Then,*

- $K_0 * F \in M_\beta^\infty(\mathbb{R}^N)$,
- $K_j * G_j \in M_\beta^\infty(\mathbb{R}^N)$,

for β sufficiently small.

Indeed, we have for $0 \leqslant \beta < N$ and for every $x \in \mathbb{R}^N$,

$$|x|^\beta \big|K_0 * F(x)\big| \leqslant A\bigg(\int_{\mathbb{R}^N} |x-y|^\beta \big|K_0(x-y)\big|\big|F(y)\big|\,dy + \int_{\mathbb{R}^N} \big|K_0(x-y)\big||y|^\beta\big|F(y)\big|\,dy\bigg).$$

On one hand, by Theorem 8,

$$|.|^\beta K_0 \in L^p(\mathbb{R}^N),$$

for

$$\frac{N}{N-\beta} < p < \frac{N}{N-\beta-2}$$

if $0 \leqslant \beta < N-2$, and for

$$p > \frac{N}{N-\beta}$$

if $N-2 \leqslant \beta < N$. For such a p, by Theorem 3, F is in $L^{p'}(\mathbb{R}^N)$, so, we get by Young's inequality,

$$\big\|\big(|.|^\beta K_0\big) * F\big\|_{L^\infty(\mathbb{R}^N)} \leqslant \big\||.|^\beta K_0\big\|_{L^p(\mathbb{R}^N)}\|F\|_{L^{p'}(\mathbb{R}^N)} < +\infty.$$

On the other hand, by Corollary 24,

$$K_0 \in L^q(\mathbb{R}^N)$$

for every $1 < q < \frac{N}{N-2}$, and by Proposition 28, there is some real number $\alpha > 0$ such that

$$\forall \beta \in [0, \alpha[, \quad \int_{\mathbb{R}^N} |.|^\beta\big(|F| + |G|\big) < +\infty.$$

Then, consider $\beta \in [0, \frac{2\alpha}{N}[$: there is $1 < q < \frac{N}{N-2}$ such that

$$\beta q' < \alpha.$$

As F tends to 0 at infinity by Lemma 14, we deduce

$$\int_{\mathbb{R}^N} |.|^{\beta q'}|F|^{q'} \leqslant A\int_{\mathbb{R}^N} |.|^{\beta q'}|F| < +\infty.$$

Thus, for every $\beta \in [0, \frac{2\alpha}{N}[$, we get

$$\big\| K_0 * \big(|.|^\beta F\big)\big\|_{L^\infty(\mathbb{R}^N)} \leqslant \|K_0\|_{L^q(\mathbb{R}^N)} \big\| |.|^\beta F \big\|_{L^{q'}(\mathbb{R}^N)}.$$

So, the function $K_0 * (|.|^\beta F)$ is bounded on $\mathbb{R}^N$, such as the function $|.|^\beta K_0 * F$: the proof being identical for the functions $|.|^\beta K_j * G_j$ by replacing F by G_j, we omit it.

By Eq. (10) and Step 1.1, η belongs to $M^\infty_\beta(\mathbb{R}^N)$ for β sufficiently small.

To prove the remaining results, we turn to the function $\nabla\eta$ which satisfies the equation

$$\nabla\eta = \nabla K_0 * F + 2c \sum_{j=1}^{N} \nabla K_j * G_j \tag{37}$$

and we establish similarly

Step 1.2. *Let $j \in \{1, \dots, N\}$. Then,*

- $\nabla K_0 * F \in M^\infty_\beta(\mathbb{R}^N)$,
- $\nabla K_j * G_j \in M^\infty_\beta(\mathbb{R}^N)$,

for β sufficiently small.

Indeed, we have for $0 \leqslant \beta < N+1$ and for every $x \in \mathbb{R}^N$,

$$|x|^\beta \big|\nabla K_0 * F(x)\big| \leqslant A \int_{\mathbb{R}^N} \big(|x-y|^\beta \big|\nabla K_0(x-y)\big|\big|F(y)\big| + \big|\nabla K_0(x-y)\big||y|^\beta\big|F(y)\big|\big)\, dy.$$

On one hand, by Theorem 8,

$$|.|^\beta \nabla K_0 \in L^p(\mathbb{R}^N),$$

for

$$\frac{N}{N+1-\beta} < p < \frac{N}{N-1-\beta}$$

if $0 \leqslant \beta < N-1$, and for

$$p > \frac{N}{N+1-\beta}$$

if $N-1 \leqslant \beta < N+1$. For such a p, by Theorem 3, F is in $L^{p'}(\mathbb{R}^N)$, so, we get by Young's inequality,

$$\big\|\big(|.|^\beta \nabla K_0\big) * F\big\|_{L^\infty(\mathbb{R}^N)} \leqslant \big\||.|^\beta \nabla K_0\big\|_{L^p(\mathbb{R}^N)} \|F\|_{L^{p'}(\mathbb{R}^N)} < +\infty.$$

On the other hand, by Corollary 24,

$$\nabla K_0 \in L^q(\mathbb{R}^N)$$

for $1 \leqslant q < \frac{N}{N-1}$, and by Proposition 28, there is some real number $\alpha > 0$ such that

$$\forall \beta \in [0, \alpha[, \quad \int_{\mathbb{R}^N} |.|^\beta \big(|F| + |G|\big) < +\infty.$$

Then, consider $\beta \in [0, \frac{\alpha}{N}[$: there is $1 \leqslant q < \frac{N}{N-1}$ such that

$$\beta q' < \alpha.$$

As F tends to 0 at infinity by Lemma 14, we deduce

$$\int_{\mathbb{R}^N} |.|^{\beta q'} |F|^{q'} \leqslant A \int_{\mathbb{R}^N} |.|^{\beta q'} |F| < +\infty.$$

Thus, for every $\beta \in [0, \frac{\alpha}{N}[$, we get

$$\left\| \nabla K_0 * \left(|.|^\beta F\right) \right\|_{L^\infty(\mathbb{R}^N)} \leqslant \|\nabla K_0\|_{L^q(\mathbb{R}^N)} \left\| |.|^\beta F \right\|_{L^{q'}(\mathbb{R}^N)}.$$

Hence, $\nabla K_0 * (|.|^\beta F)$ is bounded on $\mathbb{R}^N$, such as $|.|^\beta \nabla K_0 * F$: the proof being identical for $|.|^\beta \nabla K_j * G_j$ by replacing F by G_j, we omit it.

By Eq. (37) and Step 1.2, $\nabla\eta$ belongs to $M_\beta^\infty(\mathbb{R}^N)$ for β sufficiently small.

We then turn to the function $\nabla(\psi\theta)$ and study Eq. (13). The study of the terms involving the kernels K_j and $L_{j,k}$ is strictly identical to Step 1.1, and gives

Step 1.3. *Let* $(j,k) \in \{1, \dots, N\}^2$. *Then,*

- $K_j * F \in M_\beta^\infty(\mathbb{R}^N)$,
- $L_{j,k} * G_k \in M_\beta^\infty(\mathbb{R}^N)$,

for β *sufficiently small.*

It only remains to evaluate the functions $R_{j,k} * G_k$.

Step 1.4. *Let* $(j,k) \in \{1, \dots, N\}^2$. *Then,*

$$R_{j,k} * G_k \in M_\beta^\infty(\mathbb{R}^N),$$

for β *sufficiently small.*

Indeed, on one hand, by Steps 1.1 and 1.2, the functions $|.|^\beta \eta$ and $|.|^\beta \nabla\eta$ are bounded on $\mathbb{R}^N$ for β sufficiently small.

On the other hand, $\nabla(\psi\theta)$ is C^∞ on $\mathbb{R}^N$ and is given by

$$\nabla(\psi\theta) = \frac{iv.\nabla v}{|v|^2}$$

at infinity. However, by Theorem 3, ∇v and $d^2 v$ are bounded on $\mathbb{R}^N$, and by Lemma 14,

$$\left|v(x)\right| = \rho(x) \underset{|x| \to +\infty}{\to} 1,$$

so, $\nabla(\psi\theta)$ and $d^2(\psi\theta)$ are bounded on $\mathbb{R}^N$.

At last, G is C^∞ on $\mathbb{R}^N$ and is equal to

$$G = \eta\nabla(\psi\theta)$$

at infinity, so, $|.|^\beta G$ and $|.|^\beta \nabla G$ are bounded on $\mathbb{R}^N$ for β sufficiently small. As G and ∇G belong to all the spaces $L^p(\mathbb{R}^N)$ by Step 1 of the proof of Proposition 18, it follows from Proposition 26 that $|.|^\beta R_{j,k} * G_k$ is bounded for β sufficiently small.

By Eq. (13) and Steps 1.3 and 1.4, $\nabla(\psi\theta)$ belongs to $M_\beta^\infty(\mathbb{R}^N)$ for β sufficiently small.

We achieve the proof of Proposition 29 by deducing that

$$\nabla v \in M_\beta^\infty(\mathbb{R}^N)$$

for β sufficiently small. Indeed, by Theorem 3, ∇v is C^∞ on $\mathbb{R}^N$ and satisfies at infinity

$$|\nabla v|^2 = |\nabla\rho|^2 + \rho^2|\nabla\theta|^2 = \frac{|\nabla\eta|^2}{4\rho^2} + \rho^2\big|\nabla(\psi\theta)\big|^2.$$

Since

$$\rho(x) \underset{|x|\to+\infty}{\to} 1$$

by Lemma 14, we infer from the study of $\nabla\eta$ and $\nabla(\psi\theta)$ that $|.|^\beta\nabla v$ is bounded on $\mathbb{R}^N$ for β sufficiently small. □

3.2.2. Inductive argument for the decay of the functions η, $\nabla\eta$, $\nabla(\psi\theta)$ and ∇v

We then improve by the inductive argument of Lemma 6 the decay rate of the functions η, $\nabla\eta$, $\nabla(\psi\theta)$ and ∇v.

Proposition 30. *Assume there is some real number $\alpha > 0$ such that*

$$\big(\eta, \nabla\eta, \nabla(\psi\theta), \nabla v\big) \in M_\beta^\infty(\mathbb{R}^N)^4,$$

for

$$\beta \in [0, \alpha[.$$

Then,

$$\big(\eta, \nabla(\psi\theta), \nabla v\big) \in M_\beta^\infty(\mathbb{R}^N)^3,$$

for

$$\beta \in [0, \min\{N, 2\alpha\}[,$$

and

$$\nabla\eta \in M_\beta^\infty(\mathbb{R}^N),$$

for

$$\beta \in [0, \min\{N+1, 2\alpha\}[.$$

Proof. The proof is quite similar to the previous one: we first use the quadratic form of F and G.

Step 2.1. *The function*

$$|.|^\beta\big(|F| + |G|\big)$$

is bounded for every

$$\beta \in [0, 2\alpha[.$$

By formulae (8) and (9), F and G are C^∞ on $\mathbb{R}^N$ and are given by

$$F = 2|\nabla v|^2 + 2\eta^2 - 2c\eta\partial_1(\psi\theta),$$

and

$$G = \eta\nabla(\psi\theta),$$

at infinity. Step 2.1 then follows directly from the assumptions of Proposition 30.

Now, we study the function η by Eq. (10).

Step 2.2. *Let $j \in \{1, \ldots, N\}$ and $\beta \in [0, \min\{N, 2\alpha\}[$. Then,*

- $K_0 * F \in M_\beta^\infty(\mathbb{R}^N)$,

- $K_j * G_j \in M_\beta^\infty(\mathbb{R}^N)$.

Indeed, we have likewise for $0 \leqslant \beta < N$ and for every $x \in \mathbb{R}^N$,

$$|x|^\beta \left|K_0 * F(x)\right| \leqslant A\left(\int_{\mathbb{R}^N} |x-y|^\beta \left|K_0(x-y)\right|\left|F(y)\right| dy + \int_{\mathbb{R}^N} \left|K_0(x-y)\right||y|^\beta \left|F(y)\right| dy \right).$$

On one hand, we have already proved in the proof of Step 1.1 that for every $\beta \in [0, N[$,

$$\left\| \left(|.|^\beta K_0\right) * F \right\|_{L^\infty(\mathbb{R}^N)} < +\infty.$$

On the other hand, by Corollary 24,

$$K_0 \in L^q(\mathbb{R}^N)$$

for $1 < q < \frac{N}{N-2}$: so, we get for every $\beta \in [0, 2\alpha[$,

$$\left\| K_0 * \left(|.|^\beta F\right) \right\|_{L^\infty(\mathbb{R}^N)} \leqslant \|K_0\|_{L^q(\mathbb{R}^N)} \left\| |.|^\beta F \right\|_{L^{q'}(\mathbb{R}^N)}.$$

By Step 2.1, there is some real number $1 < q < \frac{N}{N-2}$ such that

$$\int_{\mathbb{R}^N} |.|^{\beta q'} |F|^{q'} < +\infty.$$

Thus, the function $K_0 * (|.|^\beta F)$ is bounded on $\mathbb{R}^N$, such as the function $|.|^\beta K_0 * F$: the proof being identical for the functions $|.|^\beta K_j * G_j$ by replacing F by G_j, we omit it.

By Step 2.2 and Eq. (10), Proposition 30 holds for the function η.

Then, we estimate the function $\nabla\eta$ by Eq. (37).

Step 2.3. *Let $j \in \{1, \dots, N\}$ and $\beta \in [0, \min\{2\alpha, N+1\}[$. Then,*

- $\nabla K_0 * F \in M_\beta^\infty(\mathbb{R}^N)$,
- $\nabla K_j * G_j \in M_\beta^\infty(\mathbb{R}^N)$.

In Step 1.2, we have shown that

$$\left(|.|^\beta \nabla K_0\right) * F \in L^\infty(\mathbb{R}^N)$$

for $\beta \in [0, N+1[$. We also deduce from Corollary 24 that for $q \in [1, \frac{N}{N-1}[$ sufficiently small and for every $\beta \in [0, 2\alpha[$,

$$\left\| \nabla K_0 * \left(|.|^\beta F\right) \right\|_{L^\infty(\mathbb{R}^N)} \leqslant \|\nabla K_0\|_{L^q(\mathbb{R}^N)} \left\| |.|^\beta F \right\|_{L^{q'}(\mathbb{R}^N)} < +\infty.$$

Similarly, the functions $\nabla K_j * (|.|^\beta G_j)$ and $(|.|^\beta \nabla K_j) * G_j$ are bounded for $\beta \in [0, \min\{N+1, 2\alpha\}[$, which completes the proof of Step 2.3.

The result of Proposition 30 for the function $\nabla\eta$ follows from Step 2.3 and Eq. (37), and we can turn to the function $\nabla(\psi\theta)$, which satisfies Eq. (13). The study of the terms involving the kernels K_j and $L_{j,k}$ is strictly identical to those of Steps 2.2 and 2.3.

Step 2.4. *Let $(j,k) \in \{1, \dots, N\}^2$. Then,*

- $K_j * F \in M_\beta^\infty(\mathbb{R}^N)$,

- $L_{j,k} * G_k \in M^\infty_\beta(\mathbb{R}^N)$,

for every $\beta \in [0, \min\{N, 2\alpha\}[$.

Thus, it only remains to evaluate the functions $R_{j,k} * G_k$.

Step 2.5. *Let* $(j,k) \in \{1,\dots,N\}^2$ *and* $\beta \in [0, \min\{N, 2\alpha\}[$. *Then,*

$$R_{j,k} * G_k \in M^\infty_\beta(\mathbb{R}^N).$$

Indeed, by Steps 2.2 and 2.3, the functions $|.|^\beta \eta$ and $|.|^\beta \nabla\eta$ are bounded on $\mathbb{R}^N$ for $\beta \in [0, \min\{N, 2\alpha\}[$: so, the functions $|.|^\beta G$ and $|.|^\beta \nabla G$ are also bounded on $\mathbb{R}^N$ for β in this range. Since G and ∇G belong to the spaces $L^p(\mathbb{R}^N)$ for $1 \leqslant p \leqslant +\infty$ by Step 1 of the proof of Proposition 18, by Proposition 26, the functions $|.|^\beta R_{j,k} * G_k$ are bounded for β in this range.

Subsequently, by Steps 2.4 and 2.5, and Eq. (13), $\nabla(\psi\theta)$ is in $M^\infty_\beta(\mathbb{R}^N)$ for $\beta \in [0, \min\{N, 2\alpha\}[$.

We conclude the proof of Proposition 30 by showing that

$$\nabla v \in M^\infty_\beta(\mathbb{R}^N)$$

for $\beta \in [0, \min\{N, 2\alpha\}[$. Indeed, by Theorem 3, ∇v is C^∞ on $\mathbb{R}^N$ and satisfies at infinity

$$|\nabla v|^2 = |\nabla\rho|^2 + \rho^2|\nabla\theta|^2 = \frac{|\nabla\eta|^2}{4\rho^2} + \rho^2\big|\nabla(\psi\theta)\big|^2.$$

Since

$$\rho(x) \underset{|x|\to+\infty}{\to} 1$$

by Lemma 14, it follows from the study of $\nabla\eta$ and $\nabla(\psi\theta)$ that $|.|^\beta \nabla v$ is bounded on $\mathbb{R}^N$ for $0 \leqslant \beta < \min\{N, 2\alpha\}$. □

3.2.3. Inductive argument for the decay of the derivatives of the functions η, $\nabla(\psi\theta)$ and ∇v

We deduce from Propositions 29 and 30 that

$$\big(\eta, \nabla(\psi\theta), \nabla v\big) \in M^\infty_\beta(\mathbb{R}^N)^3,$$

for every $\beta \in [0, N[$ and

$$\nabla\eta \in M^\infty_\beta(\mathbb{R}^N),$$

for every $\beta \in [0, N+1[$. We now estimate the decay of the derivatives of η, $\nabla(\psi\theta)$ and ∇v.

Proposition 31. *Let* $\alpha \in \mathbb{N}^N$. *Then,*

$$\big(\eta, \partial^\alpha\nabla(\psi\theta), \partial^\alpha\nabla v\big) \in M^\infty_\beta(\mathbb{R}^N)^3,$$

for every $\beta \in [0, N[$ *and*

$$\partial^\alpha\nabla\eta \in M^\infty_\beta(\mathbb{R}^N),$$

for every $\beta \in [0, N+1[$.

Proof. The proof is by induction on $|\alpha| \in \mathbb{N}$: the case $\alpha = 0$ follows from Propositions 29 and 30.

Now, assume that Proposition 31 holds for every $|\alpha| \leqslant p$ and fix $\alpha \in \mathbb{N}^N$ such that $|\alpha| = p+1$. As in the proof of Proposition 30, we first estimate F and G.

Step 3.1. *The function*

$$|.|^{\beta}\big(|\partial^{\gamma} F| + |\partial^{\gamma} G|\big)$$

is bounded for every $\beta \in [0, N[$ and for every $\gamma \in \mathbb{N}^N$ such that $|\gamma| = p+1$.

Step 3.1 relies on Leibnitz's formula and on the quadratic form of F and G.

F is a C^{∞} function on $\mathbb{R}^N$ given by

$$F = 2|\nabla v|^2 + 2\eta^2 - 2c\eta\partial_1(\psi\theta)$$

at infinity. By Leibnitz's formula, we compute

$$\partial^{\gamma} F = 2\sum_{\delta \leqslant \gamma} c_{\delta,\gamma}\big[\partial^{\gamma-\delta}\nabla v.\partial^{\delta}\nabla v + \partial^{\gamma-\delta}\eta.\partial^{\delta}\eta - c\partial^{\gamma-\delta}\eta.\partial^{\delta}\partial_1(\psi\theta)\big],$$

where the coefficients $c_{\delta,\gamma}$ are positive integers.

On one hand, by the assumption of induction,

$$|.|^{\beta}\big(|\partial^{\delta}\nabla v| + |\partial^{\delta}\eta| + \big|\partial^{\delta}\partial_1(\psi\theta)\big|\big) \in L^{\infty}(\mathbb{R}^N)$$

for $\delta \leqslant \gamma$ and $\delta \neq \gamma$, and for $\beta \in [0, N[$.

On the other hand, by Theorem 3, $\partial^{\gamma}\nabla v$, $\partial^{\gamma}\eta$ and $\partial^{\gamma}\partial_1(\psi\theta)$ are bounded on $\mathbb{R}^N$, so,

$$|.|^{\beta}|\partial^{\gamma} F| \in L^{\infty}(\mathbb{R}^N)$$

for every $\beta \in [0, N[$.

Likewise, G is a C^{∞} function on $\mathbb{R}^N$ given by

$$G = \eta\nabla(\psi\theta)$$

at infinity, so, by the same argument, $|.|^{\beta}\partial^{\gamma} G$ is bounded on $\mathbb{R}^N$ for $\beta \in [0, N[$.

We then study the function $\partial^{\alpha}\nabla\eta$, which satisfies

$$\partial^{\alpha}\nabla\eta = \nabla K_0 * \partial^{\alpha} F + 2c\sum_{j=1}^{N}\nabla K_j * \partial^{\alpha} G_j. \tag{38}$$

Step 3.2. *Let $j \in \{1, \dots, N\}$ and $\beta \in [0, N[$. Then,*

- $\nabla K_0 * \partial^{\alpha} F \in M_{\beta}^{\infty}(\mathbb{R}^N)$,
- $\nabla K_j * \partial^{\alpha} G_j \in M_{\beta}^{\infty}(\mathbb{R}^N)$.

By Step 3.1, the proof is similar to the proof of Step 2.3: by Step 1 of the proof of Proposition 18, $\partial^{\alpha} F$ and $\partial^{\alpha} G_j$ are in all the spaces $L^p(\mathbb{R}^N)$ for $1 \leqslant p \leqslant +\infty$ as well as F and G_j. So, we omit it.

Thus, $\partial^{\alpha}\nabla\eta$ belongs to $M_{\beta}^{\infty}(\mathbb{R}^N)$ for every $\beta \in [0, N[$.

Now, we turn to the function $\partial^{\alpha}\partial_j(\psi\theta)$, which satisfies

$$\partial^{\alpha}\partial_j(\psi\theta) = \frac{c}{2}K_j * \partial^{\alpha} F + c^2\sum_{k=1}^{N} L_{j,k} * \partial^{\alpha} G_k + \sum_{k=1}^{N} R_{j,k} * \partial^{\alpha} G_k. \tag{39}$$

By Step 3.1, the study of the terms involving the kernels K_j and $L_{j,k}$ is strictly identical to Steps 2.2, 2.3, 2.4 or 3.2.

Step 3.3. *Let* $(j,k) \in \{1,\ldots,N\}^2$. *Then,*

- $K_j * \partial^\alpha F \in M_\beta^\infty(\mathbb{R}^N)$,
- $L_{j,k} * \partial^\alpha G_k \in M_\beta^\infty(\mathbb{R}^N)$,

for every $\beta \in [0,N[$.

It only remains to evaluate the functions $R_{j,k} * \partial^\alpha G_k$.

Step 3.4. *Let* $(j,k) \in \{1,\ldots,N\}^2$ *and* $\beta \in [0,N[$. *Then,*

$$R_{j,k} * \partial^\alpha G_k \in M_\beta^\infty(\mathbb{R}^N).$$

Indeed, let $H_k = \partial^\alpha G_k$: H_k belongs to all the spaces $L^p(\mathbb{R}^N)$ for $1 \leqslant p \leqslant +\infty$ by Step 1 of the proof of Proposition 18, and $|.|^\beta H_k$ is bounded on $\mathbb{R}^N$ for every $\beta \in [0,N[$ by Step 3.1.

In order to apply Proposition 26, we claim that $|.|^\beta \nabla H_k$ is bounded on $\mathbb{R}^N$ for every $\beta \in [0,N[$: it follows from Leibniz's formula as well as in the proof of Step 3.1. Indeed, by formula (9), we have at infinity

$$\nabla G_k = \nabla\eta.\partial_k(\psi\theta) + \eta.\nabla\partial_k(\psi\theta).$$

By Leibniz's formula, we get

$$\nabla H_k = \sum_{\delta \leqslant \alpha} c_{\delta,\alpha}\big(\partial^\delta\nabla\eta.\partial^{\alpha-\delta}\partial_k(\psi\theta) + \partial^\delta\eta.\partial^{\alpha-\delta}\nabla\partial_k(\psi\theta)\big).$$

The terms involving the highest derivatives are $\partial^\alpha\nabla\eta.\partial_k(\psi\theta)$, $\nabla\eta.\partial^\alpha\partial_k(\psi\theta)$, $\partial^\alpha\eta.\nabla\partial_k(\psi\theta)$, $\eta.\nabla\partial_k(\psi\theta)$. All of them belong to $M_\beta^\infty(\mathbb{R}^N)$ for $\beta \in [0,N[$ because of the assumption of induction and of Step 1 of the proof of Proposition 18. The other terms are also in $M_\beta^\infty(\mathbb{R}^N)$ for $\beta \in [0,N[$ by the same argument. Therefore, $|.|^\beta \nabla H_k$ is bounded on $\mathbb{R}^N$ for every $\beta \in [0,N[$ and we can apply Proposition 26 to achieve the proof of Step 3.4.

Subsequently, by Steps 3.3 and 3.4, and Eq. (39), $\partial^\alpha\nabla(\psi\theta)$ is in $M_\beta^\infty(\mathbb{R}^N)$ for $\beta \in [0,N[$.

Then, by Steps 3.2, 3.3 and 3.4, we claim that

$$\partial^\alpha\nabla v \in M_\beta^\infty(\mathbb{R}^N)$$

for $\beta \in [0,N[$. Indeed, ∇v is C^∞ on $\mathbb{R}^N$ and is given by

$$\nabla v = \frac{\nabla\eta}{2\rho}e^{i\psi\theta} + i\rho\nabla(\psi\theta)e^{i\psi\theta}$$

at infinity: the claim follows from Theorem 3, Lemma 14, Steps 3.2, 3.3 and 3.4, the chain rule theorem and Leibniz's formula once more.

At last, we improve Step 3.1 so as to improve the estimate for the function $\partial^\alpha\nabla\eta$.

Step 3.5. *The function*

$$|.|^\beta\big(|\partial^\gamma F| + |\partial^\gamma G|\big)$$

is bounded for every $\beta \in [0,2N[$ *and for every* $\gamma \in \mathbb{N}^N$ *such that* $|\gamma| = p+1$.

The proof is similar to the proof of Step 3.1.

For instance, F is a C^∞ function on $\mathbb{R}^N$ given by

$$F = 2|\nabla v|^2 + 2\eta^2 - 2c\eta\partial_1(\psi\theta)$$

at infinity. By Leibnitz's formula, we compute again

$$\partial^\gamma F = 2\sum_{\delta\leqslant\gamma} c_{\delta,\gamma}\big[\partial^{\gamma-\delta}\nabla v.\partial^\delta\nabla v + \partial^{\gamma-\delta}\eta.\partial^\delta\eta - c\partial^{\gamma-\delta}\eta.\partial^\delta\partial_1(\psi\theta)\big].$$

On one hand, by the assumption of induction, we know

$$|.|^\beta\big(|\partial^\delta\nabla v| + |\partial^\delta\eta| + \big|\partial^\delta\partial_1(\psi\theta)\big|\big) \in L^\infty(\mathbb{R}^N)$$

for $\delta\leqslant\gamma$ and $\delta\neq\gamma$, and for $\beta\in[0,N[$.

On the other hand, by Steps 3.2, 3.3 and 3.4,

$$|.|^\beta\big(|\partial^\gamma\nabla v| + |\partial^\gamma\eta| + \big|\partial^\gamma\partial_1(\psi\theta)\big|\big) \in L^\infty(\mathbb{R}^N)$$

for every $\beta\in[0,N[$, so,

$$|.|^\beta|\partial^\gamma F| \in L^\infty(\mathbb{R}^N)$$

for every $\beta\in[0,2N[$.

The proof is identical for $\partial^\gamma G$.

We then deduce from Eq. (38)

Step 3.6. *Let $j\in\{1,\dots,N\}$ and $\beta\in[0,N+1[$. Then,*

- $\nabla K_0 * \partial^\alpha F \in M^\infty_\beta(\mathbb{R}^N)$,
- $\nabla K_j * \partial^\alpha G_j \in M^\infty_\beta(\mathbb{R}^N)$.

The proof is identical to the proof of Step 3.2 by replacing Step 3.1 by Step 3.5, so, we omit it.

By Eq. (38), $\partial^\alpha\nabla\eta$ belongs to $M^\infty_\beta(\mathbb{R}^N)$ for every $\beta\in[0,N+1[$, which achieves the inductive argument of the proof of Proposition 31. □

3.2.4. Critical decay of the functions η, $\nabla(\psi\theta)$ and ∇v

At last, we study the critical case, i.e. the case $\beta = N$ or $\beta = N+1$.

Proposition 32. *Let $\alpha\in\mathbb{N}^N$. Then,*

$$\big(\eta, \partial^\alpha\nabla(\psi\theta), \partial^\alpha\nabla v\big) \in M^\infty_N(\mathbb{R}^N)^3,$$

and

$$\partial^\alpha\nabla\eta \in M^\infty_{N+1}(\mathbb{R}^N).$$

Proof. The proof is similar to the proofs of Propositions 29, 30 and 31. We first recall some estimates for the functions F and G.

Step 4.1. *The function*

$$|.|^\beta\big(|\partial^\alpha F| + |\partial^\alpha G|\big)$$

is bounded on $\mathbb{R}^N$ for every $\alpha\in\mathbb{N}^N$ and $\beta\in[0,2N[$.

The proof of Step 4.1 is the same as the proof of Step 3.5, so, we omit it.

We then turn to the function η, and so, we study Eq. (10).

Step 4.2. *Let* $j \in \{1, \dots, N\}$. *Then,*

- $K_0 * F \in M_N^\infty(\mathbb{R}^N)$,
- $K_j * G_j \in M_N^\infty(\mathbb{R}^N)$.

Indeed, we have for every $x \in \mathbb{R}^N$,

$$|x|^N \left|K_0 * F(x)\right| \leqslant A\left(\int_{\mathbb{R}^N} |x-y|^N \left|K_0(x-y)\right|\left|F(y)\right| dy + \int_{\mathbb{R}^N} \left|K_0(x-y)\right||y|^N\left|F(y)\right| dy\right).$$

On one hand, by Theorem 9 and Step 1 of the proof of Proposition 18,

$$\left\|\left(|.|^N K_0\right) * F\right\|_{L^\infty(\mathbb{R}^N)} \leqslant \left\||.|^N K_0\right\|_{L^\infty(\mathbb{R}^N)} \|F\|_{L^1(\mathbb{R}^N)} < +\infty.$$

On the other hand, by Corollary 24,

$$K_0 \in L^q(\mathbb{R}^N)$$

for $1 < q < \frac{N}{N-2}$: so,

$$\left\|K_0 * \left(|.|^N F\right)\right\|_{L^\infty(\mathbb{R}^N)} \leqslant \|K_0\|_{L^q(\mathbb{R}^N)} \left\||.|^N F\right\|_{L^{q'}(\mathbb{R}^N)}.$$

By Step 4.1, there is some real number $1 < q < \frac{N}{N-2}$ such that

$$\left\||.|^N F\right\|_{L^{q'}(\mathbb{R}^N)} < +\infty,$$

so, the function $K_0 * (|.|^N F)$ is bounded on $\mathbb{R}^N$, such as the function $|.|^N K_0 * F$: the proof being identical for the functions $|.|^N K_j * G_j$ by replacing F by G_j, we omit it.

By Step 4.2 and Eq. (10), Proposition 32 holds for the function η.

For the functions $\partial^\alpha \nabla \eta$, we study Eq. (38).

Step 4.3. *Let* $j \in \{1, \dots, N\}$. *Then,*

- $\nabla K_0 * \partial^\alpha F \in M_{N+1}^\infty(\mathbb{R}^N)$,
- $\nabla K_j * \partial^\alpha G_j \in M_{N+1}^\infty(\mathbb{R}^N)$.

Indeed, we have for every $x \in \mathbb{R}^N$,

$$|x|^{N+1}\left|\nabla K_0 * \partial^\alpha F(x)\right| \leqslant A \int_{\mathbb{R}^N} \left(|x-y|^{N+1}\left|\nabla K_0(x-y)\right|\left|\partial^\alpha F(y)\right| + \left|\nabla K_0(x-y)\right||y|^{N+1}\left|\partial^\alpha F(y)\right|\right) dy.$$

On one hand, by Theorem 9 and Step 1 of the proof of Proposition 18,

$$\left\|\left(|.|^{N+1}\nabla K_0\right) * \partial^\alpha F\right\|_{L^\infty(\mathbb{R}^N)} \leqslant \left\||.|^{N+1}\nabla K_0\right\|_{L^\infty(\mathbb{R}^N)} \|\partial^\alpha F\|_{L^1(\mathbb{R}^N)} < +\infty.$$

On the other hand, by Corollary 24,

$$\nabla K_0 \in L^q(\mathbb{R}^N)$$

for $1 \leqslant q < \frac{N}{N-1}$: so,

$$\left\|\nabla K_0 * \left(|.|^{N+1}\partial^\alpha F\right)\right\|_{L^\infty(\mathbb{R}^N)} \leqslant \|\nabla K_0\|_{L^q(\mathbb{R}^N)} \left\||.|^{N+1}\partial^\alpha F\right\|_{L^{q'}(\mathbb{R}^N)}.$$

By Step 4.1, there is some real number $1 < q < \frac{N}{N-2}$ such that

$$\left\||.|^{N+1}\partial^\alpha F\right\|_{L^{q'}(\mathbb{R}^N)} < +\infty,$$

so, the function $\nabla K_0 * (|.|^{N+1}\partial^\alpha F)$ is bounded on $\mathbb{R}^N$, such as the function $|.|^{N+1}\nabla K_0 * \partial^\alpha F$: the proof being identical for the functions $|.|^{N+1}\nabla K_j * \partial^\alpha G_j$ by replacing $\partial^\alpha F$ by $\partial^\alpha G_j$, we omit it.

By Step 4.3 and Eq. (38), Proposition 32 also holds for the function $\partial^\alpha \nabla\eta$.

We then deduce a similar estimate for $\partial^\alpha \partial_j(\psi\theta)$ by Eq. (39): we first study the terms involving the kernels K_j and $L_{j,k}$.

Step 4.4. *Let $(j,k) \in \{1,\ldots,N\}^2$. Then,*

- $K_j * \partial^\alpha F \in M_N^\infty(\mathbb{R}^N)$,
- $L_{j,k} * \partial^\alpha G_k \in M_N^\infty(\mathbb{R}^N)$.

The proof is identical to the proof of Steps 4.2 and 4.3, so, we omit it.

Finally, it only remains to evaluate the functions $R_{j,k} * G_k$.

Step 4.5. *Let $(j,k) \in \{1,\ldots,N\}^2$. Then,*

$$R_{j,k} * \partial^\alpha G_k \in M_N^\infty(\mathbb{R}^N).$$

Indeed, by Step 4.1 and Step 1 of the proof of Proposition 18, $\partial^\alpha G$ and $\partial^\alpha \nabla G$ belong to $L^1(\mathbb{R}^N)$, and $|.|^N\partial^\alpha G$ and $|.|^{N+1}\partial^\alpha\nabla G$ are bounded on $\mathbb{R}^N$: Step 4.5 then follows from Proposition 27.

Steps 4.4 and 4.5 yield the critical decay of $\partial^\alpha\nabla(\psi\theta)$, and we can achieve the proofs of Proposition 32 and of Theorem 11 by proving the critical decay of the functions $\partial^\alpha \nabla v$. Indeed, ∇v is C^∞ on $\mathbb{R}^N$ and is given by

$$\nabla v = \frac{\nabla\eta}{2\rho}e^{i\psi\theta} + i\rho\nabla(\psi\theta)e^{i\psi\theta}$$

at infinity: the critical decay of $\partial^\alpha \nabla v$ then follows from Theorem 3, Lemma 14, Steps 4.3, 4.4 and 4.5, the chain rule theorem and Leibnitz's formula. □

3.3. Asymptotic decay for the function v

In the last section, we complete the proof of Theorem 1: we have already shown the convergence at infinity of v towards a complex number of modulus one in Corollary 4. We are now in position to prove the second part of Theorem 1.

Proposition 33. *The function $|.|^{N-1}(v-1)$ is bounded on $\mathbb{R}^N$.*

Proof. Indeed, by Theorem 11, the function $|.|^N\nabla v$ is bounded on $\mathbb{R}^N$. Since

$$\forall x \in \mathbb{R}^N \setminus \{0\}, \quad v(x) - 1 = -\int_{|x|}^{+\infty} \partial_r v\left(\frac{sx}{|x|}\right) ds,$$

we get

$$\forall x \in \mathbb{R}^N \setminus \{0\}, \quad \left|v(x) - 1\right| \leqslant A \int_{|x|}^{+\infty} \frac{ds}{s^N} \leqslant \frac{A}{|x|^{N-1}},$$

which achieves the proofs of Proposition 33 and of Theorem 1. □

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