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Asymptotic behaviour and correctors for linear Dirichlet problems with simultaneously varying operators and domains

Comportement asymptotique et correcteurs pour des problèmes de Dirichlet linéaires avec des opérateurs et des domaines qui varient simultanément

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Abstract

We consider a sequence of Dirichlet problems in varying domains (or, more generally, of relaxed Dirichlet problems involving measures in $\mathcal{M}_0^+(\Omega)$) for second order linear elliptic operators in divergence form with varying matrices of coefficients. When the matrices H-converge to a matrix A^0, we prove that there exist a subsequence and a measure μ^0 in $\mathcal{M}_0^+(\Omega)$ such that the limit problem is the relaxed Dirichlet problem corresponding to A^0 and μ^0. We also prove a corrector result which provides an explicit approximation of the solutions in the H^1-norm, and which is obtained by multiplying the corrector for the H-converging matrices by some special test function which depends both on the varying matrices and on the varying domains.

Résumé

Nous considérons une suite de problèmes de Dirichlet dans des ouverts variables (ou plus généralement une suite de problèmes de Dirichlet relaxés définis par des mesures de $\mathcal{M}_0^+(\Omega)$) pour des opérateurs elliptiques linéaires du deuxième ordre sous forme divergence avec des matrices de coefficients elles aussi variables. Quand les matrices H-convergent vers une matrice A^0, nous démontrons qu'il existe une sous suite et une mesure μ^0 de $\mathcal{M}_0^+(\Omega)$ telles qu'à la limite on obtienne le problème de Dirichlet relaxé correspondant à A^0 et μ^0. Nous démontrons également un résultat de correcteur qui donne une approximation explicite des solutions en norme H^1 ; ce correcteur est obtenu en multipliant le correcteur pour la H-convergence des matrices par une fonction test spéciale qui dépend à la fois des matrices variables et des ouverts variables.

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1. Introduction

In this paper we consider a sequence of linear Dirichlet problems

$$\begin{cases} u^\varepsilon \in H_0^1(\Omega^\varepsilon), \\ -\operatorname{div}(A^\varepsilon Du^\varepsilon) = f \quad \text{in } \mathcal{D}'(\Omega^\varepsilon), \end{cases} \tag{1.1}$$

where the matrices A^ε and the domains Ω^ε both depend on the parameter ε. We assume that the open sets Ω^ε are all contained in a fixed bounded open subset Ω of $\mathbf{R}^n$, and that the matrices A^ε, defined on Ω with measurable coefficients, are coercive and bounded, uniformly with respect to ε. Our goal is to study the behaviour of the solutions u^ε as ε tends to zero.

In the special case $\Omega^\varepsilon = \Omega$ it is known (see Section 3) that there exist a subsequence, still denoted by (A^ε), and a matrix A^0, called the H-limit of (A^ε), such that for every $f \in H^{-1}(\Omega)$ the solutions v^ε of the problems

$$\begin{cases} v^\varepsilon \in H_0^1(\Omega), \\ -\operatorname{div}(A^\varepsilon Dv^\varepsilon) = f \quad \text{in } \mathcal{D}'(\Omega), \end{cases}$$

converge weakly in $H_0^1(\Omega)$ to the solution v^0 of

$$\begin{cases} v^0 \in H_0^1(\Omega), \\ -\operatorname{div}(A^0 Dv^0) = f \quad \text{in } \mathcal{D}'(\Omega), \end{cases}$$

and satisfy also

$$A^\varepsilon Dv^\varepsilon \rightharpoonup A^0 Dv^0 \quad \text{weakly in } L^2(\Omega, \mathbf{R}^n).$$

Without making any further hypothesis on the open sets Ω^ε, we prove in the present paper that there exists a subsequence, still denoted by (Ω^ε), such that for every $f \in H^{-1}(\Omega)$ the solutions u^ε of (1.1) converge to the solution u^0 of the problem

$$\begin{cases} u^0 \in H_0^1(\Omega) \cap L^2(\Omega, \mu^0), \\ \displaystyle\int_\Omega A^0 Du^0 Dy \, \mathrm{d}x + \int_\Omega u^0 y \, \mathrm{d}\mu^0 = \langle f, y \rangle \quad \forall y \in H_0^1(\Omega) \cap L^2(\Omega, \mu^0), \end{cases} \tag{1.2}$$

where μ^0 belongs to $\mathcal{M}_0^+(\Omega)$, a class of nonnegative Borel measures which vanish on all sets of capacity zero, but can take the value $+\infty$ on some subsets of Ω (see Section 2).

Problems like (1.2) are called relaxed Dirichlet problems (see Section 4) and have been extensively studied to describe the limits of the solutions of (1.1) when the matrices A^ε do not depend on ε. On the other hand, problems (1.1) can be written as relaxed Dirichlet problems (see Remark 4.1) by considering the measures μ^ε defined by

$$\mu^\varepsilon(B) = \begin{cases} 0, & \text{if } \operatorname{cap}(B \setminus \Omega^\varepsilon) = 0, \\ +\infty, & \text{otherwise.} \end{cases} \tag{1.3}$$

Actually in the paper we consider not only the case of Dirichlet problems (1.1), which correspond to the measures μ^ε defined by (1.3), but more in general we study the case of a sequence of relaxed Dirichlet problems with arbitrary $\mu^\varepsilon \in \mathcal{M}_0^+(\Omega)$.

In the limit problem (1.2) the measure μ^0 does not depend on f, but, as shown in Section 6, it depends both on the sequence of sets (Ω^ε) and on the sequence of matrices (A^ε) (and not only on its H-limit A^0). Nevertheless the sequence (Ω^ε) has a stronger influence than the sequence (A^ε): indeed the limit measures corresponding to the same sequence (Ω^ε) but to different sequences (A_i^ε) are equivalent (see Theorem 8.1).

In Section 5 we give a fairly general and flexible method to construct the limit measure μ^0 using suitable test functions ω^ε associated to Ω^ε and A^ε. We then pass to the limit in the sequence of problems (1.1) by a duality argument and obtain (1.2).

In Section 7 we continue the study of the behaviour of the solutions u^ε of (1.1) by giving a corrector result. By this we mean the following: when the solution u^0 of the limit problem (1.2) can be written as

$$u^0 = \psi\omega^0, \tag{1.4}$$

where ω^0 is the limit of the above test functions ω^ε and ψ is sufficiently smooth (actually in $H^2(\Omega)\cap W^{1,\infty}(\Omega)$), we prove that

$$u^\varepsilon = \left(\psi + \sum_{j=1}^{n} D_j\psi\, z_j^\varepsilon\right)\omega^\varepsilon + r^\varepsilon \quad \text{with } r^\varepsilon \to 0 \text{ strongly in } H_0^1(\Omega), \tag{1.5}$$

where the functions z_j^ε depend only on the matrices A^ε. This provides an approximation of u^ε in the norm of $H_0^1(\Omega)$ by means of functions that are constructed explicitly.

When (1.4) is not satisfied with a smooth ψ, a similar but more technical result holds (see Theorem 7.2). We also prove a local version of this corrector result.

Moreover, we prove (global and local) convergence and corrector results when also the right-hand side of (1.1) depends on ε and converges strongly in a convenient sense (see Section 10).

Let us finally note that the case where the matrices A^ε and the domains Ω^ε are periodic, with periods converging to zero with different speeds, has been studied in detail by Ansini and Braides in [1]. Results similar to those proved in the present paper have been obtained by Kovalevsky in [18] for a class of nonlinear monotone elliptic equations under some geometric assumptions on the sets Ω^ε, and more recently by Calvo Jurado and Casado Diaz in [6] in the general case.

2. Preliminaries on capacity and measures

In this section we first introduce a few notation. Then we recall some known results on measures, capacity, and fine properties of Sobolev functions.

2.1. Notation

Throughout the paper Ω is a bounded open subset of $\mathbf{R}^n$, $n \geqslant 1$. The space $\mathcal{D}'(\Omega)$ of distributions in Ω is the dual of the space $C_c^\infty(\Omega)$. The space $W_0^{1,p}(\Omega)$, $1 \leqslant p < +\infty$, is the closure of $C_c^\infty(\Omega)$ in the Sobolev space $W^{1,p}(\Omega)$, while $W^{-1,q}(\Omega)$, $1 \leqslant q < +\infty$, is the space of all distributions of the form $f = f_0 + \sum_j D_j f_j$, with $f_0, f_1, \ldots, f_n \in L^q(\Omega)$ (if $1/p + 1/q = 1$, then $W^{-1,q}(\Omega)$ is the dual of $W_0^{1,p}(\Omega)$). In the Hilbert case $p = q = 2$ these spaces are denoted by $H_0^1(\Omega)$, $H^1(\Omega)$, and $H^{-1}(\Omega)$, respectively. The norm in $H_0^1(\Omega)$ is defined by

$$\|u\|_{H_0^1(\Omega)} = \left(\int_\Omega |Du|^2\,\mathrm{d}x\right)^{1/2},$$

while the duality pairing between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$ is denoted by $\langle\cdot,\cdot\rangle$. We shall sometimes use also the Sobolev space $H^2(\Omega) = W^{2,2}(\Omega)$.

The *adjoint* of a matrix A is denoted by $\bar{A}$. Since complex numbers are not used in this paper, *the bar never denotes complex conjugation*. If w is an object related to the matrix A, then $\overline{w}$ denotes the corresponding object related to the adjoint $\bar{A}$.

Throughout the paper ε varies in a stricly decreasing sequence of positive real numbers which converges to 0. When we write $\varepsilon > 0$, we consider only the elements of this sequence, while when we write $\varepsilon \geqslant 0$ we also consider its limit $\varepsilon = 0$.

2.2. Capacity and measures

For every subset E of Ω the *capacity* of E in Ω, denoted by $\mathrm{cap}(E)$, is defined as the infimum of $\int_\Omega |Du|^2\,\mathrm{d}x$ over the set of all functions $u \in H_0^1(\Omega)$ such that $u \geqslant 1$ a.e. in a neighbourhood of E. We say that a property $\mathcal{P}(x)$ holds *quasi everywhere* (abbreviated as *q.e.*) in a set E if it holds for all $x \in E$ except for a subset N of E with $\mathrm{cap}(N) = 0$. The expression *almost everywhere* (abbreviated as *a.e.*) refers, as usual, to the analogous property for the Lebesgue measure.

A function $u : \Omega \to \mathbf{R}$ is said to be *quasi continuous* if for every $\varepsilon > 0$ there exists a set $E \subseteq \Omega$, with $\mathrm{cap}(E) < \varepsilon$, such that the restriction of u to $\Omega \setminus E$ is continuous. A subset U of Ω is said to be *quasi open* if for every $\varepsilon > 0$ there exists an open set $V \subseteq \Omega$, with $\mathrm{cap}(V \vartriangle U) < \varepsilon$, where $\vartriangle$ denotes the symmetric difference.

Every $u \in H^1(\Omega)$ has a *quasi continuous representative*, which is uniquely defined up to a set of capacity zero. In the sequel we shall always identify u with its quasi continuous representative, so that the pointwise values of a function $u \in H^1(\Omega)$ are defined quasi everywhere in Ω. If $u \in H^1(\Omega)$, then

$$u \geqslant 0 \quad \text{a.e. in } \Omega \quad \Longleftrightarrow \quad u \geqslant 0 \quad \text{q.e. in } \Omega. \tag{2.1}$$

If a sequence (u_j) converges to u strongly in $H_0^1(\Omega)$, then a subsequence of (u_j) converges to u q.e. in Ω. For all these properties concerning quasi continuous representatives of Sobolev functions we refer to [16], Section 4.8, [17], Section 4, [19], Section 7.2.4, or [27], Chapter 3.

The characteristic function 1_E of a set $E \subseteq \Omega$ is defined by $1_E(x) = 1$ if $x \in E$ and $1_E(x) = 0$ if $x \in \Omega \setminus E$. The following lemma (see [9], Lemma 1.5, or [11], Lemma 1.1) concerns the pointwise approximation of the characteristic function of a quasi open set.

Lemma 2.1. *For every quasi open set U of Ω there exists an increasing sequence (z_k) of nonnegative functions of $H_0^1(\Omega)$ converging to 1_U pointwise q.e. in Ω.*

By a *nonnegative Borel measure* on Ω we mean a countably additive set function defined on the Borel subsets of Ω with values in $[0, +\infty]$. By a *nonnegative Radon measure* on Ω we mean a nonnegative Borel measure which is finite on every compact subset of Ω. Every nonnegative Borel measure μ on Ω can be extended to a Borel regular outer measure on Ω by setting for every subset E of Ω

$$\mu(E) = \inf\bigl\{\mu(B):\ B \text{ Borel},\ E \subseteq B \subseteq \Omega\bigr\}.$$

If μ is a nonnegative Borel measure on Ω, we shall use $L^r(\Omega, \mu)$, $1 \leqslant r \leqslant +\infty$, to denote the usual Lebesgue space with respect to the measure μ. We adopt the standard notation $L^r(\Omega)$ when μ is the Lebesgue measure.

We will consider the cone $\mathcal{M}_0^+(\Omega)$ of all nonnegative Borel measures μ on Ω such that

(a) $\mu(B) = 0$ for every Borel set $B \subseteq \Omega$ with $\mathrm{cap}(B) = 0$,

(b) $\mu(B) = \inf\{\mu(U):\ U \text{ quasi open},\ B \subseteq U\}$ for every Borel set $B \subseteq \Omega$.

If $E \subseteq \Omega$ and $\mathrm{cap}(E) = 0$, then E is contained in a Borel set $B \subseteq \Omega$ with $\mathrm{cap}(B) = 0$. Therefore E is μ-measurable by (a). Property (b) is a weak regularity property of the measure μ. It is always satisfied if μ is a nonnegative Radon measure. Since any quasi open set differs from a Borel set by a set of capacity zero, every quasi open set is μ-measurable for every nonnegative Borel measure μ which satisfies (a).

Let us explicitly observe that the notation is not fixed in the literature and that in other works (see, e.g., [14]) $\mathcal{M}_0(\Omega)$ denotes the set of nonnegative Borel measures which only satisfy (a), while the set that we call $\mathcal{M}_0^+(\Omega)$ in the present paper is sometimes denoted by $\mathcal{M}_0^*(\Omega)$ (see, e.g., [10]).

For every quasi open set $U \subseteq \Omega$ we define the Borel measure μ_U by

$$\mu_U(B) = \begin{cases} 0, & \text{if } \operatorname{cap}(B \setminus U) = 0, \\ +\infty, & \text{otherwise.} \end{cases} \tag{2.2}$$

Roughly speaking, μ_U is identically zero on U and identically $+\infty$ on $\Omega \setminus U$. It is easy to see that this measure belongs to the class $\mathcal{M}_0^+(\Omega)$. Indeed, property (a) follows immediately from the definition, and it is enough to verify (b) only for every Borel set with $\mu_U(B) < +\infty$; in this case $\operatorname{cap}(B \setminus U) = 0$, and this implies that $V = U \cup B$ is quasi open (since U is quasi open), contains B, and $\mu_U(V) = 0$ (since $\operatorname{cap}(V \setminus U) = \operatorname{cap}(B \setminus U) = 0$), so that (b) is satisfied. The measures μ_U will be used to transform a sequence of Dirichlet problems on varying domains into a sequence of relaxed Dirichlet problems on a fixed domain (see Remark 4.1 and the proof of Corollary 5.5).

If $\mu \in \mathcal{M}_0^+(\Omega)$, then the space $H^1(\Omega) \cap L^2(\Omega, \mu)$ is well defined, since every function u in $H^1(\Omega)$ is defined μ-almost everywhere and is μ-measurable in Ω (recall that u is quasi continuous, so that $\{u > t\}$ is quasi open for every $t \in \mathbf{R}$). It is easy to see that $H^1(\Omega) \cap L^2(\Omega, \mu)$ is a Hilbert space for the scalar product

$$(u, v)_{H^1(\Omega)\cap L^2(\Omega,\mu)} = \int_\Omega Du\, Dv\, \mathrm{d}x + \int_\Omega uv\, \mathrm{d}x + \int_\Omega uv\, \mathrm{d}\mu \tag{2.3}$$

(see [5], Proposition 2.1).

The space of all (signed) Radon measures on Ω will be denoted by $\mathcal{M}(\Omega)$, while $\mathcal{M}_\mathrm{b}(\Omega)$ will be the space of all $\mu \in \mathcal{M}(\Omega)$ with $|\mu|(\Omega) < +\infty$, where $|\mu|$ denotes the total variation of μ. A subset $\mathcal{A}$ of $\mathcal{M}(\Omega)$ is bounded if for every compact set $K \subseteq \Omega$ we have

$$\sup_{\mu \in \mathcal{A}} |\mu|(K) < +\infty.$$

Every Radon measure on Ω will be identified with an element of $\mathcal{D}'(\Omega)$ in the usual way. Therefore μ belongs to $\mathcal{M}(\Omega) \cap W^{-1,q}(\Omega)$ if and only if there exist $f_0, f_1, \ldots, f_n \in L^q(\Omega)$ such that

$$\int_\Omega \varphi\, \mathrm{d}\mu = \int_\Omega f_0 \varphi\, \mathrm{d}x - \sum_{j=1}^n \int_\Omega f_j D_j \varphi\, \mathrm{d}x \quad \forall \varphi \in C_c^\infty(\Omega).$$

Note that, by Riesz' theorem, every nonnegative element of $W^{-1,p}(\Omega)$ is a nonnegative Radon measure on Ω.

The cone of all nonnegative elements of $H^{-1}(\Omega)$ will be denoted by $H^{-1}(\Omega)^+$. It is well known that every element of $H^{-1}(\Omega)^+$ is a nonnegative Radon measure which belongs also to $\mathcal{M}_0^+(\Omega)$. In other words we have the inclusion $H^{-1}(\Omega)^+ \subseteq \mathcal{M}(\Omega) \cap \mathcal{M}_0^+(\Omega)$.

3. H-convergence

In this section we recall the definition of H-convergence and the corresponding corrector result. Moreover we prove a fairly general convergence theorem for right-hand sides which do not converge strongly in $H^{-1}(\Omega)$.

Throughout the paper we fix two constants α and β such that

$$0 < \alpha \leqslant \beta < +\infty.$$

We define $M_\alpha^\beta(\Omega)$ as the set of all matrices A in $L^\infty(\Omega, \mathbf{R}^{n\times n})$ such that

$$A(x) \geqslant \alpha I, \quad \big(A(x)\big)^{-1} \geqslant \beta^{-1} I, \quad \text{for a.e. } x \in \Omega. \tag{3.1}$$

In (3.1) I is the identity matrix in $\mathbf{R}^{n\times n}$, and the inequalities are in the sense of the quadratic forms defined by$A(x)\xi\xi$ for $\xi \in \mathbf{R}^n$. Note that (3.1) implies that

$$\big|A(x)\big| \leqslant \beta \quad \text{for a.e. } x \in \Omega, \tag{3.2}$$

and that necessarily $\alpha \leqslant \beta$.

3.1. Definition of H-convergence

A sequence (A^ε) of matrices in $M_\alpha^\beta(\Omega)$ *H-converges* to a matrix A^0 in $M_\alpha^\beta(\Omega)$ if for every $f \in H^{-1}(\Omega)$ the sequence (u^ε) of the solutions to the problems

$$\begin{cases} u^\varepsilon \in H_0^1(\Omega), \\ -\operatorname{div}\big(A^\varepsilon Du^\varepsilon\big) = f \quad \text{in } \mathcal{D}'(\Omega), \end{cases} \tag{3.3}$$

satisfies

$$u^\varepsilon \rightharpoonup u^0 \quad \text{weakly in } H_0^1(\Omega),$$
$$A^\varepsilon Du^\varepsilon \rightharpoonup A^0 Du^0 \quad \text{weakly in } L^2\big(\Omega, \mathbf{R}^n\big),$$

where u^0 is the solution to the problem

$$\begin{cases} u^0 \in H_0^1(\Omega), \\ -\operatorname{div}\big(A^0 Du^0\big) = f \quad \text{in } \mathcal{D}'(\Omega). \end{cases} \tag{3.4}$$

Every sequence of matrices in $M_\alpha^\beta(\Omega)$ has a subsequence which H-converges to a matrix in $M_\alpha^\beta(\Omega)$ (see [22,24,26]).

Denoting the adjoint of A^ε by $\bar{A}^\varepsilon$, it is easy to prove that the sequence $(\bar{A}^\varepsilon)$ H-converges to $\bar{A}^0$ when the sequence (A^ε) H-converges to A^0.

If U is an open set contained in Ω, we can consider also the notion of H-convergence in U, replacing Ω by U in the definition. It is not difficult to prove that (A^ε) H-converges to A^0 in U, for every open set $U \subseteq \Omega$, if (A^ε) H-converges to A^0 in Ω.

3.2. Corrector result

Besides the compactness result mentioned above, one of the main theorems is the corrector result (see [22,26], and [3,23] in the periodic case). Let $(e_1, e_2, \dots, e_n)$ be the canonical basis of $\mathbf{R}^n$. For $j = 1, 2, \dots, n$ there exists a sequence (z_j^ε) in $H^1(\Omega)$ such that

$$z_j^\varepsilon \rightharpoonup 0 \quad \text{weakly in } H^1(\Omega), \tag{3.5}$$

$$A^\varepsilon\big(Dz_j^\varepsilon + e_j\big) \rightharpoonup A^0 e_j \quad \text{weakly in } L^2\big(\Omega, \mathbf{R}^n\big), \tag{3.6}$$

$$-\operatorname{div}\big(A^\varepsilon\big(Dz_j^\varepsilon + e_j\big)\big) \to -\operatorname{div}\big(A^0 e_j\big) \quad \text{strongly in } H^{-1}(\Omega). \tag{3.7}$$

Throughout the paper we will also assume that

$$z_j^\varepsilon \to 0 \quad \text{strongly in } L^\infty(\Omega), \tag{3.8}$$

$$z_j^\varepsilon \rightharpoonup 0 \quad \text{weakly in } W^{1,p}(\Omega) \text{ for some } p > 2; \tag{3.9}$$

using De Giorgi's and Meyers' regularity theorems, such a sequence can be constructed, for instance, by solving the problems

$$\begin{cases} z_j^\varepsilon \in H_0^1\big(\Omega'\big), \\ -\operatorname{div}\big(A^\varepsilon\big(Dz_j^\varepsilon + e_j\big)\big) = -\operatorname{div}\big(A^0 e_j\big) \quad \text{in } \mathcal{D}'\big(\Omega'\big), \end{cases}$$

where Ω' is a bounded open set with $\Omega \Subset \Omega'$, and A^ε is extended by αI on $\Omega' \setminus \Omega$. (The use of Ω' is needed here only to obtain a global $W^{1,p}(\Omega)$ bound for z_j^ε in the case where $\partial\Omega$ is not smooth.)

Let $f \in H^{-1}(\Omega)$, let (u^ε) be the sequence of the solutions to (3.3), and let u^0 be the solution to (3.4). Given $\delta > 0$, let ψ_δ be a function in $C_c^\infty(\Omega)$ which satisfies

$$\beta \int_\Omega \left|Du^0 - D\psi_\delta\right|^2 \mathrm{d}x < \delta, \tag{3.10}$$

and let v_δ^ε be defined by

$$v_\delta^\varepsilon = \psi_\delta + \sum_{j=1}^n D_j \psi_\delta \, z_j^\varepsilon. \tag{3.11}$$

Then (see [22,26])

$$\limsup_{\varepsilon \to 0} \alpha \int_\Omega \left|Du^\varepsilon - Dv_\delta^\varepsilon\right|^2 \mathrm{d}x < \delta. \tag{3.12}$$

If u^0 belongs to $C_c^\infty(\Omega)$, we can take $\psi_\delta = u^0$ in (3.10) for every $\delta > 0$, so that

$$v_\delta^\varepsilon = v^\varepsilon = u^0 + \sum_{j=1}^n D_j u^0 z_j^\varepsilon, \tag{3.13}$$

and (3.12) implies that

$$Du^\varepsilon - Dv^\varepsilon \to 0 \quad \text{strongly in } L^2\big(\Omega, \mathbf{R}^n\big), \tag{3.14}$$

which means that Du^ε is equivalent to Dv^ε (and also to $Du^0 + \sum_j D_j u^0 D z_j^\varepsilon$, using (3.5)), as far as convergences in $L^2(\Omega, \mathbf{R}^n)$ are concerned.

In the general case where u^0 only belongs to $H_0^1(\Omega)$, we obtain from (3.12) that

$$Du^\varepsilon = D\psi_\delta + \sum_{j=1}^n D_j \psi_\delta \, D z_j^\varepsilon + R_\delta^\varepsilon, \quad \text{with } \limsup_{\varepsilon \to 0} \left\| R_\delta^\varepsilon \right\|^2_{L^2(\Omega, \mathbf{R}^n)} < \frac{\delta}{\alpha}.$$

This is a corrector result: indeed it allows one to replace Du^ε by an explicit expression, up to a remainder R_δ^ε which is small in $L^2(\Omega, \mathbf{R}^n)$ for δ small, uniformly in ε. Similar corrector results also hold in the case of local solutions. Applications can be found, e.g., in [2,8].

3.3. *A convergence result*

We conclude this section with the following convergence result, which is implicitly used in various works (see, e.g., [4]). Observe that there is no boundary condition on the solutions u^ε and that the right-hand sides f^ε do not converge strongly in $H^{-1}(\Omega)$.

Theorem 3.1. *Let (A^ε) be a sequence of matrices in $M_\alpha^\beta(\Omega)$ which H-converges to a matrix A^0 in $M_\alpha^\beta(\Omega)$, and let (u^ε) be a sequence in $H^1(\Omega)$ such that*

$$\begin{cases} u^\varepsilon \rightharpoonup u^0 \quad \text{weakly in } H^1(\Omega), \\ -\operatorname{div}\big(A^\varepsilon Du^\varepsilon\big) = f^\varepsilon \quad \text{in } \mathcal{D}'(\Omega) \text{ for every } \varepsilon \geqslant 0. \end{cases} \tag{3.15}$$

Assume that $f^\varepsilon = g^\varepsilon + \mu^\varepsilon + \nu^\varepsilon$ for every $\varepsilon > 0$, where

$$\begin{cases} (g^\varepsilon) \text{ is relatively compact in } W_{\mathrm{loc}}^{-1,p}(\Omega) \text{ for some } p > 1, \\ (\mu^\varepsilon) \text{ is bounded in } \mathcal{M}(\Omega), \\ \nu^\varepsilon \geqslant 0 \text{ in } \mathcal{D}'(\Omega). \end{cases} \tag{3.16}$$

Then

$$\begin{cases} f^\varepsilon \rightharpoonup f^0 \quad \textit{weakly in } H^{-1}(\Omega) \textit{ and strongly in } W^{-1,q}_{\mathrm{loc}}(\Omega) \textit{ for every } q<2, \\ A^\varepsilon Du^\varepsilon \rightharpoonup A^0 Du^0 \quad \textit{weakly in } L^2\big(\Omega,\mathbf{R}^n\big). \end{cases} \tag{3.17}$$

In the present paper, this theorem will be used with $\mu^\varepsilon = 0$ and (g^ε) relatively compact (or even constant) in $H^{-1}(\Omega)$.

Proof. Let K be any compact set of $\mathbf{R}^n$ with $K \subseteq \Omega$, and let $\varphi \in C^\infty_c(\Omega)$ with $\varphi \geqslant 0$ on Ω and $\varphi = 1$ on K. We have

$$0 \leqslant \int\limits_K \mathrm{d}\nu^\varepsilon \leqslant \int\limits_\Omega \varphi\,\mathrm{d}\nu^\varepsilon = \int\limits_\Omega A^\varepsilon Du^\varepsilon D\varphi\,\mathrm{d}x - \langle g^\varepsilon, \varphi\rangle - \int\limits_\Omega \varphi\,\mathrm{d}\mu^\varepsilon. \tag{3.18}$$

Because of (3.2), (3.15), and (3.16), the right-hand side of (3.18) is bounded independently of ε. This implies that

$$(\nu^\varepsilon) \text{ is bounded in } \mathcal{M}(\Omega). \tag{3.19}$$

For every bounded open set U of $\mathbf{R}^n$, the embedding $W^{1,r}_0(U) \subseteq C^0_0(U)$ is compact for every $r > n$. This implies that the embedding $\mathcal{M}_b(U) \subseteq W^{-1,s}(U)$ is compact for every $s < n/(n-1)$, and therefore the embedding $\mathcal{M}(\Omega) \subseteq W^{-1,s}_{\mathrm{loc}}(\Omega)$ is compact for every $s < n/(n-1)$. Therefore (3.16) and (3.19) imply that $(\mu^\varepsilon + \nu^\varepsilon)$ is relatively compact in $W^{-1,s}_{\mathrm{loc}}(\Omega)$, which implies by (3.16) that (f^ε) is relatively compact in $W^{-1,t}_{\mathrm{loc}}(\Omega)$ for some $t > 1$. On the other hand, we deduce from (3.15) and (3.2) that (f^ε) is bounded in $H^{-1}(\Omega)$. By interpolation, (f^ε) is relatively compact in $W^{-1,q}_{\mathrm{loc}}(\Omega)$ for every $q < 2$.

Let now $\bar v^0$ be an arbitrary function in $C^\infty_c(\Omega)$, and, for every $\varepsilon > 0$, let $\bar v^\varepsilon$ be the solution to the problem

$$\begin{cases} \bar v^\varepsilon \in H^1_0(\Omega), \\ -\operatorname{div}\big(\bar A^\varepsilon D\bar v^\varepsilon\big) = -\operatorname{div}\big(\bar A^0 D\bar v^0\big) \quad \text{in } \mathcal{D}'(\Omega). \end{cases} \tag{3.20}$$

Recall that the sequence $(\bar A^\varepsilon)$ H-converges to $\bar A^0$, so that

$$\begin{cases} \bar v^\varepsilon \rightharpoonup \bar v^0 \quad \text{weakly in } H^1_0(\Omega), \\ \bar A^\varepsilon D\bar v^\varepsilon \rightharpoonup \bar A^0 D\bar v^0 \quad \text{weakly in } L^2\big(\Omega,\mathbf{R}^n\big), \\ \bar v^\varepsilon \rightharpoonup \bar v^0 \quad \text{weakly in } W^{1,p}_{\mathrm{loc}}(\Omega) \text{ for some } p > 2, \end{cases} \tag{3.21}$$

where in the last assertion we have used Meyers' regularity result (see [20]).

Let $\varphi \in C^\infty_c(\Omega)$. Using $\bar v^\varepsilon\varphi$ as test function in (3.15), and $u^\varepsilon\varphi$ as test function in (3.20), we have

$$\begin{cases} \big\langle f^\varepsilon, \bar v^\varepsilon\varphi\big\rangle = \displaystyle\int\limits_\Omega A^\varepsilon Du^\varepsilon D\bar v^\varepsilon \varphi\,\mathrm{d}x + \int\limits_\Omega A^\varepsilon Du^\varepsilon D\varphi\,\bar v^\varepsilon\,\mathrm{d}x \\ \qquad = \big\langle -\operatorname{div}\big(\bar A^0 D\bar v^0\big), u^\varepsilon\varphi\big\rangle - \displaystyle\int\limits_\Omega \bar A^\varepsilon D\bar v^\varepsilon D\varphi\, u^\varepsilon\,\mathrm{d}x + \int\limits_\Omega A^\varepsilon Du^\varepsilon D\varphi\,\bar v^\varepsilon\,\mathrm{d}x. \end{cases} \tag{3.22}$$

Passing to a subsequence, we may assume that

$$\begin{cases} A^\varepsilon Du^\varepsilon \rightharpoonup \sigma \quad \text{weakly in } L^2\big(\Omega,\mathbf{R}^n\big), \\ f^\varepsilon \rightharpoonup f \quad \text{weakly in } H^{-1}(\Omega) \text{ and strongly in } W^{-1,q}_{\mathrm{loc}}(\Omega) \text{ for every } q<2, \end{cases} \tag{3.23}$$

for some $\sigma \in L^2(\Omega,\mathbf{R}^n)$ and $f \in H^{-1}(\Omega)$. It is now easy to pass to the limit in the left and right-hand sides of (3.22) by using (3.15), (3.21), (3.23), and Rellich's compactness theorem. One obtains

$$\begin{cases} \langle f, \bar v^0\varphi\rangle = \langle -\operatorname{div}(\bar A^0 D\bar v^0), u^0\varphi\rangle - \int_\Omega \bar A^0 D\bar v^0 D\varphi\, u^0\,\mathrm{d}x + \int_\Omega \sigma D\varphi\,\bar v^0\,\mathrm{d}x \\ \qquad = \int_\Omega \bar A^0 D\bar v^0 Du^0\varphi\,\mathrm{d}x + \int_\Omega \sigma D\varphi\,\bar v^0\,\mathrm{d}x \\ \qquad = \int_\Omega A^0 Du^0 D\bar v^0\varphi\,\mathrm{d}x + \int_\Omega \sigma D\varphi\,\bar v^0\,\mathrm{d}x. \end{cases} \tag{3.24}$$

Since

$$-\operatorname{div}(\sigma) = f \quad \text{in } \mathcal{D}'(\Omega), \tag{3.25}$$

one deduces from (3.24) that

$$\int_\Omega \sigma D\bar v^0\varphi\,\mathrm{d}x = \int_\Omega A^0 Du^0 D\bar v^0\varphi\,\mathrm{d}x, \tag{3.26}$$

for every $\varphi \in C_c^\infty(\Omega)$ and every $\bar v^0 \in C_c^\infty(\Omega)$. Since, for every point $x \in \Omega$, the vector $D\bar v^0(x)$ can be chosen to coincide with any prescribed vector of $\mathbf{R}^n$, (3.26) implies that

$$\sigma = A^0 Du^0 \quad \text{a.e. in } \Omega,$$

which, together with (3.25), gives $f = f^0$. The uniqueness of the limits in (3.23) implies that the whole sequences converge, and this completes the proof of (3.17). □

4. Relaxed Dirichlet problems

In this section we recall the definition, introduced in [13,14], of relaxed Dirichlet problems associated with measures $\mu \in \mathcal{M}_0^+(\Omega)$, and prove that, under some conditions on the data, the measure μ can be reconstructed from a solution of the corresponding relaxed Dirichlet problem.

4.1. *Relaxed Dirichlet problems*

Given $A \in M_\alpha^\beta(\Omega)$, $\mu \in \mathcal{M}_0^+(\Omega)$, and $f \in H^{-1}(\Omega)$, we call *relaxed Dirichlet problem* the problem of finding u such that

$$\begin{cases} u \in H_0^1(\Omega) \cap L^2(\Omega,\mu), \\ \int_\Omega A\,Du\,Dy\,\mathrm{d}x + \int_\Omega u y\,\mathrm{d}\mu = \langle f, y\rangle \quad \forall y \in H_0^1(\Omega)\cap L^2(\Omega,\mu). \end{cases} \tag{4.1}$$

By a straightforward application of the Lax–Milgram lemma problem (4.1) has a unique solution u (see [14], Theorem 2.4) and u satisfies the estimate

$$\alpha\int_\Omega |Du|^2\,\mathrm{d}x + \int_\Omega |u|^2\,\mathrm{d}\mu \leqslant \frac{1}{\alpha}\|f\|^2_{H^{-1}(\Omega)}. \tag{4.2}$$

A connection between classical Dirichlet problems on open subsets of Ω and relaxed Dirichlet problems of the form (4.1) is given by the following remark.

Remark 4.1. Using Theorem 4.5 of [17] it is easy to check that, if $U \subseteq \Omega$ is open and μ_U is the measure introduced in (2.2), then $u \in H_0^1(\Omega) \cap L^2(\Omega, \mu_U)$ if and only if the restriction of u to U belongs to $H_0^1(U)$ and $u = 0$ q.e. in $\Omega \setminus U$. Therefore when $\mu = \mu_U$ problem (4.1) reduces to the following boundary value problem on U:

$$\begin{cases} u \in H_0^1(U), \\ -\operatorname{div}(ADu) = f \quad \text{in } \mathcal{D}'(U), \end{cases} \tag{4.3}$$

in the sense that u is the solution of (4.1) if and only if its restriction to U is the solution of (4.3) and $u = 0$ q.e. in $\Omega \setminus U$.

The name "relaxed Dirichlet problem" is motivated by the fact that the limit of the solutions to Dirichlet problems on varying domains Ω^ε always satisfies a relaxed Dirichlet problem (see, e.g., [11,14], and also Corollary 5.5 below). Moreover, the results proved in [12,14] ensure that every relaxed Dirichlet problem on Ω can be approximated in a convenient sense by classical Dirichlet problems on a suitable sequence of open sets (Ω^ε) included in Ω.

4.2. Reconstructing the measure μ

We now want to reconstruct the measure μ from one particular solution of the relaxed Dirichlet problem (4.1). In view of the applications we consider also solutions of the equation in (4.1) which do not necessarily satisfy the homogeneous Dirichlet boundary condition on $\partial\Omega$, but we study only the case where the solution and the right-hand side are nonnegative. Let us fix

$$A \in M_\alpha^\beta(\Omega), \qquad \mu \in \mathcal{M}_0^+(\Omega), \qquad \lambda \in H^{-1}(\Omega)^+, \tag{4.4}$$

and a solution ω to the problem

$$\begin{cases} \omega \in H^1(\Omega) \cap L^2(\Omega, \mu), \\ \displaystyle\int_\Omega AD\omega Dy \,\mathrm{d}x + \int_\Omega \omega y \,\mathrm{d}\mu = \int_\Omega y \,\mathrm{d}\lambda \quad \forall y \in H_0^1(\Omega) \cap L^2(\Omega, \mu), \end{cases} \tag{4.5}$$

which satisfies

$$\omega \geqslant 0 \quad \text{q.e. in } \Omega. \tag{4.6}$$

Remark 4.2. From the Lax–Milgram lemma, there exists a solution of (4.5) which belongs to $H_0^1(\Omega)$; by the comparison principle (Theorem 2.10 of [13]) this solution satisfies (4.6), so that the set of such functions ω is not empty.

The following proposition (proved in [13], Proposition 2.6) will be frequently used throughout the paper.

Proposition 4.3. *Assume* (4.4), (4.5), *and* (4.6). *Then there exists $\nu \in H^{-1}(\Omega)^+$ such that*

$$-\operatorname{div}(AD\omega) + \nu = \lambda \quad \text{in } \mathcal{D}'(\Omega). \tag{4.7}$$

For technical reasons, the reconstruction of the measure μ from ω requires the following assumption: for every quasi open set U in Ω we have

$$\operatorname{cap}\bigl(U \cap \{\omega = 0\}\bigr) > 0 \quad \Longrightarrow \quad \lambda(U) > 0. \tag{4.8}$$

Remark 4.4. Condition (4.8) is satisfied in the following (extreme) cases:

(a) $\omega > 0$ q.e. in Ω;

(b) $\lambda(U) > 0$ for every quasi open set $U \subseteq \Omega$ with $\operatorname{cap}(U) > 0$.

Note that (b) is always satisfied if $\lambda(U) = \int_U f\,\mathrm{d}x$ with $f \in L^1_{\mathrm{loc}}(\Omega)$ and $f > 0$ a.e. in Ω, since, by Lemma 2.1 and (2.1), every quasi open set with positive capacity has positive Lebesgue measure.

Proposition 4.5. *Assume* (4.4)–(4.6), *and* (4.8). *Then*

$$u \in H^1_0(\Omega) \cap L^2(\Omega,\mu) \quad \Longrightarrow \quad u = 0 \quad \text{q.e. in } \{\omega = 0\}. \tag{4.9}$$

Moreover for every Borel set $B \subseteq \Omega$

$$\operatorname{cap}\big(B \cap \{\omega = 0\}\big) > 0 \quad \Longrightarrow \quad \mu(B) = +\infty. \tag{4.10}$$

Proof. The proof follows along the lines of Lemma 3.2 of [11], with some important variants, due to the fact that now λ is not the Lebesgue measure.

To prove (4.9) it is enough to consider a function $u \in H^1_0(\Omega) \cap L^2(\Omega,\mu)$ such that $0 \leqslant u \leqslant 1$ q.e. in Ω. For every $k \in \mathbf{N}$ let u_k be the solution of the relaxed Dirichlet problem

$$\begin{cases} u_k \in H^1_0(\Omega) \cap L^2(\Omega,\mu), \\ \displaystyle\int_\Omega A Du_k Dy\,\mathrm{d}x + \int_\Omega u_k y\,\mathrm{d}\mu + k\int_\Omega u_k y\,\mathrm{d}\lambda = k\int_\Omega u y\,\mathrm{d}\lambda \\ \forall y \in H^1_0(\Omega) \cap L^2(\Omega,\mu). \end{cases} \tag{4.11}$$

By the comparison principle (Theorem 2.10 of [13]) we have $0 \leqslant u_k \leqslant k\omega$ q.e. in Ω, hence $u_k = 0$ q.e. in $\{\omega = 0\}$.

Taking $y = u_k - u$ as test function in (4.11), from (3.1) we obtain, by using Cauchy's inequality,

$$\alpha\int_\Omega |Du_k|^2\,\mathrm{d}x + \int_\Omega |u_k|^2\,\mathrm{d}\mu + 2k\int_\Omega |u_k - u|^2\,\mathrm{d}\lambda \leqslant \frac{1}{\alpha}\int_\Omega |A\,Du|^2\,\mathrm{d}x + \int_\Omega |u|^2\,\mathrm{d}\mu.$$

It follows that (u_k) is bounded in $H^1_0(\Omega)$ and converges to u strongly in $L^2(\Omega,\lambda)$. Therefore a subsequence, still denoted by (u_k), converges weakly in $H^1_0(\Omega)$ to some function v in $H^1_0(\Omega)$ such that $v = u$ λ-a.e. in Ω. Since $u_k = 0$ q.e. in $\{\omega = 0\}$, and since suitable convex combinations of (u_k) converge to v strongly in $H^1_0(\Omega)$, we conclude that $v = 0$ q.e. in $\{\omega = 0\}$. Let $V = \{v \neq u\}$. Then V is quasi open and $\lambda(V) = 0$. It follows from (4.8) that $\operatorname{cap}(V \cap \{\omega = 0\}) = 0$. As $u = v$ in $\Omega \setminus V$ and $v = 0$ q.e. in $\{\omega = 0\}$, this implies that $u = 0$ q.e. in $\{\omega = 0\}$.

Let us prove (4.10). Let U be a quasi open subset of Ω such that $\mu(U) < +\infty$. By Lemma 2.1 there exists an increasing sequence (z_k) in $H^1_0(\Omega)$ converging to 1_U pointwise q.e. in Ω and such that $0 \leqslant z_k \leqslant 1_U$ q.e. in Ω for every $k \in \mathbf{N}$. As $\mu(U) < +\infty$, each function z_k belongs to $L^2(\Omega,\mu)$, hence $z_k = 0$ q.e. on $\{\omega = 0\}$ by the previous step. This implies that $1_U = 0$ q.e. on $\{\omega = 0\}$, hence $\operatorname{cap}(U \cap \{\omega = 0\}) = 0$.

Let us consider a Borel set B with $\operatorname{cap}(B \cap \{\omega = 0\}) > 0$. For every quasi open set U containing B we have $\operatorname{cap}(U \cap \{\omega = 0\}) > 0$, hence $\mu(U) = +\infty$ by the previous step of the proof. Then the regularity property (b) in the definition of $\mathcal{M}^+_0(\Omega)$ implies that $\mu(B) = +\infty$. □

Proposition 4.6. *Assume* (4.4)–(4.6), *and* (4.8), *and let ν be the measure of $H^{-1}(\Omega)^+$ defined in* (4.7). *Then for every Borel set $B \subseteq \Omega$ we have*

$$\mu(B) = \begin{cases} \displaystyle\int_B \frac{\mathrm{d}\nu}{\omega} & \text{if } \operatorname{cap}\big(B \cap \{\omega = 0\}\big) = 0, \\ +\infty & \text{if } \operatorname{cap}\big(B \cap \{\omega = 0\}\big) > 0, \end{cases} \tag{4.12}$$

and

$$\nu\big(B\cap\{\omega>0\}\big)=\int_B \omega\,\mathrm{d}\mu. \tag{4.13}$$

In particular, this implies that $\nu=\omega\mu$ *on* $\{\omega>0\}$.

Proof. The proof follows along the lines of Lemma 3.3 and Proposition 3.4 of [11]. For every $\eta>0$ let ν_η be the Borel measure defined by

$$\nu_\eta(B)=\int_{B\cap\{\omega>\eta\}} \omega\,\mathrm{d}\mu. \tag{4.14}$$

As $\omega\in L^2(\Omega,\mu)$, we have $\nu_\eta(\Omega)\leqslant\frac{1}{\eta}\int_\Omega\omega^2\,\mathrm{d}\mu<+\infty$. Let us prove that

$$\nu_\eta(B)=\nu\big(B\cap\{\omega>\eta\}\big), \tag{4.15}$$

for every Borel set $B\subseteq\Omega$. Since ν_η is a Radon measure, it is enough to prove that $\nu_\eta(U)=\nu(U\cap\{\omega>\eta\})$ for every open set $U\subseteq\Omega$. Let us fix an open set U, and let $U_\eta=U\cap\{\omega>\eta\}$. As U_η is quasi open, by Lemma 2.1 there exists an increasing sequence (z_k) of nonnegative functions of $H_0^1(\Omega)$ converging to 1_{U_η} pointwise q.e. in Ω. Since $\mu(U_\eta)<+\infty$, the functions z_k belong to $L^2(\Omega,\mu)$. Using z_k as test function in (4.5) and (4.7) we obtain

$$\int_\Omega z_k\,\mathrm{d}\nu=\int_\Omega \omega z_k\,\mathrm{d}\mu.$$

Taking the limit as k tends to ∞ we get $\nu(U\cap\{\omega>\eta\})=\nu_\eta(U_\eta)=\nu_\eta(U)$, which proves (4.15). When η tends to 0, we obtain (4.13) from (4.14) and (4.15) (recall that $\omega\geqslant0$ q.e. in Ω).

From (4.13) we have

$$\mu\big(B\cap\{\omega>\eta\}\big)=\int_{B\cap\{\omega>\eta\}}\frac{\mathrm{d}\nu}{\omega},$$

for every Borel set $B\subseteq\Omega$ and every $\eta>0$. Taking the limit as η tends to 0 we obtain

$$\mu(B)=\int_B\frac{\mathrm{d}\nu}{\omega}, \tag{4.16}$$

for every Borel set $B\subseteq\{\omega>0\}$. Since μ vanishes on all sets with capacity zero, (4.16) holds also when $\operatorname{cap}(B\cap\{\omega=0\})=0$. Finally, if $\operatorname{cap}(B\cap\{\omega=0\})>0$, then $\mu(B)=+\infty$ by Proposition 4.5. □

4.3. Density and uniqueness results

In the next proposition we assume, in addition, that

$$\omega\in L^\infty(\Omega). \tag{4.17}$$

The following density result will be crucial in Sections 7 and 9. The proof follows along the lines of Proposition 5.5 of [15], with one important variant, due to the fact that it is now possible that the solutions u_k of the penalized problem (4.11) do not converge to u weakly in $H_0^1(\Omega)$ (see the proof of Proposition 4.5).

Proposition 4.7. *Assume* (4.4)–(4.6), (4.8), *and* (4.17). *Then the set* $\{\omega\varphi\colon\ \varphi\in C_c^\infty(\Omega)\}$ *is dense in* $H_0^1(\Omega)\cap L^2(\Omega,\mu)$.

Proof. For every $u \in H_0^1(\Omega) \cap L^2(\Omega,\mu)$ we have to construct a sequence (φ_k) in $C_c^\infty(\Omega)$ such that $(\omega\varphi_k)$ converges to u both in $H_0^1(\Omega)$ and in $L^2(\Omega,\mu)$. Clearly it is enough to consider the case $u \geqslant 0$ q.e. in Ω. For every $j \in \mathbf{N}$ let $v_j = u \wedge (j\omega)$. Since $\omega \geqslant 0$ q.e. in Ω and $u = 0$ q.e. in $\{\omega = 0\}$ by Proposition 4.5, the sequence (v_j) is nondecreasing and converges to u q.e. in Ω. By Lemma 1.6 of [9] there exists a sequence (u_j) in $H_0^1(\Omega)$, converging to u strongly in $H_0^1(\Omega)$, such that $0 \leqslant u_j \leqslant v_j \leqslant u$ q.e. in Ω for every $j \in \mathbf{N}$. By the dominated convergence theorem it turns out that (u_j) converges to u in $L^2(\Omega,\mu)$ too.

It is thus sufficient to consider the case where $u \in H_0^1(\Omega)$ is such that $0 \leqslant u \leqslant c\omega$ q.e. in Ω for some constant $c > 0$. Since $\{(u - c\,\varepsilon)^+ > 0\} \subseteq \{\omega > \varepsilon\}$, and $(u - c\,\varepsilon)^+$ converges to u in $H_0^1(\Omega) \cap L^2(\Omega,\mu)$ as ε tends to 0, we may also assume that there exists $\varepsilon > 0$ such that $\{u > 0\} \subseteq \{\omega > \varepsilon\}$. Then $u/\omega = u/(\omega \vee \varepsilon)$. Since $\omega \in H^1(\Omega) \cap L^\infty(\Omega)$, we have $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$, and thus $u/\omega \in H_0^1(\Omega) \cap L^\infty(\Omega)$. Therefore there exists a sequence (φ_k) in $C_c^\infty(\Omega)$, bounded in $L^\infty(\Omega)$, which converges to $z = u/\omega$ strongly in $H_0^1(\Omega)$ and q.e. in Ω, hence μ-a.e. in Ω. Since $\omega \in H^1(\Omega) \cap L^\infty(\Omega)$, the sequence $(\omega\varphi_k)$ converges to $\omega z = u$ strongly in $H_0^1(\Omega)$. As $\omega \in L^2(\Omega,\mu)$ and (φ_k) is bounded in $L^\infty(\Omega,\mu)$ and converges to $z = u/\omega$ μ-a.e. in Ω, by the dominated convergence theorem the sequence $(\omega\varphi_k)$ converges to $\omega z = u$ strongly in $L^2(\Omega,\mu)$. $\square$

The following uniqueness result will be crucial in Theorems 5.1 and 5.4. The proof follows along the lines of Lemma 3.5 of [11], with one important variant, due to the fact that now the condition $\int_\Omega u^2\,\mathrm{d}\lambda = 0$ does not imply that $u = 0$ q.e. in Ω.

Proposition 4.8. *Assume* (4.4)–(4.6), (4.8), *and* (4.17). *Let u be a solution of the problem*

$$\begin{cases} u \in H_0^1(\Omega) \cap L^\infty(\Omega), \\ \displaystyle\int_\Omega A D\varphi D u\, \omega \,\mathrm{d}x - \int_\Omega A D\omega D\varphi\, u \,\mathrm{d}x + \int_\Omega u\varphi \,\mathrm{d}\lambda = 0 \quad \forall \varphi \in C_c^\infty(\Omega). \end{cases} \tag{4.18}$$

Then $u = 0$ q.e. in Ω.

Proof. Since u and ω belong to $H^1(\Omega) \cap L^\infty(\Omega)$, it is easy to see that the equation in (4.18) is satisfied also for $\varphi \in H_0^1(\Omega) \cap L^\infty(\Omega)$. Using $\varphi = u$ as test function in this equation we obtain

$$\int_\Omega A D u D u\, \omega \,\mathrm{d}x - \frac{1}{2}\int_\Omega A D\omega D\big(u^2\big) \,\mathrm{d}x + \int_\Omega u^2 \,\mathrm{d}\lambda = 0. \tag{4.19}$$

Using $y = u^2$ as test function in (4.7), from (4.19) we get

$$\int_\Omega A D u D u\, \omega \,\mathrm{d}x + \frac{1}{2}\int_\Omega u^2 \,\mathrm{d}\nu + \frac{1}{2}\int_\Omega u^2 \,\mathrm{d}\lambda = 0.$$

This implies

$$Du = 0 \quad \text{a.e. in } \{\omega > 0\}, \tag{4.20}$$

$$u = 0 \quad \lambda\text{-a.e. in } \Omega. \tag{4.21}$$

Let $U = \{u \neq 0\}$. Then U is quasi open and $\lambda(U) = 0$ by (4.21). Therefore (4.8) implies that $u = 0$ q.e. in $\{\omega = 0\}$, and consequently $Du = 0$ a.e. in $\{\omega = 0\}$. By (4.20) we conclude that $Du = 0$ a.e. in Ω. Since $u \in H_0^1(\Omega)$, this yields $u = 0$ q.e. in Ω. $\square$

5. A global convergence result

For every $\varepsilon \geqslant 0$ we consider a matrix A^ε in $M_\alpha^\beta(\Omega)$ and a measure μ^ε in $\mathcal{M}_0^+(\Omega)$, that will remain fixed throughout the rest of the paper. We assume that

$$(A^\varepsilon) \quad H\text{-converges to } A^0. \tag{5.1}$$

In this section we use a duality argument to prove that, under suitable hypotheses on (μ^ε) (which are always satisfied by a subsequence), the solutions u^ε of the relaxed Dirichlet problems (4.1) for $A = A^\varepsilon$ and $\mu = \mu^\varepsilon$ converge to the solution u^0 of the relaxed Dirichlet problem for $A = A^0$ and $\mu = \mu^0$.

5.1. *Definition of special test functions*

For every $\varepsilon \geqslant 0$ we define the functions w^ε and $\overline{w}^\varepsilon$ as the unique solutions to the problems

$$\begin{cases} w^\varepsilon \in H_0^1(\Omega) \cap L^2(\Omega, \mu^\varepsilon), \\ \displaystyle\int_\Omega A^\varepsilon Dw^\varepsilon Dy \, dx + \int_\Omega w^\varepsilon y \, d\mu^\varepsilon = \int_\Omega y \, dx \quad \forall y \in H_0^1(\Omega) \cap L^2(\Omega, \mu^\varepsilon), \end{cases} \tag{5.2}$$

$$\begin{cases} \overline{w}^\varepsilon \in H_0^1(\Omega) \cap L^2(\Omega, \mu^\varepsilon), \\ \displaystyle\int_\Omega \bar{A}^\varepsilon D\overline{w}^\varepsilon Dy \, dx + \int_\Omega \overline{w}^\varepsilon y \, d\mu^\varepsilon = \int_\Omega y \, dx \quad \forall y \in H_0^1(\Omega) \cap L^2(\Omega, \mu^\varepsilon). \end{cases} \tag{5.3}$$

By the comparison principle (Theorem 2.10 of [13]) we have

$$w^\varepsilon \geqslant 0 \quad \text{and} \quad \overline{w}^\varepsilon \geqslant 0 \quad \text{q.e. in } \Omega. \tag{5.4}$$

Moreover, by the maximum principle, we have also

$$\sup_{\varepsilon \geqslant 0} \|w^\varepsilon\|_{L^\infty(\Omega)} < +\infty \quad \text{and} \quad \sup_{\varepsilon \geqslant 0} \|\overline{w}^\varepsilon\|_{L^\infty(\Omega)} < +\infty \tag{5.5}$$

(see [11], Section 3). By Proposition 4.3 there exists two measures ν^ε and $\bar{\nu}^\varepsilon$ in $H^{-1}(\Omega)^+$ such that

$$-\operatorname{div}\big(A^\varepsilon Dw^\varepsilon\big) + \nu^\varepsilon = 1, \quad -\operatorname{div}\big(\bar{A}^\varepsilon D\overline{w}^\varepsilon\big) + \bar{\nu}^\varepsilon = 1 \quad \text{in } \mathcal{D}'(\Omega). \tag{5.6}$$

Finally, from (4.2) we obtain

$$\sup_{\varepsilon \geqslant 0} \int_\Omega |Dw^\varepsilon|^2 \, dx < +\infty, \qquad \sup_{\varepsilon \geqslant 0} \int_\Omega |D\overline{w}^\varepsilon|^2 \, dx < +\infty, \tag{5.7}$$

$$\sup_{\varepsilon \geqslant 0} \int_\Omega |w^\varepsilon|^2 \, d\mu^\varepsilon < +\infty, \qquad \sup_{\varepsilon \geqslant 0} \int_\Omega |\overline{w}^\varepsilon|^2 \, d\mu^\varepsilon < +\infty. \tag{5.8}$$

5.2. *The main convergence result*

Given, for every $\varepsilon \geqslant 0$, f^ε and $\bar{f}^\varepsilon$ in $H^{-1}(\Omega)$, we consider the solutions u^ε and $\bar{u}^\varepsilon$ to the following problems

$$\begin{cases} u^\varepsilon \in H_0^1(\Omega) \cap L^2(\Omega, \mu^\varepsilon), \\ \displaystyle\int_\Omega A^\varepsilon Du^\varepsilon Dy \, dx + \int_\Omega u^\varepsilon y \, d\mu^\varepsilon = \langle f^\varepsilon, y \rangle \quad \forall y \in H_0^1(\Omega) \cap L^2(\Omega, \mu^\varepsilon), \end{cases} \tag{5.9}$$

$$\begin{cases} \bar{u}^\varepsilon \in H_0^1(\Omega) \cap L^2(\Omega, \mu^\varepsilon), \\ \int_\Omega \bar{A}^\varepsilon D\bar{u}^\varepsilon Dy \, dx + \int_\Omega \bar{u}^\varepsilon y \, d\mu^\varepsilon = \langle \bar{f}^\varepsilon, y \rangle \quad \forall y \in H_0^1(\Omega) \cap L^2(\Omega, \mu^\varepsilon). \end{cases} \tag{5.10}$$

Theorem 5.1. *Assume* (5.1) *and let* w^ε *and* $\overline{w}^\varepsilon$ *be the solutions to* (5.2) *and* (5.3). *The following conditions are equivalent*:

(a) $w^\varepsilon \rightharpoonup w^0$ *weakly in* $H_0^1(\Omega)$;
(b) $\overline{w}^\varepsilon \rightharpoonup \overline{w}^0$ *weakly in* $H_0^1(\Omega)$;
(c) *for every* (f^ε) *and* (u^ε) *satisfying* (5.9), *if* $f^\varepsilon \to f^0$ *strongly in* $H^{-1}(\Omega)$, *then* $u^\varepsilon \rightharpoonup u^0$ *weakly in* $H_0^1(\Omega)$;
(d) *for every* $(\bar{f}^\varepsilon)$ *and* $(\bar{u}^\varepsilon)$ *satisfying* (5.10), *if* $\bar{f}^\varepsilon \to \bar{f}^0$ *strongly in* $H^{-1}(\Omega)$, *then* $\bar{u}^\varepsilon \rightharpoonup \bar{u}^0$ *weakly in* $H_0^1(\Omega)$.

Proof. (a) $\Rightarrow$ (d). Assume (a). By (4.2) it is enough to prove (d) when $\bar{f}^\varepsilon = \bar{f}^0 = \bar{f} \in L^\infty(\Omega)$. Since the equation is linear, it suffices to consider the case $0 \leqslant \bar{f} \leqslant 1$ a.e. in Ω, so that $0 \leqslant \bar{u}^\varepsilon \leqslant \overline{w}^\varepsilon$ q.e. in Ω by the comparison principle (Theorem 2.10 of [13]).

By (4.2) the sequence $(\bar{u}^\varepsilon)$ is bounded in $H_0^1(\Omega)$ and by (5.5) it is bounded in $L^\infty(\Omega)$. Extracting a subsequence, we may assume that

$$\bar{u}^\varepsilon \rightharpoonup \bar{u} \quad \text{weakly in } H_0^1(\Omega), \tag{5.11}$$

for some function $\bar{u} \in H_0^1(\Omega) \cap L^\infty(\Omega)$. We want to show that $\bar{u} = \bar{u}^0$. Since the limit does not depend on the subsequence, this will prove that the whole sequence $(\bar{u}^\varepsilon)$ converges to $\bar{u}^0$.

By Proposition 4.3 we have

$$-\operatorname{div}\big(\bar{A}^\varepsilon D\bar{u}^\varepsilon\big) + \bar{\gamma}^\varepsilon = \bar{f} \quad \text{in } \mathcal{D}'(\Omega), \tag{5.12}$$

for some $\bar{\gamma}^\varepsilon \in H^{-1}(\Omega)^+$. By Theorem 3.1, from (5.6) and (5.12) we deduce that

$$\begin{cases} A^\varepsilon Dw^\varepsilon \rightharpoonup A^0 Dw^0 \quad \text{weakly in } L^2\big(\Omega, \mathbf{R}^n\big), \\ \bar{A}^\varepsilon D\bar{u}^\varepsilon \rightharpoonup \bar{A}^0 D\bar{u} \quad \text{weakly in } L^2\big(\Omega, \mathbf{R}^n\big). \end{cases} \tag{5.13}$$

Let $\varphi \in C_c^\infty(\Omega)$. Using $y = w^\varepsilon \varphi$ as test function in (5.10) and $y = \bar{u}^\varepsilon \varphi$ as test function in (5.2), by difference we obtain

$$\int_\Omega \bar{A}^\varepsilon D\bar{u}^\varepsilon D\varphi \, w^\varepsilon \, dx - \int_\Omega A^\varepsilon Dw^\varepsilon D\varphi \, \bar{u}^\varepsilon \, dx = \int_\Omega \bar{f} w^\varepsilon \varphi \, dx - \int_\Omega \bar{u}^\varepsilon \varphi \, dx, \tag{5.14}$$

for every $\varepsilon \geqslant 0$. Since (w^ε) converges to w^0 strongly in $L^2(\Omega)$ by (a) and $(\bar{u}^\varepsilon)$ converges to $\bar{u}$ strongly in $L^2(\Omega)$ by (5.11), using (5.13) we can pass to the limit in each term of (5.14) and we obtain

$$\int_\Omega \bar{A}^0 D\bar{u} D\varphi \, w^0 \, dx - \int_\Omega A^0 Dw^0 D\varphi \, \bar{u} \, dx = \int_\Omega \bar{f} w^0 \varphi \, dx - \int_\Omega \bar{u} \varphi \, dx. \tag{5.15}$$

Since (5.14), with $\varepsilon = 0$, and (5.15) hold for every $\varphi \in C_c^\infty(\Omega)$, the difference $u = \bar{u}^0 - \bar{u}$ belongs to $H_0^1(\Omega) \cap L^\infty(\Omega)$ and satisfies (4.18) with $A = A^0$, $\omega = w^0$, and $\lambda = 1$. This implies $\bar{u} = \bar{u}^0$ q.e. in Ω by Proposition 4.8.

(d) $\Rightarrow$ (b). It is enough to take $\bar{f}^\varepsilon = \bar{f}^0 = 1$ in condition (d).

(b) $\Rightarrow$ (c). Since $(\bar{A}^\varepsilon)$ H-converges to $\bar{A}^0$, we can replace A^ε by $\bar{A}^\varepsilon$ and $\bar{f}^\varepsilon$ by f^ε in the proof of the implication (a) $\Rightarrow$ (d).

(c) $\Rightarrow$ (a). It is enough to take $f^\varepsilon = f^0 = 1$ in condition (c). $\square$

5.3. A compactness result

We now prove that the equivalent conditions of Theorem 5.1 are always satisfied by a subsequence.

Theorem 5.2. *Assume* (5.1). *For every sequence* $(\mu^\varepsilon)_{\varepsilon>0}$ *in* $\mathcal{M}_0^+(\Omega)$ *there exist a subsequence, still denoted by* (μ^ε), *and a measure* μ^0 *in* $\mathcal{M}_0^+(\Omega)$, *such that the equivalent conditions* (a)–(d) *of Theorem* 5.1 *are satisfied.*

Proof. By (5.7) the sequence (w^ε) is bounded in $H_0^1(\Omega)$. Passing to a subsequence, we may assume that (w^ε) converges weakly in $H_0^1(\Omega)$ to some function $w \in H_0^1(\Omega)$. By (5.4) we have $w \geqslant 0$ q.e. in Ω. Now we want to construct a measure $\mu^0 \in \mathcal{M}_0^+(\Omega)$ such that w coincides with the solution w^0 of (5.2) for $\varepsilon = 0$.

By (5.6) and Theorem 3.1 the sequence $(A^\varepsilon Dw^\varepsilon)$ converges to $A^0 Dw$ weakly in $L^2(\Omega, \mathbf{R}^n)$. Therefore (ν^ε) converges to ν weakly in $H^{-1}(\Omega)$, where $\nu \in H^{-1}(\Omega)^+$ is defined by

$$-\operatorname{div}\big(A^0 Dw\big) + \nu = 1 \quad \text{in } \mathcal{D}'(\Omega). \tag{5.16}$$

Let us define the measure μ^0 by

$$\mu^0(B) = \begin{cases} \displaystyle\int_B \frac{\mathrm{d}\nu}{w} & \text{if } \operatorname{cap}\big(B \cap \{w = 0\}\big) = 0, \\ +\infty & \text{if } \operatorname{cap}\big(B \cap \{w = 0\}\big) > 0. \end{cases} \tag{5.17}$$

Using (5.16), from Proposition 3.4 of [11] we obtain that $\mu^0 \in \mathcal{M}_0^+(\Omega)$ and that w coincides with the unique solution w^0 to problem (5.2) for $\varepsilon = 0$. This shows that condition (a) of Theorem 5.1 is satisfied. □

5.4. More general test functions

We introduce now a more general family of test functions (ω^ε). While it is very difficult to compute explicitly the functions w^ε defined by (5.2), in some interesting situations it will be very easy to construct explicitly the new family (ω^ε), from which one can determine immediately the limit measure μ^0.

For every $\varepsilon \geqslant 0$ let $\lambda^\varepsilon \in H^{-1}(\Omega)^+$ and let ω^ε be a solution of the problem

$$\begin{cases} \omega^\varepsilon \in H^1(\Omega) \cap L^2(\Omega, \mu^\varepsilon), \\ \displaystyle\int_\Omega A^\varepsilon D\omega^\varepsilon Dy \, \mathrm{d}x + \int_\Omega \omega^\varepsilon y \, \mathrm{d}\mu^\varepsilon = \int_\Omega y \, \mathrm{d}\lambda^\varepsilon \quad \forall y \in H_0^1(\Omega) \cap L^2(\Omega, \mu^\varepsilon). \end{cases} \tag{5.18}$$

We assume that

$$\lambda^\varepsilon \in H^{-1}(\Omega)^+ \quad \text{for every } \varepsilon \geqslant 0, \tag{5.19}$$

$$\lambda^\varepsilon \to \lambda^0 \quad \text{strongly in } H^{-1}(\Omega), \tag{5.20}$$

$$\omega^\varepsilon \geqslant 0 \quad \text{q.e. in } \Omega \text{ for every } \varepsilon \geqslant 0, \tag{5.21}$$

$$\omega^\varepsilon \rightharpoonup \omega^0 \quad \text{weakly in } H^1(\Omega). \tag{5.22}$$

Moreover we assume that for every quasi open set U in Ω we have

$$\operatorname{cap}\big(U \cap \big\{\omega^0 = 0\big\}\big) > 0 \quad \Longrightarrow \quad \lambda^0(U) > 0, \tag{5.23}$$

and that

$$\omega^0 \in L^\infty(\Omega). \tag{5.24}$$

Remark 5.3. If condition (a) of Theorem 5.1 is satisfied, then the functions w^ε, $\varepsilon \geqslant 0$, defined by (5.2) satisfy conditions (5.18)–(5.24) with $\lambda^\varepsilon = 1$ for every $\varepsilon \geqslant 0$ (see Remark 4.4). Other sequences $(\omega^\varepsilon)_{\varepsilon \geqslant 0}$ and $(\lambda^\varepsilon)_{\varepsilon \geqslant 0}$ satisfying (5.18)–(5.24), with $\omega^0 = 1$, are constructed in [7] when $\mu^0 \in H^{-1}(\Omega)^+$.

If conditions (5.18)–(5.24) are satisfied in Ω, then they are satisfied in every open set $U \subseteq \Omega$.

Theorem 5.4. *Assume that* (5.1) *holds and that* $(\omega^\varepsilon)_{\varepsilon \geqslant 0}$ *and* $(\lambda^\varepsilon)_{\varepsilon \geqslant 0}$ *satisfy* (5.18)–(5.24). *Then the equivalent conditions* (a)–(d) *of Theorem* 5.1 *are fulfilled.*

Proof. We will prove that condition (b) holds. By (5.7) the sequence $(\overline{w}^\varepsilon)$ is bounded in $H_0^1(\Omega)$ and by (5.5) it is bounded in $L^\infty(\Omega)$. Extracting a subsequence, we may assume that

$$\overline{w}^\varepsilon \rightharpoonup \overline{w} \quad \text{weakly in } H_0^1(\Omega), \tag{5.25}$$

for some function $\overline{w} \in H_0^1(\Omega) \cap L^\infty(\Omega)$. We will show that $\overline{w} = \overline{w}^0$. Since the limit does not depend on the subsequence, this will prove that the whole sequence $(\overline{w}^\varepsilon)$ converges to $\overline{w}^0$.

By Proposition 4.3 and Theorem 3.1 we have

$$\begin{cases} A^\varepsilon D\omega^\varepsilon \rightharpoonup A^0 D\omega^0 & \text{weakly in } L^2\big(\Omega, \mathbf{R}^n\big), \\ \bar{A}^\varepsilon D\overline{w}^\varepsilon \rightharpoonup \bar{A}^0 D\overline{w} & \text{weakly in } L^2\big(\Omega, \mathbf{R}^n\big). \end{cases} \tag{5.26}$$

Let $\varphi \in C_c^\infty(\Omega)$. Using $y = \omega^\varepsilon \varphi$ as test function in (5.3) and $y = \overline{w}^\varepsilon \varphi$ as test function in (5.18), by difference we obtain

$$\int_\Omega \bar{A}^\varepsilon D\overline{w}^\varepsilon D\varphi\, \omega^\varepsilon \,\mathrm{d}x - \int_\Omega A^\varepsilon D\omega^\varepsilon D\varphi\, \overline{w}^\varepsilon \,\mathrm{d}x = \int_\Omega \omega^\varepsilon \varphi \,\mathrm{d}x - \int_\Omega \overline{w}^\varepsilon \varphi \,\mathrm{d}\lambda^\varepsilon, \tag{5.27}$$

for every $\varepsilon \geqslant 0$. Since (ω^ε) converges to ω^0 strongly in $L^2_{\mathrm{loc}}(\Omega)$ by (5.22) and $(\overline{w}^\varepsilon)$ converges to $\overline{w}$ strongly in $L^2(\Omega)$ by (5.25), using (5.26) we can pass to the limit in each term of (5.27) and we obtain

$$\int_\Omega \bar{A}^0 D\overline{w} D\varphi\, \omega^0 \,\mathrm{d}x - \int_\Omega A^0 D\omega^0 D\varphi\, \overline{w} \,\mathrm{d}x = \int_\Omega \omega^0 \varphi \,\mathrm{d}x - \int_\Omega \overline{w} \varphi \,\mathrm{d}\lambda^0. \tag{5.28}$$

Since (5.27), with $\varepsilon = 0$, and (5.28) hold for every $\varphi \in C_c^\infty(\Omega)$, the difference $\overline{w}^0 - \overline{w}$ belongs to $H_0^1(\Omega) \cap L^\infty(\Omega)$ and satisfies (4.18) with $A = A^0$, $\lambda = \lambda^0$ and $\omega = \omega^0$. This implies $\overline{w} = \overline{w}^0$ q.e. in Ω by Proposition 4.8. □

5.5. Dirichlet problems on varying domains

We conclude this section by considering the particular case of classical Dirichlet problems on varying domains. Let $(\Omega^\varepsilon)_{\varepsilon > 0}$ be a sequence of open sets, with $\Omega^\varepsilon \subseteq \Omega$, and let μ^0 be a measure in $\mathcal{M}_0^+(\Omega)$. For every $\varepsilon > 0$ let w^ε and $\overline{w}^\varepsilon$ be the unique solutions to the problems

$$\begin{cases} w^\varepsilon \in H_0^1\big(\Omega^\varepsilon\big), \\ -\operatorname{div}\big(A^\varepsilon D w^\varepsilon\big) = 1 & \text{in } \mathcal{D}'\big(\Omega^\varepsilon\big), \end{cases} \tag{5.29}$$

$$\begin{cases} \overline{w}^\varepsilon \in H_0^1\big(\Omega^\varepsilon\big), \\ -\operatorname{div}\big(\bar{A}^\varepsilon D \overline{w}^\varepsilon\big) = 1 & \text{in } \mathcal{D}'\big(\Omega^\varepsilon\big), \end{cases} \tag{5.30}$$

and let w^0 and $\overline{w}^0$ be the solutions of (5.2) and (5.3) with $\varepsilon = 0$.

Given f^ε and $\bar{f}^\varepsilon$ in $H^{-1}(\Omega)$, for $\varepsilon > 0$, we consider the solutions u^ε and $\bar{u}^\varepsilon$ to the following problems

$$\begin{cases} u^\varepsilon \in H_0^1(\Omega^\varepsilon), \\ -\operatorname{div}(A^\varepsilon D u^\varepsilon) = f^\varepsilon \quad \text{in } \mathcal{D}'(\Omega^\varepsilon), \end{cases} \tag{5.31}$$

$$\begin{cases} \bar{u}^\varepsilon \in H_0^1(\Omega^\varepsilon), \\ -\operatorname{div}(\bar{A}^\varepsilon D \bar{u}^\varepsilon) = \bar{f}^\varepsilon \quad \text{in } \mathcal{D}'(\Omega^\varepsilon). \end{cases} \tag{5.32}$$

Given f^0 and $\bar{f}^0$ in $H^{-1}(\Omega)$, let u^0 and $\bar{u}^0$ be the solutions of (5.9) and (5.10) with $\varepsilon = 0$. All functions in $H_0^1(\Omega^\varepsilon)$ are considered as functions in $H_0^1(\Omega)$ which are equal to 0 q.e. in $\Omega \setminus \Omega^\varepsilon$. (Observe that u^ε, $\bar{u}^\varepsilon$, f^ε, and $\bar{f}^\varepsilon$ are defined in the whole of Ω, for $\varepsilon \geqslant 0$.)

Corollary 5.5. *Assume* (5.1) *and let w^ε and $\overline{w}^\varepsilon$ be the solutions of* (5.29) *and* (5.30) *for $\varepsilon > 0$, and of* (5.2) *and* (5.3) *for $\varepsilon = 0$. The following conditions are equivalent*:

(a) $w^\varepsilon \rightharpoonup w^0$ *weakly in* $H_0^1(\Omega)$;
(b) $\overline{w}^\varepsilon \rightharpoonup \overline{w}^0$ *weakly in* $H_0^1(\Omega)$;
(c) *for every* (f^ε) *and* (u^ε) *satisfying* (5.31) *for $\varepsilon > 0$ and* (5.9) *for $\varepsilon = 0$, if $f^\varepsilon \to f^0$ strongly in $H^{-1}(\Omega)$, then $u^\varepsilon \rightharpoonup u^0$ weakly in $H_0^1(\Omega)$*;
(d) *for every* $(\bar{f}^\varepsilon)$ *and* $(\bar{u}^\varepsilon)$ *satisfying* (5.32) *for $\varepsilon > 0$ and* (5.10) *for $\varepsilon = 0$, if $\bar{f}^\varepsilon \to \bar{f}^0$ strongly in $H^{-1}(\Omega)$, then $\bar{u}^\varepsilon \rightharpoonup \bar{u}^0$ weakly in $H_0^1(\Omega)$.*

Proof. For every $\varepsilon > 0$ let μ_{Ω^ε} be the measures introduced in (2.2) with $U = \Omega^\varepsilon$. By Remark 4.1 the functions w^ε and $\overline{w}^\varepsilon$ defined in (5.29) and (5.30) coincide with the solutions of (5.2) and (5.3) with $\mu^\varepsilon = \mu_{\Omega^\varepsilon}$. For the same reason the functions u^ε and $\bar{u}^\varepsilon$ defined in (5.31) and (5.32) coincide with the solutions of (5.9) and (5.10) with $\mu^\varepsilon = \mu_{\Omega^\varepsilon}$. The conclusion now follows from Theorem 5.1. □

Remark 5.6. Let (λ^ε) be a sequence in $H^{-1}(\Omega)^+$ and, for every $\varepsilon > 0$, let ω^ε be a function in $H^1(\Omega)$ such that $\omega^\varepsilon = 0$ q.e. in $\Omega \setminus \Omega^\varepsilon$ and

$$-\operatorname{div}(A^\varepsilon D\omega^\varepsilon) = \lambda^\varepsilon \quad \text{in } \mathcal{D}'(\Omega^\varepsilon).$$

Let $\lambda^0 \in H^{-1}(\Omega)^+$ and let ω^0 be a solution of (5.18) with $\varepsilon = 0$. If conditions (5.19)–(5.24) are satisfied, then the equivalent conditions (a)–(d) of Corollary 5.5 are satisfied. To prove this fact, it is enough to use Remark 4.1 and Theorem 5.4.

6. An example

In this section we present an example, not yet considered in the literature, which shows that the measure μ^0 which appears in the limit problem depends not only on the sequence (Ω^ε) and on A^0, but also on the sequence (A^ε). To identify the measure μ^0 we apply Corollary 5.5 and Remark 5.6. Another interesting example is given in Section 5 of [1].

To simplify the exposition, we assume $n \geqslant 3$ (the case $n = 2$ requires obvious modifications, as done, e.g., in [7]). Let us fix an exponent γ with

$$1 < \gamma < \frac{n}{n-2}. \tag{6.1}$$

For every $\varepsilon > 0$ and $i \in \mathbf{Z}^n$, we consider the point $x_i^\varepsilon = \varepsilon i$ and the open balls B_i^ε and C_i^ε with centre x_i^ε and radii $\varepsilon^{n/(n-2)}$ and ε^γ, respectively. By (6.1) we have $B_i^\varepsilon \subseteq C_i^\varepsilon$ for $0 < \varepsilon < 1$, and the sets $(C_i^\varepsilon)_{i\in\mathbf{Z}^n}$ are pairwise disjoint for $0 < \varepsilon < 2^{1/(1-\gamma)}$. We define

$$B^\varepsilon = \bigcup_{i\in\mathbf{Z}^n} B_i^\varepsilon, \qquad C^\varepsilon = \bigcup_{i\in\mathbf{Z}^n} C_i^\varepsilon.$$

For two given constants $a, b \in [\alpha, \beta]$, we define the matrices A^ε, for $\varepsilon \geqslant 0$, by

$$A^\varepsilon(x) = \begin{cases} aI & \text{for } x \in \Omega \setminus C^\varepsilon, \\ bI & \text{for } x \in \Omega \cap C^\varepsilon, \end{cases} \tag{6.2}$$

where we set $C^0 = \emptyset$, so that $A^0(x) = aI$ for every $x \in \Omega$. Since (A^ε) converges in measure to A^0 by (6.1), it is easy to prove that (A^ε) H-converges to A^0. Finally, for every $\varepsilon > 0$ we define

$$\Omega^\varepsilon = \Omega \setminus \overline{B}^\varepsilon.$$

We will determine $\mu^0 \in \mathcal{M}_0^+(\Omega)$ such that the equivalent conditions (a)–(d) of Corollary 5.5 are satisfied. More precisely, using Remark 5.6 we will construct, for $\varepsilon \geqslant 0$, a measure λ^ε in $H^{-1}(\Omega)^+$ and, for $\varepsilon > 0$, a function ω^ε in $H^1(\Omega)$ such that $\omega^\varepsilon = 0$ q.e. in $\overline{B}^\varepsilon$ and

$$-\operatorname{div}\bigl(A^\varepsilon D\omega^\varepsilon\bigr) = \lambda^\varepsilon \quad \text{in } \mathcal{D}'\bigl(\Omega^\varepsilon\bigr), \tag{6.3}$$

for which we will prove that conditions (5.19)–(5.24) are satisfied, where ω^0 is a solution of (5.18) with $\varepsilon = 0$.

For every $\varepsilon > 0$ and $i \in \mathbf{Z}^n$, let $\omega_i^\varepsilon \in H^1(C_i^\varepsilon \setminus \overline{B}_i^\varepsilon)$ be the solution of the equation $\Delta\omega_i^\varepsilon = 0$ on $C_i^\varepsilon \setminus \overline{B}_i^\varepsilon$ which satisfies the boundary conditions $\omega_i^\varepsilon = 0$ on ∂B_i^ε and $\omega_i^\varepsilon = 1$ on ∂C_i^ε. By an explicit computation we find that

$$\omega_i^\varepsilon(x) = c^\varepsilon - c^\varepsilon \varepsilon^n |x - x_i^\varepsilon|^{2-n} \quad \text{for } x \in C_i^\varepsilon \setminus \overline{B}_i^\varepsilon, \tag{6.4}$$

where

$$c^\varepsilon = \frac{1}{1 - \varepsilon^{n-\gamma(n-2)}} \longrightarrow 1 \tag{6.5}$$

by (6.1). For $0 < \varepsilon < 2^{1/(1-\gamma)}$ we define ω^ε as the function which is equal to ω_i^ε on $(C_i^\varepsilon \setminus \overline{B}_i^\varepsilon) \cap \Omega$, and is extended by 0 on $\Omega \cap \overline{B}^\varepsilon$ and by 1 on $\Omega \setminus C^\varepsilon$. By an explicit computation we find that

$$\int_{C_i^\varepsilon \setminus B_i^\varepsilon} |D\omega_i^\varepsilon|^2 \, dx = (n-2) S_{n-1} c^\varepsilon \varepsilon^n,$$

where S_{n-1} is the $(n-1)$-dimensional measure of the boundary of the unit ball in $\mathbf{R}^n$. This yields

$$\int_\Omega |D\omega^\varepsilon|^2 \, dx \leqslant (n-2) S_{n-1} c^\varepsilon N^\varepsilon \varepsilon^n, \tag{6.6}$$

where N^ε is the number of indices $i \in \mathbf{Z}^n$ such that the distance from x_i^ε to Ω is less than ε. Since

$$\lim_{\varepsilon\to 0} N^\varepsilon \varepsilon^n = \operatorname{meas}\bigl(\overline{\Omega}\bigr) < +\infty, \tag{6.7}$$

we deduce from (6.5) and (6.6) that (ω^ε) is bounded in $H^1(\Omega)$. As (ω^ε) converges to 1 in measure, we conclude that (ω^ε) converges to $\omega^0 = 1$ weakly in $H^1(\Omega)$, i.e., condition (5.22) is fulfilled.

Let σ^ε denote the $(n-1)$-dimensional measure on ∂C^ε and let λ^ε be the measure on Ω defined by

$$\lambda^\varepsilon = b(n-2) c^\varepsilon \varepsilon^{n-\gamma(n-1)} \sigma^\varepsilon.$$

Since, by (6.4),

$$\frac{\partial \omega_i^\varepsilon}{\partial \nu} = (n-2)c^\varepsilon \varepsilon^{n-\gamma(n-1)} \quad \text{on } \partial C_i^\varepsilon,$$

we obtain that $-b\Delta\omega^\varepsilon = \lambda^\varepsilon$ in $\mathcal{D}'(\Omega^\varepsilon)$. As $D\omega^\varepsilon = 0$ a.e. in $\Omega^\varepsilon \setminus C^\varepsilon$, we have $A^\varepsilon D\omega^\varepsilon = bD\omega^\varepsilon$ a.e. in Ω^ε by (6.2), and we conclude that (6.3) holds. From the properties of σ^ε and from (6.5) it follows that $\lambda^\varepsilon \in H^{-1}(\Omega)^+$ and that

$$\lim_{\varepsilon\to 0} \int_\Omega \varphi \,\mathrm{d}\lambda^\varepsilon = b(n-2)S_{n-1} \int_\Omega \varphi \,\mathrm{d}x, \tag{6.8}$$

for every $\varphi \in C_c^\infty(\Omega)$. We now define $\mu^0 = \lambda^0 = b(n-2)S_{n-1}$. Then condition (5.19) is satisfied and $\omega^0 = 1$ is a solution to problem (5.18) for $\varepsilon = 0$. It remains to prove that (λ^ε) converges to λ^0 strongly in $H^{-1}(\Omega)$.

To this aim, for every $\varepsilon > 0$ and $i \in \mathbf{Z}^n$, we consider the open balls D_i^ε and E_i^ε with centre x_i^ε and radii $\varepsilon/4$ and $\varepsilon/2$, respectively. Then we introduce the functions v_i^ε defined by

$$v_i^\varepsilon(x) = \begin{cases} bc^\varepsilon \varepsilon^n |x - x_i^\varepsilon|^{2-n} & \text{if } x \in E_i^\varepsilon \setminus C_i^\varepsilon, \\ bc^\varepsilon \varepsilon^{n-\gamma(n-2)} & \text{if } x \in C_i^\varepsilon. \end{cases}$$

By computing the normal derivatives of v_i^ε on both sides of ∂C_i^ε we obtain that

$$-\Delta v_i^\varepsilon = \lambda^\varepsilon \quad \text{on } E_i^\varepsilon, \tag{6.9}$$

for $0 < \varepsilon < 2^{1/(1-\gamma)}$. We fix a cut-off function $\varphi_i^\varepsilon \in C_c^\infty(E_i^\varepsilon)$ such that $\varphi_i^\varepsilon = 1$ on D_i^ε, and $0 \leqslant \varphi_i^\varepsilon \leqslant 1$, $|D\varphi_i^\varepsilon| \leqslant c/\varepsilon$, and $|\Delta\varphi_i^\varepsilon| \leqslant c/\varepsilon^2$ on E_i^ε, where c is a suitable constant independent of ε and i. Finally, we define $v^\varepsilon \in H^1(\Omega)$ by

$$v^\varepsilon = \sum_{i\in\mathbf{Z}^n} \varphi_i^\varepsilon v_i^\varepsilon.$$

By (6.9) we have

$$-\Delta v^\varepsilon = \lambda^\varepsilon - g^\varepsilon, \tag{6.10}$$

where

$$g^\varepsilon = 2\sum_{i\in\mathbf{Z}^n} D\varphi_i^\varepsilon Dv_i^\varepsilon + \sum_{i\in\mathbf{Z}^n} \Delta\varphi_i^\varepsilon v_i^\varepsilon.$$

From the definition of v_i^ε and from the estimates for $D\varphi_i^\varepsilon$ and $\Delta\varphi_i^\varepsilon$ we obtain that the sequence (g^ε) is bounded in $L^\infty(\Omega)$. Therefore, passing to a subsequence, we may assume that

$$g^\varepsilon \rightharpoonup g \quad \text{weakly in } L^2(\Omega) \text{ and strongly in } H^{-1}(\Omega). \tag{6.11}$$

Moreover we have, for $0 < \varepsilon < 4^{1/(1-\gamma)}$,

$$\int_\Omega |Dv^\varepsilon|^2 \,\mathrm{d}x \leqslant 2\sum_{i\in\mathbf{Z}^n} \left\{ \int_{E_i^\varepsilon\setminus C_i^\varepsilon} |Dv_i^\varepsilon|^2 \,\mathrm{d}x + \frac{c^2}{\varepsilon^2} \int_{E_i^\varepsilon\setminus D_i^\varepsilon} |v_i^\varepsilon|^2 \,\mathrm{d}x \right\} \leqslant MN^\varepsilon \varepsilon^n \big(\varepsilon^{n-\gamma(n-2)} + \varepsilon^2\big),$$

for a suitable constant M independent of ε. Taking (6.1) and (6.7) into account, we conclude that (Dv^ε) converges to 0 strongly in $L^2(\Omega, \mathbf{R}^n)$, hence (Δv^ε) converges to 0 strongly in $H^{-1}(\Omega)$. By (6.10) and (6.11) this implies that (λ^ε) converges to g strongly in $H^{-1}(\Omega)$, and by (6.8) we have $g = b(n-2)S_{n-1} = \lambda^0$. Since the limit does not depend on the subsequence, we deduce that the whole sequence (λ^ε) converges to λ^0 strongly in $H^{-1}(\Omega)$.

In conclusion, for $\varepsilon \geqslant 0$ we have built ω^ε and λ^ε such that conditions (5.19)–(5.24) are satisfied. Therefore, by Remark 5.6, if (f^ε) converges to f^0 strongly in $H^{-1}(\Omega)$, then the solutions u^ε of the classical Dirichlet problems

$$\begin{cases} u^\varepsilon \in H_0^1\big(\Omega^\varepsilon\big), \\ -\operatorname{div}\big(A^\varepsilon Du^\varepsilon\big) = f^\varepsilon \quad \text{in } \mathcal{D}'\big(\Omega^\varepsilon\big), \end{cases}$$

extended by 0 on $\Omega \setminus \Omega^\varepsilon$, converge weakly in $H_0^1(\Omega)$ to the solution u^0 of the problem

$$\begin{cases} u^0 \in H_0^1(\Omega), \\ -\operatorname{div}\big(A^0 Du^0\big) + \mu^0 u^0 = f^0 \quad \text{in } \mathcal{D}'(\Omega), \end{cases}$$

where $\mu^0 = b(n-2)S_{n-1}$ and $A^0 = aI$.

If we change the constant b in the definition (6.2) of A^ε, the H-limit A^0 does not change, but the measure μ^0 changes. This shows that μ^0 depends on the whole sequence (A^ε), and not only on A^0.

7. Global and local corrector results

In this section we prove a corrector result for the solutions of problems (5.9) in the special case $f^\varepsilon = f^0 = f$, with $f \in L^\infty(\Omega)$. In Section 10 we shall consider the case where (f^ε) converges to f^0 strongly in $H^{-1}(\Omega)$, together with the case of more general data.

Assume that $(\omega^\varepsilon)_{\varepsilon \geqslant 0}$ and $(\lambda^\varepsilon)_{\varepsilon \geqslant 0}$ satisfy (5.18)–(5.24). In order to obtain the corrector result we assume, in addition, that

$$\sup_{\varepsilon \geqslant 0} \|\omega^\varepsilon\|_{L^\infty(\Omega)} < +\infty, \tag{7.1}$$

$$\sup_{\varepsilon \geqslant 0} \int_\Omega |\omega^\varepsilon|^2 \, d\mu^\varepsilon < +\infty. \tag{7.2}$$

Remark 7.1. If conditions (5.18)–(5.24), (7.1), and (7.2) are satisfied in Ω, then they are satisfied in every open set $U \subseteq \Omega$.

The functions w^ε introduced in (5.2) satisfy conditions (7.1) and (7.2), as stated in (5.5) and (5.8).

7.1. Global corrector result

For $j = 1, 2, \ldots, n$ let us fix a sequence (z_j^ε) in $H^1(\Omega)$ satisfying (3.5)–(3.9). Let u^0 be the solution of (5.9) with $\varepsilon = 0$ and $f^0 = f \in L^\infty(\Omega)$. Let us fix $\delta > 0$ and $\psi_\delta \in H^2(\Omega) \cap W^{1,\infty}(\Omega)$ such that

$$\beta \int_\Omega \big|Du^0 - D\big(\psi_\delta \omega^0\big)\big|^2 \, dx + \int_\Omega \big|u^0 - \psi_\delta \omega^0\big|^2 \, d\mu^0 < \delta. \tag{7.3}$$

Such a ψ_δ exists since the set $\{\omega^0 \varphi\colon \varphi \in C_c^\infty(\Omega)\}$ is dense in $H_0^1(\Omega) \cap L^2(\Omega, \mu^0)$ by Proposition 4.7.

For every $\varepsilon > 0$ let v_δ^ε be the function defined by

$$v_\delta^\varepsilon = \left(\psi_\delta + \sum_{j=1}^n D_j \psi_\delta \, z_j^\varepsilon\right)\omega^\varepsilon. \tag{7.4}$$

By (3.5), (3.8), (5.22), and (7.1) we have

$$v_\delta^\varepsilon \rightharpoonup \psi_\delta \omega^0 \quad \text{weakly in } H^1(\Omega) \text{ and weakly* in } L^\infty(\Omega). \tag{7.5}$$

Moreover we have

$$Dv_\delta^\varepsilon = \left(\psi_\delta + \sum_{j=1}^n D_j \psi_\delta \, z_j^\varepsilon\right) D\omega^\varepsilon + \sum_{j=1}^n D_j \psi_\delta (e_j + Dz_j^\varepsilon)\omega^\varepsilon + \sum_{j=1}^n DD_j \psi_\delta \, z_j^\varepsilon \omega^\varepsilon.$$

The last sum in the right-hand side converges to 0 strongly in $L^2(\Omega, \mathbf{R}^n)$ by (3.8) and (7.1), while $(D_j \psi_\delta \, z_j^\varepsilon D\omega^\varepsilon)$ converges to 0 strongly in $L^2(\Omega, \mathbf{R}^n)$ by (3.8) and (5.22). Therefore

$$Dv_\delta^\varepsilon = \psi_\delta D\omega^\varepsilon + \sum_{j=1}^n D_j\psi_\delta(e_j + Dz_j^\varepsilon)\omega^\varepsilon + H_\delta^\varepsilon, \tag{7.6}$$

where (H_δ^ε) converges to 0 strongly in $L^2(\Omega, \mathbf{R}^n)$ as ε tends to 0.

Since $\psi_\delta \in W^{1,\infty}(\Omega)$ and (z_j^ε) is bounded in $L^\infty(\Omega)$, from (7.2) we deduce that

$$\sup_{\varepsilon>0} \int_\Omega |v_\delta^\varepsilon|^2 \, d\mu^\varepsilon < +\infty. \tag{7.7}$$

Theorem 7.2. *Assume* (5.1), (5.18)–(5.24), (7.1), *and* (7.2). *Let $\delta > 0$ and let ψ_δ be a function in $H^2(\Omega) \cap W^{1,\infty}(\Omega)$ which satisfies* (7.3). *Assume that the functions v_δ^ε defined by* (7.4) *belong to $H_0^1(\Omega)$. Then for every $f \in L^\infty(\Omega)$ the solutions u^ε of problems* (5.9) *with $f^\varepsilon = f$ satisfy the estimate*

$$\limsup_{\varepsilon\to 0} \left\{ \alpha \int_\Omega |Du^\varepsilon - Dv_\delta^\varepsilon|^2 \, dx + \int_\Omega |u^\varepsilon - v_\delta^\varepsilon|^2 \, d\mu^\varepsilon \right\} < \delta. \tag{7.8}$$

Remark 7.3. In the special case $u^0 = \psi\omega^0$, for some $\psi \in H^2(\Omega) \cap W^{1,\infty}(\Omega)$, we can take $\psi_\delta = \psi$ for every $\delta > 0$ in (7.3), so that

$$v_\delta^\varepsilon = v^\varepsilon = \left(\psi + \sum_{j=1}^n D_j\psi \, z_j^\varepsilon \right) \omega^\varepsilon. \tag{7.9}$$

Therefore when $v^\varepsilon \in H_0^1(\Omega)$, (7.8) implies

$$\lim_{\varepsilon\to 0} \left\{ \alpha \int_\Omega |Du^\varepsilon - Dv^\varepsilon|^2 \, dx + \int_\Omega |u^\varepsilon - v^\varepsilon|^2 \, d\mu^\varepsilon \right\} = 0, \tag{7.10}$$

which is a corrector result.

When the measures μ^ε are fixed and equal to 0 (so that we can choose $\omega^\varepsilon = \omega^0 = 1$ and $\psi = u^0$), formulas (7.9) and (7.10) provide the classical corrector result for H-converging operators stated in (3.14) (see [22,26] and, in the periodic case, [3,23]). When the matrices A^ε are fixed and equal to some matrix A^0 (so that we can choose $z_j^\varepsilon = 0$), formulas (7.9) and (7.10) with $\omega^\varepsilon = w^\varepsilon$ defined by (5.2) provide the corrector result of [11,15]; with a different choice of ω^ε, which leads to $\omega^0 = 1$, the same formulas give also the corrector result of [7] in the periodic case. When both A^ε and μ^ε depend on ε, but $\omega^0 = 1$, so that we have $\psi = u^0$, the combination of H-converging operators and varying domains results in the multiplication of the corresponding correctors.

In the general case, ψ_δ and v_δ^ε depend on δ and we obtain from (7.8) that

$$Du^\varepsilon = Dv_\delta^\varepsilon + R_\delta^\varepsilon \quad \text{with} \ \limsup_{\varepsilon\to 0} \|R_\delta^\varepsilon\|^2_{L^2(\Omega,\mathbf{R}^n)} < \frac{\delta}{\alpha},$$

which is still a corrector result, but in a more technical form.

7.2. Local convergence and corrector results

We consider now the case where the functions u^ε are solutions of the problems

$$\begin{cases} u^\varepsilon \in H^1(\Omega) \cap L^2(\Omega, \mu^\varepsilon), \\ \displaystyle\int_\Omega A^\varepsilon Du^\varepsilon Dy \, dx + \int_\Omega u^\varepsilon y \, d\mu^\varepsilon = \int_\Omega fy \, dx \quad \forall y \in H_0^1(\Omega) \cap L^2(\Omega, \mu^\varepsilon), \end{cases} \tag{7.11}$$

but are not required to satisfy the boundary condition $u^\varepsilon = 0$ on $\partial\Omega$. We still consider the case of fixed data $f \in L^\infty(\Omega)$. More general data will be studied in Section 10.

The following theorem is a local version of the convergence result given in Theorem 5.4. It will be proved in Section 9.

Theorem 7.4. *Assume* (5.1), (5.18)–(5.24), (7.1), *and* (7.2). *Let* $f \in L^\infty(\Omega)$ *and, for every* $\varepsilon > 0$, *let* u^ε *be a solution of* (7.11). *Assume that*

$$u^\varepsilon \rightharpoonup u^0 \quad \textit{weakly in } H^1(\Omega), \tag{7.12}$$

for some function $u^0 \in H^1(\Omega)$, *and that*

$$\sup_{\varepsilon>0} \|u^\varepsilon\|_{L^\infty(\Omega)} < +\infty, \tag{7.13}$$

$$\sup_{\varepsilon>0} \int_\Omega |u^\varepsilon|^2 \, d\mu^\varepsilon < +\infty. \tag{7.14}$$

Then u^0 *is a solution of* (7.11) *for* $\varepsilon = 0$.

The following lemma, which will be proved in Section 9, shows that (under the other assumptions of Theorem 7.4) conditions (7.13) and (7.14) are always satisfied in every open set $U \Subset \Omega$, and also in Ω if every u^ε belongs to $H^1_0(\Omega)$.

Lemma 7.5. *Assume* (5.1), (5.18)–(5.24), (7.1), *and* (7.2). *Let* $f \in L^\infty(\Omega)$ *and, for every* $\varepsilon > 0$, *let* u^ε *be a solution of* (7.11). *Assume that* (7.12) *holds for some function* $u^0 \in H^1(\Omega)$. *Then we have*

$$\sup_{\varepsilon>0} \|u^\varepsilon\|_{L^\infty(U)} < +\infty, \tag{7.15}$$

$$\sup_{\varepsilon>0} \int_U |u^\varepsilon|^2 \, d\mu^\varepsilon < +\infty, \tag{7.16}$$

for every open set $U \Subset \Omega$. *If, in addition,* $u^\varepsilon \in H^1_0(\Omega)$ *for every* $\varepsilon > 0$, *then* (7.15) *and* (7.16) *also hold for* $U = \Omega$.

In the next corollary $H^1_c(\Omega)$ denotes the space of all functions $u \in H^1(\Omega)$ with compact support in Ω. The first assertion of the corollary follows immediately from Theorem 7.4 and Lemma 7.5, while the last assertion is easily obtained by approximating any nonnegative function $y \in H^1_0(\Omega) \cap L^2(\Omega, \mu^0)$ by the sequence $(|\varphi_j| \wedge y)$, where $\varphi_j \in C^\infty_c(\Omega)$ converges to y in $H^1_0(\Omega)$.

Corollary 7.6. *Under the assumptions of Lemma* 7.5, u^0 *is a solution to the problem*

$$\begin{cases} u^0 \in H^1(\Omega) \cap L^2_{\mathrm{loc}}(\Omega, \mu^0), \\ \displaystyle\int_\Omega A^0 Du^0 Dy \, dx + \int_\Omega u^0 y \, d\mu^0 = \int_\Omega f y \, dx \quad \forall y \in H^1_c(\Omega) \cap L^2(\Omega, \mu^0). \end{cases} \tag{7.17}$$

If, in addition, $u^0 \in L^2(\Omega, \mu^0)$, *then the last line of* (7.17) *holds for every* $y \in H^1_0(\Omega) \cap L^2(\Omega, \mu^0)$.

Let us fix an open set $U \Subset \Omega$ and a function $\zeta \in C_c^\infty(\Omega)$ such that $\zeta = 1$ in U. Given $u^0 \in H^1_{\mathrm{loc}}(\Omega) \cap L^2_{\mathrm{loc}}(\Omega, \mu^0)$, by Proposition 4.7 we can approximate the function ζu^0 in $H^1_0(\Omega) \cap L^2(\Omega, \mu^0)$ by functions of the form $\psi\omega^0$ with $\psi \in C_c^\infty(\Omega)$. Therefore for every $\delta > 0$ there exists $\psi_\delta \in H^2(U) \cap W^{1,\infty}(U)$ such that

$$\beta \int_U \left|Du^0 - D\left(\psi_\delta\omega^0\right)\right|^2 \mathrm{d}x + \int_U \left|u^0 - \psi_\delta\omega^0\right|^2 \mathrm{d}\mu^0 < \delta. \tag{7.18}$$

The following theorem is a local version of the corrector result given in Theorem 7.2.

Theorem 7.7. *Under the hypotheses of Lemma* 7.5*, let U be an open set with $U \Subset \Omega$, let $\delta > 0$, let ψ_δ be a function in $H^2(U) \cap W^{1,\infty}(U)$ which satisfies* (7.18)*, and let v^ε_δ be the functions defined in U by* (7.4)*. Then*

$$\limsup_{\varepsilon\to 0}\left\{\alpha \int_V |Du^\varepsilon - Dv^\varepsilon_\delta|^2 \,\mathrm{d}x + \int_V |u^\varepsilon - v^\varepsilon_\delta|^2 \,\mathrm{d}\mu^\varepsilon\right\} < \delta, \tag{7.19}$$

for every open set $V \Subset U$.

Theorems 7.2 and 7.7 can be deduced from the following theorem, which will be proved in Section 9. Indeed, by Theorem 5.4 and Lemma 7.5, the assumptions of Theorem 7.2 imply all assumptions of Theorem 7.4, so that (7.8) follows from (3.1), (3.2), and (7.21) with $\varphi = 1$. Similarly, the assumptions of Theorem 7.7 imply, by Lemma 7.5, that all assumptions of Theorem 7.4 are satisfied in every open set $U \Subset \Omega$, so that we can apply Theorem 7.8 with Ω replaced by U and with $\varphi \in C_c^\infty(U)$ such that $\varphi = 1$ in V and $\varphi \geqslant 0$ in $U \setminus V$.

Theorem 7.8. *Under the hypotheses of Theorem* 7.4*, let ψ be a function in $H^2(\Omega) \cap W^{1,\infty}(\Omega)$, and let v^ε be defined by*

$$v^\varepsilon = \left(\psi + \sum_{j=1}^n D_j\psi\, z^\varepsilon_j\right)\omega^\varepsilon. \tag{7.20}$$

Then for every $\varphi \in C_c^\infty(\Omega)$ we have

$$\left\{\begin{aligned} &\lim_{\varepsilon\to 0}\left\{\int_\Omega A^\varepsilon D(u^\varepsilon - v^\varepsilon) D(u^\varepsilon - v^\varepsilon)\varphi \,\mathrm{d}x + \int_\Omega |u^\varepsilon - v^\varepsilon|^2 \varphi \,\mathrm{d}\mu^\varepsilon\right\} \\ &\quad = \int_\Omega A^0 D\left(u^0 - \psi\omega^0\right) D\left(u^0 - \psi\omega^0\right)\varphi \,\mathrm{d}x + \int_\Omega \left|u^0 - \psi\omega^0\right|^2 \varphi \,\mathrm{d}\mu^0. \end{aligned}\right. \tag{7.21}$$

If the functions u^ε and v^ε belong to $H^1_0(\Omega) \cap L^2(\Omega, \mu^\varepsilon)$ for every $\varepsilon > 0$, then (7.21) *also holds with $\varphi = 1$.*

8. A comparison theorem

In this section we state and prove a comparison result for the limit measures μ^0_1 and μ^0_2 corresponding to the same sequence of measures (μ^ε) but to two different sequences of H-convergent matrices (A^ε_1) and (A^ε_2). This result has its own interest and will be crucial in the proof of the corrector results stated in the previous section.

For every $\varepsilon \geqslant 0$ let A^ε_1 and A^ε_2 be two matrices in $M^\beta_\alpha(\Omega)$. We assume that

$$(A^\varepsilon_i) \quad H\text{-converges to } A^0_i \text{ for } i = 1, 2. \tag{8.1}$$

For every $\varepsilon > 0$ let μ^ε be a measure in $\mathcal{M}_0^+(\Omega)$, and let μ_1^0 and μ_2^0 be two measures in $\mathcal{M}_0^+(\Omega)$. For $i = 1, 2$ and $\varepsilon > 0$ let w_i^ε be the solutions of the problems

$$\begin{cases} w_i^\varepsilon \in H_0^1(\Omega) \cap L^2(\Omega, \mu^\varepsilon), \\ \displaystyle\int_\Omega A_i^\varepsilon D w_i^\varepsilon D y \,\mathrm{d}x + \int_\Omega w_i^\varepsilon y \,\mathrm{d}\mu^\varepsilon = \int_\Omega y \,\mathrm{d}x \quad \forall y \in H_0^1(\Omega) \cap L^2(\Omega, \mu^\varepsilon), \end{cases} \tag{8.2}$$

and let w_i^0 be the solutions of the problems

$$\begin{cases} w_i^0 \in H_0^1(\Omega) \cap L^2\big(\Omega, \mu_i^0\big), \\ \displaystyle\int_\Omega A_i^0 D w_i^0 D y \,\mathrm{d}x + \int_\Omega w_i^0 y \,\mathrm{d}\mu_i^0 = \int_\Omega y \,\mathrm{d}x \quad \forall y \in H_0^1(\Omega) \cap L^2\big(\Omega, \mu_i^0\big). \end{cases} \tag{8.3}$$

We assume that

$$w_i^\varepsilon \rightharpoonup w_i^0 \quad \text{weakly in } H_0^1(\Omega) \text{ for } i = 1, 2. \tag{8.4}$$

Note that, by Theorem 5.2, these hypotheses are always satisfied by a subsequence.

In this section we shall prove the following comparison theorem.

Theorem 8.1. *Assume* (8.1) *and* (8.4). *Then*

$$\frac{\alpha^2}{\beta^2}\mu_2^0 \leqslant \mu_1^0 \leqslant \frac{\beta^2}{\alpha^2}\mu_2^0 \quad \textit{in } \Omega, \tag{8.5}$$

$$\operatorname{cap}\big(\big\{w_1^0 > 0\big\} \triangle \big\{w_2^0 > 0\big\}\big) = 0. \tag{8.6}$$

In particular we have $L^2(\Omega, \mu_1^0) = L^2(\Omega, \mu_2^0)$.

In order to prove Theorem 8.1, for $\varepsilon \geqslant 0$ and $i = 1, 2$ we consider the measures $\nu_i^\varepsilon \in H^{-1}(\Omega)^+$ defined by

$$-\operatorname{div}\big(A_i^\varepsilon D w_i^\varepsilon\big) + \nu_i^\varepsilon = 1 \quad \text{in } \mathcal{D}'(\Omega) \tag{8.7}$$

(see Proposition 4.3). By Proposition 4.6 we have

$$\nu_i^0 = w_i^0 \mu_i^0 \quad \text{on } \big\{w_i^0 > 0\big\}. \tag{8.8}$$

By Theorem 3.1 we have

$$A_i^\varepsilon D w_i^\varepsilon \rightharpoonup A_i^0 D w_i^0 \quad \text{weakly in } L^2\big(\Omega, \mathbf{R}^n\big). \tag{8.9}$$

Therefore

$$\nu_i^\varepsilon \rightharpoonup \nu_i^0 \quad \text{weakly in } H^{-1}(\Omega). \tag{8.10}$$

As $\nu_i^\varepsilon \geqslant 0$, by Theorem 1 of [21] we have

$$\psi \nu_i^\varepsilon \to \psi \nu_i^0 \quad \text{strongly in } W^{-1,q}(\Omega), \tag{8.11}$$

for every $\psi \in C_c^\infty(\Omega)$ and for every $q < 2$.

Let ζ_i^ε be the solution of the problem

$$\begin{cases} \zeta_i^\varepsilon \in H_0^1(\Omega), \\ -\operatorname{div}\big(A_i^\varepsilon D \zeta_i^\varepsilon\big) = -\operatorname{div}\big(A_i^0 D w_i^0\big) \quad \text{in } \mathcal{D}'(\Omega). \end{cases} \tag{8.12}$$

By the definition of H-convergence we have

$$\zeta_i^\varepsilon \rightharpoonup w_i^0 \quad \text{weakly in } H_0^1(\Omega), \tag{8.13}$$

$$A_i^\varepsilon D\zeta_i^\varepsilon \rightharpoonup A_i^0 D w_i^0 \quad \text{weakly in } L^2\big(\Omega, \mathbf{R}^n\big). \tag{8.14}$$

Lemma 8.2. *For every $\varphi \in C_c^\infty(\Omega)$ and $i = 1, 2$ we have*

$$\lim_{\varepsilon\to 0}\left\{\int_\Omega A_1^\varepsilon D\big(w_1^\varepsilon - \zeta_1^\varepsilon\big) D\big(w_2^\varepsilon - \zeta_2^\varepsilon\big)\varphi\,\mathrm{d}x + \int_\Omega w_1^\varepsilon w_2^\varepsilon \varphi\,\mathrm{d}\mu^\varepsilon\right\} = \int_\Omega w_2^0 \varphi\,\mathrm{d}\nu_1^0, \tag{8.15}$$

$$\lim_{\varepsilon\to 0}\left\{\int_\Omega A_i^\varepsilon D\big(w_i^\varepsilon - \zeta_i^\varepsilon\big) D\big(w_i^\varepsilon - \zeta_i^\varepsilon\big)\varphi\,\mathrm{d}x + \int_\Omega |w_i^\varepsilon|^2 \varphi\,\mathrm{d}\mu^\varepsilon\right\} = \int_\Omega w_i^0 \varphi\,\mathrm{d}\nu_i^0. \tag{8.16}$$

Proof. Let us first prove (8.15). For every $\varepsilon > 0$ we write

$$\int_\Omega A_1^\varepsilon D\big(w_1^\varepsilon - \zeta_1^\varepsilon\big) D\big(w_2^\varepsilon - \zeta_2^\varepsilon\big)\varphi\,\mathrm{d}x + \int_\Omega w_1^\varepsilon w_2^\varepsilon \varphi\,\mathrm{d}\mu^\varepsilon = I^\varepsilon + II^\varepsilon + III^\varepsilon, \tag{8.17}$$

where

$$I^\varepsilon = \int_\Omega A_1^\varepsilon D w_1^\varepsilon D w_2^\varepsilon \varphi\,\mathrm{d}x + \int_\Omega w_1^\varepsilon w_2^\varepsilon \varphi\,\mathrm{d}\mu^\varepsilon,$$

$$II^\varepsilon = -\int_\Omega A_1^\varepsilon D w_1^\varepsilon D\zeta_2^\varepsilon \varphi\,\mathrm{d}x,$$

$$III^\varepsilon = -\int_\Omega A_1^\varepsilon D\zeta_1^\varepsilon D\big(w_2^\varepsilon - \zeta_2^\varepsilon\big)\varphi\,\mathrm{d}x.$$

Using $y = w_2^\varepsilon \varphi$ as test function in (8.2) we get

$$I^\varepsilon = \int_\Omega w_2^\varepsilon \varphi\,\mathrm{d}x - \int_\Omega A_1^\varepsilon D w_1^\varepsilon D\varphi\, w_2^\varepsilon\,\mathrm{d}x.$$

Since (w_2^ε) converges to w_2^0 strongly in $L^2(\Omega)$ by (8.4) and since $(A_1^\varepsilon D w_1^\varepsilon)$ converges to $A_1^0 D w_1^0$ weakly in $L^2(\Omega, \mathbf{R}^n)$ by (8.9), we have

$$\begin{cases} \displaystyle\lim_{\varepsilon\to 0} I^\varepsilon = \int_\Omega w_2^0 \varphi\,\mathrm{d}x - \int_\Omega A_1^0 D w_1^0 D\varphi\, w_2^0\,\mathrm{d}x \\ \displaystyle\qquad\quad = \int_\Omega A_1^0 D w_1^0 D w_2^0 \varphi\,\mathrm{d}x + \int_\Omega w_2^0 \varphi\,\mathrm{d}\nu_1^0, \end{cases} \tag{8.18}$$

where in the last equality we used (8.7) for $\varepsilon = 0$. Note that we cannot use $w_2^0\varphi$ as test function in (8.3) for $i = 1$ because we do not know yet that $w_2^0\varphi \in L^2(\Omega, \mu_1^0)$.

From (8.7) we obtain

$$\begin{cases} \displaystyle II^\varepsilon = -\int_\Omega A_1^\varepsilon D w_1^\varepsilon D\big(\zeta_2^\varepsilon \varphi\big)\,\mathrm{d}x + \int_\Omega A_1^\varepsilon D w_1^\varepsilon D\varphi\, \zeta_2^\varepsilon\,\mathrm{d}x \\ \displaystyle\quad = \big\langle \nu_1^\varepsilon, \zeta_2^\varepsilon \varphi\big\rangle - \int_\Omega \zeta_2^\varepsilon \varphi\,\mathrm{d}x + \int_\Omega A_1^\varepsilon D w_1^\varepsilon D\varphi\, \zeta_2^\varepsilon\,\mathrm{d}x. \end{cases} \tag{8.19}$$

Since (ζ_2^ε) converges to w_2^0 strongly in $L^2(\Omega)$ by (8.13) and $(A_1^\varepsilon Dw_1^\varepsilon)$ converges to $A_1^0 Dw_1^0$ weakly in $L^2(\Omega,\mathbf{R}^n)$ by (8.9), we have

$$\lim_{\varepsilon\to 0}\left\{-\int_\Omega \zeta_2^\varepsilon\varphi\,\mathrm{d}x+\int_\Omega A_1^\varepsilon Dw_1^\varepsilon D\varphi\,\zeta_2^\varepsilon\,\mathrm{d}x\right\}=-\int_\Omega w_2^0\varphi\,\mathrm{d}x+\int_\Omega A_1^0 Dw_1^0 D\varphi\, w_2^0\,\mathrm{d}x. \tag{8.20}$$

We will prove in Lemma 8.3 that

$$\lim_{\varepsilon\to 0}\langle \nu_1^\varepsilon, \zeta_2^\varepsilon\varphi\rangle=\langle \nu_1^0, w_2^0\varphi\rangle. \tag{8.21}$$

From (8.19), (8.20), and (8.21) it follows that

$$\begin{cases} \displaystyle\lim_{\varepsilon\to 0} II^\varepsilon=\int_\Omega w_2^0\varphi\,\mathrm{d}\nu_1^0-\int_\Omega w_2^0\varphi\,\mathrm{d}x+\int_\Omega A_1^0 Dw_1^0 D\varphi\, w_2^0\,\mathrm{d}x\\ \displaystyle\qquad\qquad =-\int_\Omega A_1^0 Dw_1^0 Dw_2^0\varphi\,\mathrm{d}x, \end{cases} \tag{8.22}$$

where the last equality is obtained by using $w_2^0\varphi$ as test function in (8.7) for $\varepsilon=0$.

From (8.12) it follows that

$$\begin{cases} \displaystyle III^\varepsilon=-\int_\Omega A_1^\varepsilon D\zeta_1^\varepsilon D\big((w_2^\varepsilon-\zeta_2^\varepsilon)\varphi\big)\,\mathrm{d}x+\int_\Omega A_1^\varepsilon D\zeta_1^\varepsilon D\varphi\big(w_2^\varepsilon-\zeta_2^\varepsilon\big)\,\mathrm{d}x\\ \displaystyle\quad\;\; =-\int_\Omega A_1^0 Dw_1^0 D\big((w_2^\varepsilon-\zeta_2^\varepsilon)\varphi\big)\,\mathrm{d}x+\int_\Omega A_1^\varepsilon D\zeta_1^\varepsilon D\varphi\big(w_2^\varepsilon-\zeta_2^\varepsilon\big)\,\mathrm{d}x. \end{cases}$$

Since $(w_2^\varepsilon-\zeta_2^\varepsilon)$ converges to 0 weakly in $H_0^1(\Omega)$ and strongly in $L^2(\Omega)$ by (8.4) and (8.13), while $(A_1^\varepsilon D\zeta_1^\varepsilon)$ converges to $A_1^0 D\zeta_1^0$ weakly in $L^2(\Omega,\mathbf{R}^n)$ by (8.14), we have

$$\lim_{\varepsilon\to 0} III^\varepsilon=0. \tag{8.23}$$

Equality (8.15) now follows from (8.17), (8.18), (8.22), and (8.23).

Let us prove now (8.16) for a given $i=1,2$. To this aim for every $\varepsilon>0$ we define $\hat A_1^\varepsilon=\hat A_2^\varepsilon=A_i^\varepsilon$ and $\hat\mu^\varepsilon=\mu^\varepsilon$, so that $\hat w_1^\varepsilon=\hat w_2^\varepsilon=w_i^\varepsilon$, $\hat\zeta_1^\varepsilon=\hat\zeta_2^\varepsilon=\zeta_i^\varepsilon$, and $\hat\nu_1^0=\hat\nu_2^0=\nu_i^0$. Applying (8.15) in this new setting gives (8.16). □

Lemma 8.3. *For every $\varphi\in C_c^\infty(\Omega)$ we have*

$$\lim_{\varepsilon\to 0}\langle \nu_1^\varepsilon, \zeta_2^\varepsilon\varphi\rangle=\langle \nu_1^0, w_2^0\varphi\rangle. \tag{8.24}$$

Proof. Given $\delta>0$, let $\zeta^0\in C_c^\infty(\Omega)$ be a function such that

$$\|\zeta^0-w_2^0\|_{H_0^1(\Omega)}<\delta, \tag{8.25}$$

and let ζ^ε be the solution of the problem

$$\begin{cases} \zeta^\varepsilon\in H_0^1(\Omega),\\ -\operatorname{div}\big(A_2^\varepsilon D\zeta^\varepsilon\big)=-\operatorname{div}\big(A_2^0 D\zeta^0\big)\quad \text{in } \mathcal{D}'(\Omega). \end{cases} \tag{8.26}$$

Using $\zeta^\varepsilon-\zeta_2^\varepsilon$ as test function in (8.12) and (8.26), from (3.1) and (3.2) we obtain

$$\|\zeta^\varepsilon-\zeta_2^\varepsilon\|_{H_0^1(\Omega)}\leqslant\frac{\beta}{\alpha}\|\zeta^0-w_2^0\|_{H_0^1(\Omega)}. \tag{8.27}$$

As (ν_1^ε) is bounded in $H^{-1}(\Omega)$, form (8.25) and (8.27) we obtain that there exists a constant M, independent of δ, such that

$$\begin{cases} \left|\langle \nu_1^\varepsilon, \zeta_2^\varepsilon \varphi\rangle - \langle \nu_1^0, w_2^0\varphi\rangle\right| \leqslant \left|\langle \nu_1^\varepsilon, (\zeta_2^\varepsilon - \zeta^\varepsilon)\varphi\rangle\right| + \left|\langle \nu_1^\varepsilon, \zeta^\varepsilon\varphi\rangle - \langle \nu_1^0, \zeta^0\varphi\rangle\right| + \left|\langle \nu_1^0, (\zeta^0 - w_2^0)\varphi\rangle\right| \\ \qquad \leqslant M\delta + \left|\langle \nu_1^\varepsilon, \zeta^\varepsilon\varphi\rangle - \langle \nu_1^0, \zeta^0\varphi\rangle\right|. \end{cases} \tag{8.28}$$

By Meyers' estimate, there exists $p > 2$ such that $(\zeta^\varepsilon\varphi)$ is bounded in $W_0^{1,p}(\Omega)$. As (ζ^ε) converges to ζ^0 weakly in $H_0^1(\Omega)$ by the definition of H-convergence, we conclude that $(\zeta^\varepsilon\varphi)$ converges to $\zeta^0\varphi$ weakly in $W_0^{1,p}(\Omega)$. Since by (8.11) the sequence $(\psi\nu_1^\varepsilon)$ converges to $\psi\nu_1^0$ strongly in $W^{-1,q}(\Omega)$ for $1/p + 1/q = 1$ and for every $\psi \in C_c^\infty(\Omega)$, we obtain that

$$\lim_{\varepsilon\to 0}\langle \nu_1^\varepsilon, \zeta^\varepsilon\varphi\rangle = \lim_{\varepsilon\to 0}\langle \psi\nu_1^\varepsilon, \zeta^\varepsilon\varphi\rangle = \langle \psi\nu_1^0, \zeta^0\varphi\rangle = \langle \nu_1^0, \zeta^0\varphi\rangle,$$

where ψ is any function in $C_c^\infty(\Omega)$ which is equal to 1 in a neighbourhood of $\operatorname{supp}(\varphi)$.

Therefore by (8.28)

$$\limsup_{\varepsilon\to 0}\left|\langle \nu_1^\varepsilon, \zeta_2^\varepsilon\varphi\rangle - \langle \nu_1^0, w_2^0\varphi\rangle\right| \leqslant M\delta.$$

As $\delta > 0$ is arbitrary, we obtain (8.24). □

Lemma 8.4. *For every $\varphi \in C_c^\infty(\Omega)$, with $\varphi \geqslant 0$ in Ω, and for every $t > 0$ we have*

$$\int_\Omega w_2^0\varphi\,\mathrm{d}\nu_1^0 \leqslant \frac{\beta}{\alpha}\left\{\frac{t}{2}\int_\Omega w_1^0\varphi\,\mathrm{d}\nu_1^0 + \frac{1}{2t}\int_\Omega w_2^0\varphi\,\mathrm{d}\nu_2^0\right\}. \tag{8.29}$$

Proof. By (3.1) and (3.2) we have the estimates

$$\begin{cases} \displaystyle\int_\Omega A_1^\varepsilon D(w_1^\varepsilon - \zeta_1^\varepsilon)D(w_2^\varepsilon - \zeta_2^\varepsilon)\varphi\,\mathrm{d}x + \int_\Omega w_1^\varepsilon w_2^\varepsilon\varphi\,\mathrm{d}\mu^\varepsilon \\ \displaystyle\quad \leqslant \beta\int_\Omega \left|D(w_1^\varepsilon - \zeta_1^\varepsilon)\right|\left|D(w_2^\varepsilon - \zeta_2^\varepsilon)\right|\varphi\,\mathrm{d}x + \int_\Omega w_1^\varepsilon w_2^\varepsilon\varphi\,\mathrm{d}\mu^\varepsilon \\ \displaystyle\quad \leqslant \frac{t}{2}\left\{\beta\int_\Omega \left|D(w_1^\varepsilon - \zeta_1^\varepsilon)\right|^2\varphi\,\mathrm{d}x + \int_\Omega |w_1^\varepsilon|^2\varphi\,\mathrm{d}\mu^\varepsilon\right\} + \frac{1}{2t}\left\{\beta\int_\Omega \left|D(w_2^\varepsilon - \zeta_2^\varepsilon)\right|^2\varphi\,\mathrm{d}x + \int_\Omega |w_2^\varepsilon|^2\varphi\,\mathrm{d}\mu^\varepsilon\right\} \\ \displaystyle\quad \leqslant \frac{\beta}{\alpha}\frac{t}{2}\left\{\int_\Omega A_1^\varepsilon D(w_1^\varepsilon - \zeta_1^\varepsilon)D(w_1^\varepsilon - \zeta_1^\varepsilon)\varphi\,\mathrm{d}x + \int_\Omega |w_1^\varepsilon|^2\varphi\,\mathrm{d}\mu^\varepsilon\right\} \\ \displaystyle\qquad + \frac{\beta}{\alpha}\frac{1}{2t}\left\{\int_\Omega A_2^\varepsilon D(w_2^\varepsilon - \zeta_2^\varepsilon)D(w_2^\varepsilon - \zeta_2^\varepsilon)\varphi\,\mathrm{d}x + \int_\Omega |w_2^\varepsilon|^2\varphi\,\mathrm{d}\mu^\varepsilon\right\}. \end{cases}$$

Inequality (8.29) is obtained by applying Lemma 8.2. □

Lemma 8.5. *The following inequality holds*:

$$w_2^0\nu_1^0 \leqslant \frac{\beta^2}{\alpha^2}w_1^0\nu_2^0 \quad \text{in } \Omega. \tag{8.30}$$

Proof. Let $\nu = \nu_1^0 + \nu_2^0$. From Lemma 8.4 it follows that for every $t > 0$

$$w_2^0\frac{\mathrm{d}\nu_1^0}{\mathrm{d}\nu} \leqslant \frac{\beta}{\alpha}\left\{\frac{t}{2}w_1^0\frac{\mathrm{d}\nu_1^0}{\mathrm{d}\nu} + \frac{1}{2t}w_2^0\frac{\mathrm{d}\nu_2^0}{\mathrm{d}\nu}\right\} \quad \nu\text{-a.e. in } \Omega. \tag{8.31}$$

If we minimize with respect to t we obtain

$$w_2^0 \frac{\mathrm{d}\nu_1^0}{\mathrm{d}\nu} \leqslant \frac{\beta^2}{\alpha^2} w_1^0 \frac{\mathrm{d}\nu_2^0}{\mathrm{d}\nu} \quad \nu\text{-a.e. in } \Omega,$$

which implies (8.30). □

Proof of Theorem 8.1. We prove only the second inequality in (8.5) and

$$\operatorname{cap}\bigl(\bigl\{w_2^0 > 0\bigr\} \setminus \bigl\{w_1^0 > 0\bigr\}\bigr) = 0. \tag{8.32}$$

The other inequality and the equality $\operatorname{cap}(\{w_1^0 > 0\} \setminus \{w_2^0 > 0\}) = 0$ are proved by exchanging the roles of A_1^ε and A_2^ε.

By (8.8) we have $\nu_2^0 = w_2^0 \mu_2^0$ on $\{w_2^0 > 0\}$, so that (8.30) gives

$$\nu_1^0 \leqslant \frac{\beta^2}{\alpha^2} w_1^0 \mu_2^0 \quad \text{on } \bigl\{w_2^0 > 0\bigr\}. \tag{8.33}$$

If $y \in H_0^1(\Omega) \cap L^2(\Omega, \mu_2^0)$, then $y = 0$ q.e. on $\{w_2^0 = 0\}$ (see Proposition 4.5). From (8.7) and (8.33) it follows that

$$\int_\Omega A_1^0 D w_1^0 D y \,\mathrm{d}x + \frac{\beta^2}{\alpha^2} \int_\Omega w_1^0 y \,\mathrm{d}\mu_2^0 \geqslant \int_\Omega y \,\mathrm{d}x \tag{8.34}$$

for every $y \in H_0^1(\Omega) \cap L^2(\Omega, \mu_2^0)$ with $y \geqslant 0$ q.e. in Ω.

Let w be the solution of the problem

$$\begin{cases} w \in H_0^1(\Omega) \cap L^2\bigl(\Omega, \mu_2^0\bigr), \\ \displaystyle\int_\Omega A_1^0 D w D y \,\mathrm{d}x + \frac{\beta^2}{\alpha^2} \int_\Omega w y \,\mathrm{d}\mu_2^0 = \int_\Omega y \,\mathrm{d}x \quad \forall y \in H_0^1(\Omega) \cap L^2\bigl(\Omega, \mu_2^0\bigr). \end{cases} \tag{8.35}$$

As $0 \leqslant (w - w_1^0)^+ \leqslant w$ q.e. in Ω, the function $y = (w - w_1^0)^+$ can be taken as test function in (8.35) and (8.34). By difference we obtain

$$\int_\Omega A_1^0 D\bigl(w - w_1^0\bigr) D\bigl(w - w_1^0\bigr)^+ \,\mathrm{d}x + \frac{\beta^2}{\alpha^2} \int_\Omega \bigl(w - w_1^0\bigr)\bigl(w - w_1^0\bigr)^+ \,\mathrm{d}\mu_2^0 \leqslant 0,$$

which implies $(w - w_1^0)^+ = 0$ a.e. in Ω, and hence $w \leqslant w_1^0$ q.e. in Ω by (2.1). Therefore

$$\operatorname{cap}\bigl(\{w > 0\} \cap \bigl\{w_1^0 = 0\bigr\}\bigr) = 0. \tag{8.36}$$

Let us prove that

$$\operatorname{cap}\bigl(\bigl\{w_2^0 > 0\bigr\} \cap \{w = 0\}\bigr) = 0. \tag{8.37}$$

It is enough to show that

$$\operatorname{cap}\bigl(\bigl\{w_2^0 > \delta\bigr\} \cap \{w = 0\}\bigr) = 0 \tag{8.38}$$

for every $\delta > 0$. If (8.38) is not satisfied, by Proposition 4.5 we have $(\beta^2/\alpha^2)\mu_2^0(\{w_2^0 > \delta\}) = +\infty$, which contradicts the fact that $w_2^0 \in L^2(\Omega, \mu_2^0)$. This proves (8.37).

As $w \geqslant 0$ and $w_1^0 \geqslant 0$ q.e. on Ω by the comparison principle (Theorem 2.10 of [13]), from (8.36) and (8.37) it follows that

$$\operatorname{cap}\bigl(\bigl\{w_2^0 > 0\bigr\} \cap \bigl\{w_1^0 = 0\bigr\}\bigr) = \operatorname{cap}\bigl(\bigl\{w_2^0 > 0\bigr\} \setminus \bigl\{w_1^0 > 0\bigr\}\bigr) = 0, \tag{8.39}$$

which proves (8.32).

Since $\nu_1^0 = w_1^0\mu_1^0$ on $\{w_1^0 > 0\}$ by (8.8), it follows from (8.39) that $\nu_1^0 = w_1^0\mu_1^0$ on $\{w_2^0 > 0\}$, so that (8.33) yields

$$w_1^0\mu_1^0 \leqslant \frac{\beta^2}{\alpha^2} w_1^0\mu_2^0 \quad \text{on } \{w_2^0 > 0\}.$$

As $w_1^0 > 0$ q.e. on $\{w_2^0 > 0\}$, we conclude that

$$\mu_1^0 \leqslant \frac{\beta^2}{\alpha^2}\mu_2^0 \quad \text{on } \{w_2^0 > 0\}. \tag{8.40}$$

Let us finally prove that

$$\mu_1^0 \leqslant \frac{\beta^2}{\alpha^2}\mu_2^0 \quad \text{on } \{w_2^0 = 0\}. \tag{8.41}$$

Let B be a Borel set contained in $\{w_2^0 = 0\}$. If $\mathrm{cap}(B) = 0$, then $\mu_1^0(B) = \mu_2^0(B) = 0$, because μ_1^0 and μ_2^0 belong to $\mathcal{M}_0^+(\Omega)$. If $\mathrm{cap}(B) > 0$, then $\mu_2^0(B) = +\infty$ by Proposition 4.5. In both cases we have $\mu_1^0(B) \leqslant \frac{\beta^2}{\alpha^2}\mu_2^0(B)$, hence (8.41) is proved.

Inequality (8.5) now follows from (8.40) and (8.41). □

9. Proofs of the corrector results

In this section we prove Lemma 7.5 and Theorems 7.4 and 7.8, which give immediately all results of Section 7 (see the comments before the statement of Theorem 7.8).

We begin by the following theorem, which is proved by using the comparison result of Section 8.

Theorem 9.1. *Assume* (5.1), *and* (5.18)–(5.24). *For every $\varepsilon > 0$, let $y^\varepsilon \in H^1(\Omega) \cap L^2(\Omega, \mu^\varepsilon)$ be such that*

$$y^\varepsilon \rightharpoonup y^0 \quad \textit{weakly in } H^1(\Omega), \tag{9.1}$$

$$\sup_{\varepsilon>0} \int_\Omega |y^\varepsilon|^2 \,\mathrm{d}\mu^\varepsilon < +\infty. \tag{9.2}$$

Then $y^0 \in L^2(\Omega, \mu^0)$.

Proof. We use the notion of γ-convergence, introduced in [14] and further developed in [10], which concerns the convergence of minima and minimizers of the functionals J_f^ε defined on $H_0^1(\Omega) \cap L^2(\Omega, \mu^\varepsilon)$ by

$$J_f^\varepsilon(y) = \alpha \int_\Omega |Dy|^2 \,\mathrm{d}x + \int_\Omega |y|^2 \,\mathrm{d}\mu^\varepsilon - 2\langle f, y\rangle,$$

for any given $f \in H^{-1}(\Omega)$. Note that the minimizer of J_f^ε is the unique solution to problem (4.1) with $A = \alpha I$ and $\mu = \mu^\varepsilon$. By Theorem 4.14 of [14] there exists a subsequence, still denoted by (μ^ε), which γ-converges (with respect to the operator $-\alpha\Delta$) to a measure $\hat\mu^0 \in \mathcal{M}_0^+(\Omega)$ (the regularity property (b) of $\hat\mu^0$ is obtained by using Theorem 3.10 of [10]). By Lemma 5.5 of [10] we have

$$\alpha \int_\Omega |Dy^0|^2 \,\mathrm{d}x + \int_\Omega |y^0|^2 \,\mathrm{d}\hat\mu^0 \leqslant \liminf_{\varepsilon\to 0} \left\{ \alpha \int_\Omega |Dy^\varepsilon|^2 \,\mathrm{d}x + \int_\Omega |y^\varepsilon|^2 \,\mathrm{d}\mu^\varepsilon \right\} < +\infty. \tag{9.3}$$

Let $\widehat{w}^\varepsilon$ be the unique solution to the problem

$$\begin{cases} \widehat{w}^\varepsilon \in H_0^1(\Omega) \cap L^2(\Omega, \mu^\varepsilon), \\ \alpha \int\limits_\Omega D\widehat{w}^\varepsilon Dy \, dx + \int\limits_\Omega \widehat{w}^\varepsilon y \, d\mu^\varepsilon = \int\limits_\Omega y \, dx \quad \forall y \in H_0^1(\Omega) \cap L^2(\Omega, \mu^\varepsilon). \end{cases} \tag{9.4}$$

By Proposition 4.10 of [14] the sequence $(\widehat{w}^\varepsilon)$ converges weakly in $H_0^1(\Omega)$ to the solution $\widehat{w}^0$ of the problem

$$\begin{cases} \widehat{w}^0 \in H_0^1(\Omega) \cap L^2(\Omega, \hat{\mu}^0), \\ \alpha \int\limits_\Omega D\widehat{w}^0 Dy \, dx + \int\limits_\Omega \widehat{w}^0 y \, d\hat{\mu}^0 = \int\limits_\Omega y \, dx \quad \forall y \in H_0^1(\Omega) \cap L^2(\Omega, \hat{\mu}^0). \end{cases} \tag{9.5}$$

If we apply Theorem 8.1 with $A_1^\varepsilon = A^\varepsilon$ and $A_2^\varepsilon = \alpha I$, we obtain

$$\mu^0 \leqslant \frac{\beta^2}{\alpha^2} \hat{\mu}^0, \tag{9.6}$$

so that (9.3) implies that $y^0 \in L^2(\Omega, \mu^0)$. $\square$

Lemma 9.2. *Under the hypotheses of Theorem* 7.4, *we have*

$$A^\varepsilon Du^\varepsilon \rightharpoonup A^0 Du^0 \quad \textit{weakly in } L^2(\Omega, \mathbf{R}^n). \tag{9.7}$$

Moreover there exists $\sigma^0 \in \mathcal{M}(\Omega) \cap H^{-1}(\Omega)$, with $|\sigma^0| \in \mathcal{M}(\Omega) \cap H^{-1}(\Omega)$, such that

$$-\operatorname{div}(A^0 Du^0) + \sigma^0 = f \quad \textit{in } \mathcal{D}'(\Omega). \tag{9.8}$$

Proof. Since the positive and the negative parts $(u^\varepsilon)^+$ and $(u^\varepsilon)^-$ of u^ε belong to $H^1(\Omega) \cap L^2(\Omega, \mu^\varepsilon)$, by Theorem 2.4 of [13] for every $\varepsilon > 0$ we can consider the solutions $u_\oplus^\varepsilon$ and $u_\ominus^\varepsilon$ to the problems

$$\begin{cases} u_\oplus^\varepsilon - (u^\varepsilon)^+ \in H_0^1(\Omega) \cap L^2(\Omega, \mu^\varepsilon), \\ \int\limits_\Omega A^\varepsilon Du_\oplus^\varepsilon Dy \, dx + \int\limits_\Omega u_\oplus^\varepsilon y \, d\mu^\varepsilon = \int\limits_\Omega f^+ y \, dx \quad \forall y \in H_0^1(\Omega) \cap L^2(\Omega, \mu^\varepsilon), \end{cases} \tag{9.9}$$

$$\begin{cases} u_\ominus^\varepsilon - (u^\varepsilon)^- \in H_0^1(\Omega) \cap L^2(\Omega, \mu^\varepsilon), \\ \int\limits_\Omega A^\varepsilon Du_\ominus^\varepsilon Dy \, dx + \int\limits_\Omega u_\ominus^\varepsilon y \, d\mu^\varepsilon = \int\limits_\Omega f^- y \, dx \quad \forall y \in H_0^1(\Omega) \cap L^2(\Omega, \mu^\varepsilon). \end{cases} \tag{9.10}$$

By linearity we have

$$u^\varepsilon = u_\oplus^\varepsilon - u_\ominus^\varepsilon \quad \text{q.e. in } \Omega. \tag{9.11}$$

Using $y = u_\oplus^\varepsilon - (u^\varepsilon)^+$ as test function in (9.9), and then (3.1) and (3.2), as well as Poincaré's and Young's inequalities, we obtain

$$\sup_{\varepsilon > 0} \|u_\oplus^\varepsilon\|_{H^1(\Omega)} < +\infty. \tag{9.12}$$

Passing to a subsequence, we can assume that $(u_\oplus^\varepsilon)$ converges weakly in $H^1(\Omega)$ to some function $u_\oplus^0$. Since $u_\oplus^\varepsilon \geqslant 0$ q.e. in Ω by the comparison principle (Theorem 2.10 of [13]), by Proposition 4.3 there exists $\sigma_\oplus^\varepsilon \in H^{-1}(\Omega)^+$ such that

$$-\operatorname{div}(A^\varepsilon Du_\oplus^\varepsilon) + \sigma_\oplus^\varepsilon = f^+ \quad \text{in } \mathcal{D}'(\Omega). \tag{9.13}$$

From Theorem 3.1 we obtain that

$$A^\varepsilon Du^\varepsilon_\oplus \rightharpoonup A^0 Du^0_\oplus \quad \text{weakly in } L^2(\Omega,\mathbf{R}^n), \tag{9.14}$$

and we deduce from (9.13) that there exists $\sigma^0_\oplus \in H^{-1}(\Omega)^+$ such that

$$-\operatorname{div}(A^0 Du^0_\oplus) + \sigma^0_\oplus = f^+ \quad \text{in } \mathcal{D}'(\Omega). \tag{9.15}$$

Properties (9.7) and (9.8) now follow from (9.14) and (9.15), from the analogous results for $u^\varepsilon_\ominus$, and from (9.11). □

Proof of Lemma 7.5. By (9.11) we have $u^\varepsilon = u^\varepsilon_\oplus - u^\varepsilon_\ominus$ q.e. in Ω, where $u^\varepsilon_\oplus$ and $u^\varepsilon_\ominus$ are the solutions of (9.9) and (9.10). Let $v^\varepsilon_\oplus$ be the solution to the problem

$$\begin{cases} v^\varepsilon_\oplus - (u^\varepsilon)^+ \in H^1_0(\Omega), \\ -\operatorname{div}(A^\varepsilon D v^\varepsilon_\oplus) = f^+ \quad \text{in } \mathcal{D}'(\Omega). \end{cases}$$

By the comparison principle (Theorem 2.10 of [13]) we have $0 \leqslant u^\varepsilon_\oplus \leqslant v^\varepsilon_\oplus$ q.e. in Ω.

As $(u^\varepsilon)^+$ is bounded in $H^1(\Omega)$, the sequence $(v^\varepsilon_\oplus)$ is bounded in $H^1(\Omega)$ too. On the other hand the classical local L^∞ estimate for solutions of elliptic equations (see, e.g., [25]) asserts that for every open set $U \Subset \Omega$

$$\|v^\varepsilon_\oplus\|_{L^\infty(U)} \leqslant C_U \|v^\varepsilon_\oplus\|_{L^2(\Omega)}, \tag{9.16}$$

therefore $(v^\varepsilon_\oplus)$, and hence $(u^\varepsilon_\oplus)$, is bounded in $L^\infty(U)$. If $u^\varepsilon \in H^1_0(\Omega)$, we have also $v^\varepsilon_\oplus \in H^1_0(\Omega)$, and the global L^∞ estimate in Ω implies that $(v^\varepsilon_\oplus)$, and hence $(u^\varepsilon_\oplus)$, is bounded in $L^\infty(\Omega)$. A similar argument holds for $(u^\varepsilon_\ominus)$, so that (u^ε) is bounded in $L^\infty(U)$ (and also in $L^\infty(\Omega)$ if $u^\varepsilon \in H^1_0(\Omega)$) and (7.15) is proved.

Let φ be a function in $C^\infty_c(\Omega)$ such that $\varphi = 1$ in U. Using $y = u^\varepsilon\varphi^2$ as test function in (7.11), and then (3.1), (3.2), and the boundedness of (u^ε) in $H^1(\Omega)$, we easily obtain (7.16). If $u^\varepsilon \in H^1_0(\Omega)$, we simply use $y = u^\varepsilon$ as test function in (7.11). □

The proof of Theorems 7.4 and 7.8 will be divided in three lemmas. For every $\varepsilon > 0$ let y^ε be a function of $H^1(\Omega) \cap L^2(\Omega,\mu^\varepsilon)$ such that

$$y^\varepsilon \rightharpoonup y^0 \quad \text{weakly in } H^1(\Omega), \tag{9.17}$$

for some function y^0 in $H^1(\Omega)$. Assume that

$$\sup_{\varepsilon>0} \|y^\varepsilon\|_{L^\infty(\Omega)} < +\infty, \tag{9.18}$$

$$\sup_{\varepsilon>0} \int_\Omega |y^\varepsilon|^2 \,\mathrm{d}\mu^\varepsilon < +\infty. \tag{9.19}$$

Lemma 9.3. *Under the hypotheses of Theorem* 7.8, *let* y^ε, $\varepsilon \geqslant 0$, *be functions in* $H^1(\Omega)$ *which satisfy* (9.17), (9.18), *and* (9.19). *Then* y^0 *belongs to* $L^2(\Omega,\mu^0)$ *and for every* $\varphi \in C^\infty_c(\Omega)$ *we have*

$$\lim_{\varepsilon\to 0}\left\{\int_\Omega A^\varepsilon Dv^\varepsilon Dy^\varepsilon \varphi \,\mathrm{d}x + \int_\Omega v^\varepsilon y^\varepsilon \varphi \,\mathrm{d}\mu^\varepsilon\right\} = \int_\Omega A^0 D(\psi\,\omega^0) Dy^0 \varphi \,\mathrm{d}x + \int_\Omega y^0 \psi \omega^0 \varphi \,\mathrm{d}\mu^0. \tag{9.20}$$

If, in addition, $y^\varepsilon \in H^1_0(\Omega)$ *for every* $\varepsilon > 0$, *then* (9.20) *also holds with* $\varphi = 1$.

Proof. We prove (9.20) only in the case $\varphi \in C^\infty_c(\Omega)$, since, under the additional hypothesis $y^\varepsilon \in H^1_0(\Omega)$, the proof with $\varphi = 1$ is similar. In this proof (η^ε) will denote a sequence of real numbers converging to 0 as ε tends to 0, whose value can change from line to line.

Theorem 9.1, (9.17), and (9.19) imply that $y^0 \in L^2(\Omega, \mu^0)$.

By (7.6) for every $\varphi \in C_c^\infty(\Omega)$ we have

$$\int_\Omega A^\varepsilon Dv^\varepsilon Dy^\varepsilon \varphi \,\mathrm{d}x = \sum_{j=1}^n \int_\Omega D_j\psi A^\varepsilon(e_j + Dz_j^\varepsilon) Dy^\varepsilon \omega^\varepsilon \varphi \,\mathrm{d}x + \int_\Omega \psi A^\varepsilon D\omega^\varepsilon Dy^\varepsilon \varphi \,\mathrm{d}x + \eta^\varepsilon.$$

By (5.22) and (7.1) the sequence (ω^ε) converges to ω^0 strongly in $L^r(\Omega)$ for every $1 \leqslant r < +\infty$. Since, by (3.9), (z_j^ε) converges to 0 weakly in in $W^{1,p}(\Omega)$ for some $p > 2$, we conclude that

$$\int_\Omega D_j\psi A^\varepsilon(e_j + Dz_j^\varepsilon) Dy^\varepsilon \omega^\varepsilon \varphi \,\mathrm{d}x = \int_\Omega D_j\psi A^\varepsilon(e_j + Dz_j^\varepsilon) Dy^\varepsilon \omega^0 \varphi \,\mathrm{d}x + \eta^\varepsilon.$$

Therefore

$$\int_\Omega A^\varepsilon Dv^\varepsilon Dy^\varepsilon \varphi \,\mathrm{d}x + \int_\Omega v^\varepsilon y^\varepsilon \varphi \,\mathrm{d}\mu^\varepsilon = I^\varepsilon + II^\varepsilon + III^\varepsilon + \eta^\varepsilon, \tag{9.21}$$

where

$$I^\varepsilon = \sum_{j=1}^n \int_\Omega D_j\psi A^\varepsilon(e_j + Dz_j^\varepsilon) Dy^\varepsilon \omega^0 \varphi \,\mathrm{d}x,$$

$$II^\varepsilon = \int_\Omega \psi A^\varepsilon D\omega^\varepsilon Dy^\varepsilon \varphi \,\mathrm{d}x,$$

$$III^\varepsilon = \int_\Omega v^\varepsilon y^\varepsilon \varphi \,\mathrm{d}\mu^\varepsilon.$$

We now pass to the limit in I^ε, II^ε, and III^ε. For what concerns I^ε, we write

$$\begin{cases} I_j^\varepsilon = \displaystyle\int_\Omega D_j\psi A^\varepsilon(e_j + Dz_j^\varepsilon) Dy^\varepsilon \omega^0 \varphi \,\mathrm{d}x \\ \quad = \big\langle -\operatorname{div}\big(A^\varepsilon(e_j + Dz_j^\varepsilon)\big), D_j\psi\, y^\varepsilon \omega^0 \varphi \big\rangle - \displaystyle\int_\Omega A^\varepsilon(e_j + Dz_j^\varepsilon) DD_j\psi\, y^\varepsilon \omega^0 \varphi \,\mathrm{d}x \\ \qquad - \displaystyle\int_\Omega A^\varepsilon(e_j + Dz_j^\varepsilon) D_j\psi\, y^\varepsilon D\omega^0 \varphi \,\mathrm{d}x - \displaystyle\int_\Omega A^\varepsilon(e_j + Dz_j^\varepsilon) D_j\psi\, y^\varepsilon \omega^0 D\varphi \,\mathrm{d}x. \end{cases}$$

Properties (3.6) and (3.7) of z_j^ε, together with properties (9.17) and (9.18) of y^ε, imply that we can pass to the limit in each term of the right-hand side of the previous formula, so that

$$I^\varepsilon = \sum_{j=1}^n \int_\Omega D_j\psi A^0 e_j Dy^0 \omega^0 \varphi \,\mathrm{d}x + \eta^\varepsilon = \int_\Omega A^0 D\psi Dy^0 \omega^0 \varphi \,\mathrm{d}x + \eta^\varepsilon. \tag{9.22}$$

As for II^ε, we write

$$II^\varepsilon = \int_\Omega \psi A^\varepsilon D\omega^\varepsilon Dy^\varepsilon \varphi \,\mathrm{d}x = \int_\Omega A^\varepsilon D\omega^\varepsilon D(y^\varepsilon \psi\varphi) \,\mathrm{d}x - \int_\Omega A^\varepsilon D\omega^\varepsilon y^\varepsilon D(\psi\varphi) \,\mathrm{d}x. \tag{9.23}$$

As ω^ε satisfies (5.18), we have

$$\int_\Omega A^\varepsilon D\omega^\varepsilon D(y^\varepsilon \psi\varphi) \,\mathrm{d}x + \int_\Omega \omega^\varepsilon y^\varepsilon \psi\varphi \,\mathrm{d}\mu^\varepsilon = \int_\Omega y^\varepsilon \psi\varphi \,\mathrm{d}\lambda^\varepsilon,$$

and by (5.20) and (9.17) we conclude that

$$\int_\Omega A^\varepsilon D\omega^\varepsilon D(y^\varepsilon\psi\varphi)\,\mathrm{d}x=\int_\Omega y^0\psi\varphi\,\mathrm{d}\lambda^0-\int_\Omega \omega^\varepsilon y^\varepsilon\psi\varphi\,\mathrm{d}\mu^\varepsilon+\eta^\varepsilon. \tag{9.24}$$

By Proposition 4.3 and Theorem 3.1 $(A^\varepsilon D\omega^\varepsilon)$ converges to $A^0D\omega^0$ weakly in $L^2(\Omega,\mathbf{R}^n)$, while by (9.17) (y^ε) converges to y^0 strongly in $L^2_{\mathrm{loc}}(\Omega)$. Therefore

$$-\int_\Omega A^\varepsilon D\omega^\varepsilon y^\varepsilon D(\psi\varphi)\,\mathrm{d}x=-\int_\Omega A^0D\omega^0y^0D(\psi\varphi)\,\mathrm{d}x+\eta^\varepsilon. \tag{9.25}$$

From (9.23), (9.24), and (9.25) we obtain that

$$\begin{cases} II^\varepsilon=\displaystyle\int_\Omega y^0\psi\varphi\,\mathrm{d}\lambda^0-\int_\Omega \omega^\varepsilon y^\varepsilon\psi\varphi\,\mathrm{d}\mu^\varepsilon-\int_\Omega A^0D\omega^0y^0D(\psi\varphi)\,\mathrm{d}x+\eta^\varepsilon\\ \quad=\displaystyle\int_\Omega A^0D\omega^0Dy^0\psi\varphi\,\mathrm{d}x+\int_\Omega\omega^0y^0\psi\varphi\,\mathrm{d}\mu^0-\int_\Omega\omega^\varepsilon y^\varepsilon\psi\varphi\,\mathrm{d}\mu^\varepsilon+\eta^\varepsilon, \end{cases} \tag{9.26}$$

where the last equality follows from (5.18) for $\varepsilon=0$, since $y^0\in L^2(\Omega,\mu^0)$.

Finally, we write III^ε as

$$III^\varepsilon=\int_\Omega v^\varepsilon y^\varepsilon\varphi\,\mathrm{d}\mu^\varepsilon=\int_\Omega\psi\omega^\varepsilon y^\varepsilon\varphi\,\mathrm{d}\mu^\varepsilon+\sum_{j=1}^n\int_\Omega D_j\psi\, z_j^\varepsilon\omega^\varepsilon y^\varepsilon\varphi\,\mathrm{d}\mu^\varepsilon.$$

Since, by (3.8), (z_j^ε) converges to 0 uniformly, while, by (7.2) and (9.19), the norms of ω^ε and y^ε in $L^2(\Omega,\mu^\varepsilon)$ remain bounded, we conclude that

$$III^\varepsilon=\int_\Omega\psi\omega^\varepsilon y^\varepsilon\varphi\,\mathrm{d}\mu^\varepsilon+\eta^\varepsilon. \tag{9.27}$$

From (9.21), (9.22), (9.26), and (9.27) we obtain (9.20). □

Lemma 9.4. *Under the hypotheses of Theorem* 7.8*, for every* $\varphi\in C_c^\infty(\Omega)$ *we have*

$$\begin{cases} \displaystyle\lim_{\varepsilon\to0}\left\{\int_\Omega A^\varepsilon D(u^\varepsilon-v^\varepsilon)D(u^\varepsilon-v^\varepsilon)\varphi\,\mathrm{d}x+\int_\Omega|u^\varepsilon-v^\varepsilon|^2\varphi\,\mathrm{d}\mu^\varepsilon\right\}\\ \quad=\displaystyle\int_\Omega A^0D\big(u^0-\psi\omega^0\big)D\big(u^0-\psi\omega^0\big)\varphi\,\mathrm{d}x+\int_\Omega\big(u^0-\psi\omega^0\big)\varphi\,\mathrm{d}\sigma^0-\int_\Omega\big(u^0-\psi\,\omega^0\big)\psi\omega^0\varphi\,\mathrm{d}\mu^0, \end{cases} \tag{9.28}$$

where σ^0 *is defined by* (9.8)*. If the functions* u^ε *and* v^ε *belong to* $H_0^1(\Omega)$ *for every* $\varepsilon>0$*, then* (9.28) *also holds with* $\varphi=1$.

Proof. We prove (9.28) only in the case $\varphi\in C_c^\infty(\Omega)$, since, under the additional hypothesis $u^\varepsilon,\ v^\varepsilon\in H_0^1(\Omega)$, the proof with $\varphi=1$ is similar.

Let $y^\varepsilon=u^\varepsilon-v^\varepsilon$ and let $y^0=u^0-\psi\omega^0$. Then properties (9.17), (9.18), and (9.19) are satisfied by the definition (7.20) of v^ε and by (3.5), (3.8), (5.22), (7.1), (7.2), (7.12), (7.13), and (7.14). Using $y=y^\varepsilon\varphi$ as test function in (7.11) we get

$$\int_\Omega A^\varepsilon Du^\varepsilon Dy^\varepsilon\varphi\,\mathrm{d}x+\int_\Omega u^\varepsilon y^\varepsilon\varphi\,\mathrm{d}\mu^\varepsilon=\int_\Omega fy^\varepsilon\varphi\,\mathrm{d}x-\int_\Omega A^\varepsilon Du^\varepsilon D\varphi\, y^\varepsilon\,\mathrm{d}x. \tag{9.29}$$

Using (9.7) and (9.8) we obtain

$$\begin{cases} \lim_{\varepsilon\to 0}\left\{ \int_\Omega A^\varepsilon Du^\varepsilon Dy^\varepsilon \varphi\,\mathrm{d}x + \int_\Omega u^\varepsilon y^\varepsilon \varphi\,\mathrm{d}\mu^\varepsilon \right\} = \int_\Omega f y^0 \varphi\,\mathrm{d}x - \int_\Omega A^0 Du^0 D\varphi\, y^0\,\mathrm{d}x \\ \qquad = \int_\Omega A^0 Du^0 Dy^0 \varphi\,\mathrm{d}x + \int_\Omega y^0 \varphi\,\mathrm{d}\sigma^0. \end{cases} \tag{9.30}$$

From (9.30) and (9.20) we deduce (9.28). □

Lemma 9.5. *Under the hypotheses of Theorem* 7.8*, for every* $y \in H_0^1(\Omega) \cap L^2(\Omega, \mu^0)$ *we have*

$$\int_\Omega y\,\mathrm{d}\sigma^0 = \int_\Omega y u^0\,\mathrm{d}\mu^0, \tag{9.31}$$

where σ^0 *is defined by* (9.8).

Proof. First of all we recall that $u^0 \in H^1(\Omega) \cap L^2(\Omega, \mu^0)$ by Theorem 9.1. Let us fix $\varphi \in C_c^\infty(\Omega)$ with $\varphi \geqslant 0$ in Ω. By Lemma 9.4 we have

$$\int_\Omega A^0 D\big(u^0 - \psi\,\omega^0\big) D\big(u^0 - \psi\,\omega^0\big)\varphi\,\mathrm{d}x + \int_\Omega \big(u^0 - \psi\,\omega^0\big)\varphi\,\mathrm{d}\sigma^0 - \int_\Omega \big(u^0 - \psi\,\omega^0\big)\psi\omega^0\varphi\,\mathrm{d}\mu^0 \geqslant 0, \tag{9.32}$$

for every $\psi \in H^2(\Omega) \cap W^{1,\infty}(\Omega)$. By Proposition 4.7 the set $\{\psi\omega^0\colon\ \psi \in C_c^\infty(\Omega)\}$ is dense in $H_0^1(\Omega) \cap L^2(\Omega, \mu^0)$. Therefore (9.32) implies that

$$\int_\Omega A^0 D\big(u^0 - z\big) D\big(u^0 - z\big)\varphi\,\mathrm{d}x + \int_\Omega \big(u^0 - z\big)\varphi\,\mathrm{d}\sigma^0 - \int_\Omega \big(u^0 - z\big) z\varphi\,\mathrm{d}\mu^0 \geqslant 0, \tag{9.33}$$

for every $z \in H_0^1(\Omega) \cap L^2(\Omega, \mu^0)$.

We now use Minty's trick, and we take in (9.33) $z = u^0\zeta + ty$, with $t \in \mathbf{R}$, $y \in H_0^1(\Omega) \cap L^2(\Omega, \mu^0)$, and $\zeta \in C_c^\infty(\Omega)$ with $\zeta = 1$ on $\operatorname{supp}\varphi$. Dividing by t and passing to the limit as t tends to 0 we obtain

$$\int_\Omega y\varphi\,\mathrm{d}\sigma^0 = \int_\Omega y u^0 \varphi\,\mathrm{d}\mu^0.$$

Since $u^0 \in L^2(\Omega, \mu^0)$, we obtain (9.31) by approximating 1 by a sequence (φ_k) of functions in $C_c^\infty(\Omega)$. □

Proof of Theorem 7.4. In view of Theorem 9.1 the function u^0 belongs to $H^1(\Omega) \cap L^2(\Omega, \mu^0)$. From (9.8) we have

$$\int_\Omega A^0 Du^0 Dy\,\mathrm{d}x + \int_\Omega y\,\mathrm{d}\sigma^0 = \int_\Omega f y\,\mathrm{d}x \quad \forall y \in H_0^1(\Omega) \cap L^2\big(\Omega, \mu^0\big).$$

By Lemma 9.5 this implies (7.11) for $\varepsilon = 0$. □

Proof of Theorem 7.8. Since u^0 belongs to $H^1(\Omega) \cap L^2(\Omega, \mu^0)$ by Theorem 9.1, it is enough to apply Lemmas 9.4 and 9.5. □

10. Problems with more general data

In this section we state and prove global and local convergence and corrector results for relaxed Dirichlet problems of the form (5.9) and (7.11), when the right-hand sides f^ε and f are replaced by more general linear functionals L^ε, and when the strong convergence of (f^ε) in $H^{-1}(\Omega)$ is replaced by the strong convergence of (L^ε) "along the sequence" of spaces $(H_0^1(\Omega) \cap L^2(\Omega, \mu^\varepsilon))'$.

10.1. Strong convergence of the data

For every $\varepsilon \geqslant 0$ we consider an element of the dual space $(H_0^1(\Omega) \cap L^2(\Omega, \mu^\varepsilon))'$, i.e., a linear functional $L^\varepsilon : H_0^1(\Omega) \cap L^2(\Omega, \mu^\varepsilon) \to \mathbf{R}$ such that

$$\left|L^\varepsilon(y)\right| \leqslant C^\varepsilon \left\{ \alpha \int_\Omega |Dy|^2 \,\mathrm{d}x + \int_\Omega |y|^2 \,\mathrm{d}\mu^\varepsilon \right\}^{1/2} \quad \forall y \in H_0^1(\Omega) \cap L^2(\Omega, \mu^\varepsilon),$$

for a suitable constant $C^\varepsilon < +\infty$ (the constant α is introduced in this formula for future convenience). It is easy to prove that each functional L^ε can be represented in the form

$$L^\varepsilon(y) = \langle f^\varepsilon, y \rangle + \int_\Omega g^\varepsilon y \,\mathrm{d}\mu^\varepsilon, \tag{10.1}$$

where $f^\varepsilon \in H^{-1}(\Omega)$ and $g^\varepsilon \in L^2(\Omega, \mu^\varepsilon)$.

In this section we assume that

$$L^\varepsilon \to L^0 \quad \text{strongly along the sequence } \big(H_0^1(\Omega) \cap L^2(\Omega, \mu^\varepsilon)\big)', \tag{10.2}$$

in the sense that

$$\lim_{\varepsilon' \to 0} L^{\varepsilon'}\big(y^{\varepsilon'}\big) = L^0\big(y^0\big), \tag{10.3}$$

for every subsequence ε' of ε (see Section 2) and every sequence $(y^{\varepsilon'})$ which satisfies

$$y^{\varepsilon'} \in H_0^1(\Omega) \cap L^2\big(\Omega, \mu^{\varepsilon'}\big) \quad \forall \varepsilon' > 0, \tag{10.4}$$

$$y^{\varepsilon'} \rightharpoonup y^0 \quad \text{weakly in } H_0^1(\Omega), \tag{10.5}$$

$$\sup_{\varepsilon' > 0} \int_\Omega \big|y^{\varepsilon'}\big|^2 \,\mathrm{d}\mu^{\varepsilon'} < +\infty. \tag{10.6}$$

When (5.1), and (5.18)–(5.24) hold, Theorem 9.1 implies that $y^0 \in L^2(\Omega, \mu^0)$. Since (10.3) holds true for every sequence $(y^{\varepsilon'})$ which satisfies (10.4), (10.5), and (10.6), it is easy to prove by contradiction that there exists a constant $C < +\infty$ such that for every $\varepsilon > 0$

$$\left|L^\varepsilon(y)\right| \leqslant C \left\{ \alpha \int_\Omega |Dy|^2 \,\mathrm{d}x + \int_\Omega |y|^2 \,\mathrm{d}\mu^\varepsilon \right\}^{1/2} \quad \forall y \in H_0^1(\Omega) \cap L^2(\Omega, \mu^\varepsilon). \tag{10.7}$$

When L^ε is represented as in (10.1) with $g^\varepsilon = 0$, it is easy to see that (10.2) is satisfied if (f^ε) converges to f^0 strongly in $H^{-1}(\Omega)$ (this condition is also necessary if all measures μ^ε are zero). The case where the functions g^ε are not identically zero is of course more difficult to handle, since the measures μ^ε vary, and the corresponding spaces $L^2(\Omega, \mu^\varepsilon)$ may be different for different values of ε. This leads in a natural way to definition (10.2), where we used the word "strongly" since the test functions $y^{\varepsilon'}$ in (10.3) are only assumed to be uniformly bounded in the corresponding spaces $H_0^1(\Omega) \cap L^2(\Omega, \mu^\varepsilon)$.

In this definition the presence in (10.3) of subsequences ε' (and not just of the whole sequence ε) is due, among other reasons, to the fact that we want that the convergence of (L^ε) implies the convergence of any subsequence.

10.2. Global convergence and corrector results

By the Lax–Milgram lemma for every $\varepsilon \geqslant 0$ there exists a unique solution u^ε to the problem

$$\begin{cases} u^\varepsilon \in H_0^1(\Omega) \cap L^2(\Omega, \mu^\varepsilon), \\ \displaystyle\int_\Omega A^\varepsilon Du^\varepsilon Dy \, \mathrm{d}x + \int_\Omega u^\varepsilon y \, \mathrm{d}\mu^\varepsilon = L^\varepsilon(y) \quad \forall y \in H_0^1(\Omega) \cap L^2(\Omega, \mu^\varepsilon). \end{cases} \tag{10.8}$$

The following theorem is a generalization of Theorem 5.4.

Theorem 10.1. *Assume* (5.1), (5.18)–(5.24), *and* (10.2). *For every $\varepsilon \geqslant 0$, let u^ε be the unique solution to problem* (10.8). *Then (u^ε) converges to u^0 weakly in $H_0^1(\Omega)$.*

Proof. By (3.1), (3.2), and (10.7), using $y = u^\varepsilon$ as test function in (10.8) we obtain the estimate

$$\alpha \int_\Omega |Du^\varepsilon|^2 \, \mathrm{d}x + \int_\Omega |u^\varepsilon|^2 \, \mathrm{d}\mu^\varepsilon \leqslant C^2. \tag{10.9}$$

Extracting a subsequence, we may assume that

$$u^\varepsilon \rightharpoonup u \quad \text{weakly in } H_0^1(\Omega), \tag{10.10}$$

for some function $u \in H_0^1(\Omega)$. By Theorem 9.1 we have $u \in L^2(\Omega, \mu^0)$. We will prove that $u = u^0$. Since the limit does not depend on the subsequence, this will prove that the whole sequence (u^ε) converges to u^0.

If $y \in H_0^1(\Omega) \cap L^2(\Omega, \mu^0)$ satisfies $\int_\Omega fy \, \mathrm{d}x = 0$ for every $f \in L^\infty(\Omega)$, then $y = 0$ a.e. in Ω. By the Hahn–Banach theorem, this implies that $L^\infty(\Omega)$ is dense in the dual space of $H_0^1(\Omega) \cap L^2(\Omega, \mu^0)$. Therefore, given $\eta > 0$, there exists $f_\eta \in L^\infty(\Omega)$ such that

$$\left| L^0(y) - \int_\Omega f_\eta y \, \mathrm{d}x \right| \leqslant \eta \left\{ \alpha \int_\Omega |Dy|^2 \, \mathrm{d}x + \int_\Omega |y|^2 \, \mathrm{d}\mu^0 \right\}^{1/2} \quad \forall y \in H_0^1(\Omega) \cap L^2\big(\Omega, \mu^0\big). \tag{10.11}$$

For every $\varepsilon \geqslant 0$ let u_η^ε be the unique solution to problem (5.9) with $f^\varepsilon = f_\eta$. By Theorem 5.4 we have

$$u_\eta^\varepsilon \rightharpoonup u_\eta^0 \quad \text{weakly in } H_0^1(\Omega), \tag{10.12}$$

and taking $y = u_\eta^\varepsilon$ as test function in (5.9), with $f^\varepsilon = f_\eta$, we obtain

$$\sup_{\varepsilon > 0} \int_\Omega |u_\eta^\varepsilon|^2 \, \mathrm{d}\mu^\varepsilon < +\infty. \tag{10.13}$$

Using $y = u^\varepsilon - u_\eta^\varepsilon$ as test function in (10.8) and (5.9), with $f^\varepsilon = f_\eta$, we obtain by difference

$$\alpha \int_\Omega \big|D(u^\varepsilon - u_\eta^\varepsilon)\big|^2 \, \mathrm{d}x + \int_\Omega |u^\varepsilon - u_\eta^\varepsilon|^2 \, \mathrm{d}\mu^\varepsilon \leqslant L^\varepsilon(u^\varepsilon - u_\eta^\varepsilon) - \int_\Omega f_\eta(u^\varepsilon - u_\eta^\varepsilon) \, \mathrm{d}x, \tag{10.14}$$

for every $\varepsilon \geqslant 0$. By (10.3), (10.9), (10.10), (10.12), and (10.13) we have

$$\lim_{\varepsilon \to 0} \left\{ L^\varepsilon(u^\varepsilon - u_\eta^\varepsilon) - \int_\Omega f_\eta(u^\varepsilon - u_\eta^\varepsilon) \, \mathrm{d}x \right\} = L^0\big(u - u_\eta^0\big) - \int_\Omega f_\eta\big(u - u_\eta^0\big) \, \mathrm{d}x. \tag{10.15}$$

Let $\hat{\mu}^0$ be the measure defined in the proof of Theorem 9.1. By (9.3) we have

$$\alpha \int_\Omega \left|D\left(u - u_\eta^0\right)\right|^2 \mathrm{d}x + \int_\Omega \left|u - u_\eta^0\right|^2 \mathrm{d}\hat{\mu}^0 \leqslant \liminf_{\varepsilon\to 0}\left\{\alpha \int_\Omega \left|D(u^\varepsilon - u_\eta^\varepsilon)\right|^2 \mathrm{d}x + \int_\Omega |u^\varepsilon - u_\eta^\varepsilon|^2 \mathrm{d}\mu^\varepsilon\right\}. \tag{10.16}$$

From (9.6), (10.11), (10.14), (10.15), and (10.16) we obtain that

$$\alpha \int_\Omega \left|D\left(u - u_\eta^0\right)\right|^2 \mathrm{d}x + \int_\Omega \left|u - u_\eta^0\right|^2 \mathrm{d}\mu^0 \leqslant \frac{\beta^4}{\alpha^4}\eta^2. \tag{10.17}$$

Using (10.14) for $\varepsilon = 0$, we obtain from (10.11)

$$\alpha \int_\Omega \left|D\left(u^0 - u_\eta^0\right)\right|^2 \mathrm{d}x + \int_\Omega \left|u^0 - u_\eta^0\right|^2 \mathrm{d}\mu^0 \leqslant \eta^2. \tag{10.18}$$

From (10.17) and (10.18) we get

$$\alpha \int_\Omega \left|D\left(u - u^0\right)\right|^2 \mathrm{d}x \leqslant 4\frac{\beta^4}{\alpha^4}\eta^2.$$

Since $\eta > 0$ is arbitrary, we conclude that $u = u^0$. □

The next theorem is a generalization of Theorem 7.2.

Theorem 10.2. *Assume* (5.1), (5.18)–(5.24), (7.1), (7.2), *and* (10.2). *Let* $\delta > 0$ *and let* ψ_δ *be a function in* $H^2(\Omega) \cap W^{1,\infty}(\Omega)$ *which satisfies* (7.3). *Assume that the functions* v_δ^ε *defined by* (7.4) *belong to* $H_0^1(\Omega)$. *Then the solutions* u^ε *to the problems* (10.8) *satisfy the estimate*

$$\limsup_{\varepsilon\to 0}\left\{\alpha \int_\Omega |Du^\varepsilon - Dv_\delta^\varepsilon|^2 \mathrm{d}x + \int_\Omega |u^\varepsilon - v_\delta^\varepsilon|^2 \mathrm{d}\mu^\varepsilon\right\} < \delta. \tag{10.19}$$

Proof. Let us fix $\delta' < \delta$ such that

$$\beta \int_\Omega \left|Du^0 - D\left(\psi_\delta\omega^0\right)\right|^2 \mathrm{d}x + \int_\Omega \left|u^0 - \psi_\delta\omega^0\right|^2 \mathrm{d}\mu^0 < \delta'. \tag{10.20}$$

For $\eta > 0$, let f_η, u_η^ε, and u_η^0 be as in the proof of Theorem 10.1. Using (10.20) and (10.18), we fix $\eta > 0$ small enough such that

$$\sqrt{\delta'} + \frac{\beta}{\alpha}\eta < \sqrt{\delta}, \tag{10.21}$$

$$\beta \int_\Omega \left|Du_\eta^0 - D\left(\psi_\delta\omega^0\right)\right|^2 \mathrm{d}x + \int_\Omega \left|u_\eta^0 - \psi_\delta\omega^0\right|^2 \mathrm{d}\mu^0 < \delta'. \tag{10.22}$$

Therefore we can apply Theorem 7.2 with $f = f_\eta$ and we obtain

$$\limsup_{\varepsilon\to 0}\left\{\alpha \int_\Omega |Du_\eta^\varepsilon - Dv_\delta^\varepsilon|^2 \mathrm{d}x + \int_\Omega |u_\eta^\varepsilon - v_\delta^\varepsilon|^2 \mathrm{d}\mu^\varepsilon\right\} < \delta'. \tag{10.23}$$

As (u^ε) converges to u^0 weakly in $H_0^1(\Omega)$ by Theorem 10.1, using (10.11), (10.14), (10.15), and (10.17) we deduce that

$$\limsup_{\varepsilon\to 0}\left\{\alpha\int_\Omega |Du^\varepsilon - Du^\varepsilon_\eta|^2\,\mathrm{d}x + \int_\Omega |u^\varepsilon - u^\varepsilon_\eta|^2\,\mathrm{d}\mu^\varepsilon\right\} \leqslant \frac{\beta^2}{\alpha^2}\eta^2. \tag{10.24}$$

From (10.21), (10.23), and (10.24) we obtain (10.19). □

10.3. Local convergence and corrector results

We consider now the case where the functions u^ε are solutions to the problems

$$\begin{cases} u^\varepsilon \in H^1(\Omega)\cap L^2(\Omega,\mu^\varepsilon), \\ \displaystyle\int_\Omega A^\varepsilon Du^\varepsilon Dy\,\mathrm{d}x + \int_\Omega u^\varepsilon y\,\mathrm{d}\mu^\varepsilon = L^\varepsilon(y) \quad \forall y\in H_0^1(\Omega)\cap L^2(\Omega,\mu^\varepsilon), \end{cases} \tag{10.25}$$

but are not required to satisfy the boundary condition $u^\varepsilon = 0$ on $\partial\Omega$.

The next theorem is a generalization of Corollary 7.6.

Theorem 10.3. *Assume* (5.1), (5.18)–(5.24), (7.1), (7.2), *and* (10.2). *For every $\varepsilon > 0$, let u^ε be a solution to problem* (10.25). *Assume that*

$$u^\varepsilon \rightharpoonup u^0 \quad \text{weakly in } H^1(\Omega), \tag{10.26}$$

for some function $u^0\in H^1(\Omega)$. Then u^0 is a solution to the problem

$$\begin{cases} u^0 \in H^1(\Omega)\cap L^2_{\mathrm{loc}}\big(\Omega,\mu^0\big), \\ \displaystyle\int_\Omega A^0 Du^0 Dy\,\mathrm{d}x + \int_\Omega u^0 y\,\mathrm{d}\mu^0 = L^0(y) \quad \forall y\in H_c^1(\Omega)\cap L^2\big(\Omega,\mu^0\big), \end{cases} \tag{10.27}$$

where $H_c^1(\Omega)$ denotes the space of all functions $u\in H^1(\Omega)$ with compact support in Ω. If, in addition, $u^0\in L^2(\Omega,\mu^0)$, then the last line in (10.27) *holds for every $y\in H_0^1(\Omega)\cap L^2(\Omega,\mu^0)$.*

Proof. As we have seen in the proof of Theorem 10.1, for every $\eta > 0$ there exists $f_\eta\in L^\infty(\Omega)$ which satisfies (10.11). Let us fix an open set $U \Subset \Omega$ and let $\varphi\in C_c^\infty(\Omega)$ such that $\varphi = 1$ on U. Using $y = u^\varepsilon\varphi^2$ as test function in (10.25), and then (3.1), (3.2), (10.7), and (10.26) we obtain

$$\sup_{\varepsilon>0}\int_U |u^\varepsilon|^2\,\mathrm{d}\mu^\varepsilon < +\infty, \tag{10.28}$$

which implies that $u^0\in L^2(U,\mu^0)$ by Theorem 9.1. For every $\varepsilon\geqslant 0$ let u^ε_η be the unique solution to the problem

$$\begin{cases} u^\varepsilon_\eta - u^\varepsilon \in H_0^1(U)\cap L^2(U,\mu^\varepsilon), \\ \displaystyle\int_U A^\varepsilon Du^\varepsilon_\eta Dy\,\mathrm{d}x + \int_U u^\varepsilon_\eta y\,\mathrm{d}\mu^\varepsilon = \int_U f_\eta y\,\mathrm{d}x \quad \forall y\in H_0^1(U)\cap L^2(U,\mu^\varepsilon). \end{cases} \tag{10.29}$$

Taking $y = u^\varepsilon_\eta - u^\varepsilon$ (extended by 0 on $\Omega\cap\overline{U}$) as test function in (10.29) and (10.25), we obtain by difference

$$\alpha\int_U \big|D(u^\varepsilon - u^\varepsilon_\eta)\big|^2\,\mathrm{d}x + \int_U |u^\varepsilon - u^\varepsilon_\eta|^2\,\mathrm{d}\mu^\varepsilon \leqslant L^\varepsilon(u^\varepsilon - u^\varepsilon_\eta) - \int_U f_\eta(u^\varepsilon - u^\varepsilon_\eta)\,\mathrm{d}x, \tag{10.30}$$

for every $\varepsilon \geqslant 0$. By (10.7) this implies that $(u^\varepsilon - u^\varepsilon_\eta)$ is bounded in $H^1_0(U)$ and that the integrals $\int_U |u^\varepsilon - u^\varepsilon_\eta|^2 \,\mathrm{d}\mu^\varepsilon$ are bounded. Using (10.26) and (10.28), we conclude that (u^ε_η) is bounded in $H^1(U)$ and

$$\sup_{\varepsilon>0} \int_U |u^\varepsilon_\eta|^2 \,\mathrm{d}\mu^\varepsilon < +\infty. \tag{10.31}$$

Extracting a subsequence, we may assume that

$$u^\varepsilon_\eta \rightharpoonup u \quad \text{weakly in } H^1(U), \tag{10.32}$$

for some function $u \in H^1(U)$ with $u - u^0 \in H^1_0(U)$. By (10.31) and by Theorem 9.1 the function u belongs to $L^2(U, \mu^0)$. Using both assertions of Corollary 7.6, u is a solution to the problem

$$\begin{cases} u \in H^1(U) \cap L^2\big(U, \mu^0\big), \\ \displaystyle\int_U A^0 Du Dy \,\mathrm{d}x + \int_U uy \,\mathrm{d}\mu^0 = \int_U f_\eta y \,\mathrm{d}x \quad \forall y \in H^1_0(U) \cap L^2\big(U, \mu^0\big). \end{cases}$$

Since $u - u^0 \in H^1_0(U) \cap L^2(U, \mu^0)$, by uniqueness, we have $u = u^0_\eta$.

By (10.2), (10.26), (10.28), (10.31), and (10.32) we have

$$\lim_{\varepsilon\to 0} \left\{ L^\varepsilon(u^\varepsilon - u^\varepsilon_\eta) - \int_U f_\eta (u^\varepsilon - u^\varepsilon_\eta) \,\mathrm{d}x \right\} = L^0\big(u^0 - u^0_\eta\big) - \int_U f_\eta\big(u^0 - u^0_\eta\big) \,\mathrm{d}x. \tag{10.33}$$

Let $\hat\mu^0$ be the measure defined in the proof of Theorem 9.1. By (9.3) we have

$$\begin{cases} \alpha \displaystyle\int_U \big|D\big(u^0 - u^0_\eta\big)\big|^2 \,\mathrm{d}x + \int_U \big|u^0 - u^0_\eta\big|^2 \,\mathrm{d}\hat\mu^0 \\ \quad \leqslant \liminf\limits_{\varepsilon\to 0} \left\{ \alpha \displaystyle\int_U \big|D(u^\varepsilon - u^\varepsilon_\eta)\big|^2 \,\mathrm{d}x + \int_U |u^\varepsilon - u^\varepsilon_\eta|^2 \,\mathrm{d}\mu^\varepsilon \right\}. \end{cases} \tag{10.34}$$

From (9.6), (10.11), (10.30), (10.33), and (10.34) we obtain that

$$\alpha \int_U \big|D\big(u^0 - u^0_\eta\big)\big|^2 \,\mathrm{d}x + \int_U \big|u^0 - u^0_\eta\big|^2 \,\mathrm{d}\mu^0 \leqslant \frac{\beta^4}{\alpha^4} \eta^2. \tag{10.35}$$

Since, by (10.11), f_η converges to L^0 in the dual space of $H^1_0(\Omega) \cap L^2(\Omega, \mu^0)$ as η tends to 0, the solution u^0_η of (10.29) for $\varepsilon = 0$ converges in $H^1_0(U) \cap L^2(U, \mu^0)$, as η tends to 0, to the solution v^0 of the problem

$$\begin{cases} v^0 - u^0 \in H^1(U) \cap L^2\big(U, \mu^0\big), \\ \displaystyle\int_U A^0 Dv^0 Dy \,\mathrm{d}x + \int_U v^0 y \,\mathrm{d}\mu^0 = L^0(y) \quad \forall y \in H^1_0(U) \cap L^2\big(U, \mu^0\big). \end{cases} \tag{10.36}$$

On the other hand, by (10.35), (u^0_η) converges to u^0 in $H^1(U) \cap L^2(U, \mu^0)$ as η tends to 0. We conclude that $u^0 = v^0$ and that u^0 is the unique solution of (10.36). Since this holds for every open set $U \Subset \Omega$, this implies that u^0 is a solution of (10.27).

The final statement of the theorem can be proved as explained before Corollary 7.6. □

The next theorem is a generalization of Theorem 7.7.

Theorem 10.4. *Assume* (5.1), (5.18)–(5.24), (7.1), (7.2), *and* (10.2). *For every $\varepsilon > 0$, let u^ε be a solution to problem (10.25). Assume that*

$$u^\varepsilon \rightharpoonup u^0 \quad \textit{weakly in } H^1(\Omega),$$

for some function $u^0 \in H^1(\Omega)$. Let U be an open set with $U \Subset \Omega$, let $\delta > 0$, let ψ_δ be a function in $H^2(U) \cap W^{1,\infty}(U)$ which satisfies (7.18), *and let v^ε_δ be the functions defined in U by* (7.4). *Then*

$$\limsup_{\varepsilon\to 0}\left\{\alpha\int_V |Du^\varepsilon - Dv^\varepsilon_\delta|^2\,\mathrm{d}x + \int_V |u^\varepsilon - v^\varepsilon_\delta|^2\,\mathrm{d}\mu^\varepsilon\right\} < \delta, \tag{10.37}$$

for every open set $V \Subset U$.

Proof. Let us fix $\delta' < \delta$ such that

$$\beta\int_U \left|Du^0 - D\left(\psi_\delta\omega^0\right)\right|^2\mathrm{d}x + \int_U \left|u^0 - \psi_\delta\omega^0\right|^2\mathrm{d}\mu^0 < \delta'. \tag{10.38}$$

For $\eta > 0$, let f_η, u^ε_η, and u^0_η be as in the proof of Theorem 10.3. Since (u^0_η) converges to u^0 in $H^1(U)\cap L^2(U,\mu^0)$, we fix η small enough such that

$$\sqrt{\delta'} + \frac{\beta}{\alpha}\eta < \sqrt{\delta}, \tag{10.39}$$

$$\beta\int_U \left|Du^0_\eta - D\left(\psi_\delta\omega^0\right)\right|^2\mathrm{d}x + \int_U \left|u^0_\eta - \psi_\delta\omega^0\right|^2\mathrm{d}\mu^0 < \delta'. \tag{10.40}$$

Therefore we can apply Theorem 7.7 with $f = f_\eta$ and we obtain

$$\limsup_{\varepsilon\to 0}\left\{\alpha\int_V |Du^\varepsilon_\eta - Dv^\varepsilon_\delta|^2\,\mathrm{d}x + \int_V |u^\varepsilon_\eta - v^\varepsilon_\delta|^2\,\mathrm{d}\mu^\varepsilon\right\} < \delta', \tag{10.41}$$

for every open set $V \Subset U$. Using (10.11), (10.30), (10.33), and (10.35) we deduce that

$$\limsup_{\varepsilon\to 0}\left\{\alpha\int_U |Du^\varepsilon - Du^\varepsilon_\eta|^2\,\mathrm{d}x + \int_U |u^\varepsilon - u^\varepsilon_\eta|^2\,\mathrm{d}\mu^\varepsilon\right\} \leqslant \frac{\beta^2}{\alpha^2}\eta^2. \tag{10.42}$$

From (10.39), (10.41), and (10.42) we obtain (10.37). □

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