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Self-intersection local time of (α, d, β)-superprocess ☆

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Abstract

The existence of self-intersection local time (SILT), when the time diagonal is intersected, of the (α, d, β)-superprocess is proved for $d/2 < \alpha$ and for a renormalized SILT when $d/(2 + (1+\beta)^{-1}) < \alpha \leqslant d/2$. We also establish Tanaka-like formula for SILT.

Résumé

L'existence de temps locaux d'auto-intersection est établie pour certaines classes de superprocessus, ainsi qu'une formule analogue à celle de Tanaka.

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1. Introduction and statement of results

This paper is devoted to the proof of existence of self-intersection local time (SILT) of (α, d, β)-superprocesses for $0 < \beta < 1$. Let us introduce some notation. Let $\mathcal{B}_b(\mathbb{R}^d)$ (respectively $\mathcal{C}_b(\mathbb{R}^d)$) be the family of all bounded (respectively, bounded continuous) Borel measurable functions on $\mathbb{R}^d$, and $\mathcal{M}_F(\mathbb{R}^d)$ be the set of all finite Borel measures on $\mathbb{R}^d$. The integral of a function f with respect to a measure μ is denoted by $\mu(f)$. If E is a metric space we denote by $D([0, +\infty), E)$ the space of all càdlàg E-valued paths with the Skorohod topology. We will use c to denote a positive and finite constant whose value may vary from place to place. A constant of the form $c(a, b, \ldots)$ means that this constant depends on parameters $a, b, \ldots$.

Let $(\Omega', \mathcal{F}', \mathcal{F}'_\cdot, P')$ be a filtered probability space where the (α, d, β)-superprocess $X = \{X_t\colon t \geqslant 0\}$ is defined. That is, by X we mean a $\mathcal{M}_F(\mathbb{R}^d)$-valued, time homogeneous, strong Markov process with càdlàg sample paths, such

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that for any non-negative function $\varphi \in \mathcal{B}_b(\mathbb{R}^d)$,

$$E\big[\exp\big(-X_t(\varphi)\big)|X_0=\mu\big]=\exp\big(-\mu\big(V_t(\varphi)\big)\big),$$

where $\mu \in \mathcal{M}_F(\mathbb{R}^d)$ and $V_t(\varphi)$ denotes the unique non-negative solution of the following evolution equation

$$v_t = S_t\varphi - \int_0^t S_{t-s}\big((v_s)^{1+\beta}\big)\,\mathrm{d}s, \quad t \geqslant 0.$$

Here $\{S_t\colon t \geqslant 0\}$ denotes the semigroup corresponding to the fractional Laplacian operator Δ_α.

Another way to characterize the (α, d, β)-superprocess X is by means of the following martingale problem:

$$\begin{cases} \text{For all } \varphi \in D(\Delta_\alpha) \text{ (domain of } \Delta_\alpha) \text{ and } \mu \in \mathcal{M}_F(\mathbb{R}^d), \\ X_0 = \mu, \text{ and } M_t(\varphi) = X_t(\varphi) - X_0(\varphi) - \int_0^t X_s(\Delta_\alpha\varphi)\,\mathrm{d}s, \\ \text{is a } \mathcal{F}'_t\text{-martingale.} \end{cases} \tag{1.1}$$

If $\beta = 1$ then $M_\cdot(\varphi)$ is a continuous martingale. In this paper we are interested in the case of $0 < \beta < 1$, and here $M_t(\varphi)$ is a purely discontinuous martingale. This martingale can be expressed as

$$M_t(\varphi) = \int_0^t \int_{\mathbb{R}^d} \varphi(x) M(\mathrm{d}s, \mathrm{d}x), \tag{1.2}$$

where $M(\mathrm{d}s, \mathrm{d}x)$ is a martingale measure, it and the stochastic integral with respect to such martingale measure is defined in [11] (or in Section II.3 of [8]).

The SILT is heuristically defined by

$$\gamma_X(B) = \int_B \int_{\mathbb{R}^{2d}} \delta(x-y) X_s(\mathrm{d}x) X_t(\mathrm{d}y)\,\mathrm{d}s\,\mathrm{d}t,$$

where $B \subset [0,\infty) \times [0,\infty)$ is a bounded Borel set and δ is the Dirac delta function. Let $D = \{(t,t)\colon t > 0\}$ be the time diagonal on $\mathbb{R}_+ \times \mathbb{R}_+$. For $\beta = 1$, $B \cap D = \varnothing$ and $d \leqslant 7$, Dynkin [6] proved the existence of SILT, γ_X, for a very general class of continuous superprocesses. Also, from the Dynkin's works follows the existence of SILT when $\beta = 1$, $B \cap D \neq \varnothing$ and $d \leqslant 3$ (see [1]). For $\beta = 1$, $d = 4, 5$ and $B \cap D \neq \varnothing$, Rosen [15] proved the existence of a renormalized SILT for the $(\alpha, d, 1)$-superprocess. A Tanaka-like formula for the local time of $(\alpha, d, 2)$-superprocess was established by Adler and Lewin in [3]. The same authors derived a Tanaka-like formula for self-intersection local time for $(\alpha, d, 2)$-superprocess (see [2]). In this paper we are going to extend the above results for the case of $0 < \beta < 1$.

The usual way to give a rigorous definition of SILT is to take a sequence $(\varphi_\varepsilon)_{\varepsilon>0}$ of smooth functions that converges in distribution to δ, define the approximating SILTs

$$\gamma_{X,\varepsilon}(B) = \int_B \int_{\mathbb{R}^{2d}} \varphi_\varepsilon(x-y) X_s(\mathrm{d}x) X_t(\mathrm{d}y)\,\mathrm{d}s\,\mathrm{d}t, \quad B \subset \mathbb{R}_+ \times \mathbb{R}_+,$$

and prove that $(\gamma_{X,\varepsilon}(B))_{\varepsilon>0}$ converges, in some sense (it is usually taken $L^2(P')$, $L^{1+\beta}(P')$, $L^1(P')$, distribution or in probability), to a random variable $\gamma_X(B)$. In what follows we choose $\varphi_\varepsilon = p_\varepsilon$, where p_ε is the α-stable density, given by

$$p_\varepsilon(x,y) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \mathrm{e}^{-\mathrm{i}(z\cdot(x-y)) - \varepsilon|z|^\alpha}\,\mathrm{d}z, \quad x, y \in \mathbb{R}^d,$$

when $0 < \alpha < 2$ and

$$p_\varepsilon(x,y) = \frac{1}{(2\pi\varepsilon)^{d/2}} \mathrm{e}^{-|x-y|^2/2\varepsilon}, \quad x, y \in \mathbb{R}^d,$$

for $\alpha = 2$. In this paper we will consider the particular case when $B = \{(t, s)\colon 0 \leqslant s \leqslant t \leqslant T\}$. Here we denote $\gamma_{X,\varepsilon}(B)$ by $\gamma_{X,\varepsilon}(T)$, that is

$$\gamma_{X,\varepsilon}(T) = \int_0^T \int_0^t \int_{\mathbb{R}^{2d}} p_\varepsilon(x - y) X_s(\mathrm{d}x) X_t(\mathrm{d}y)\, \mathrm{d}s\, \mathrm{d}t, \quad \forall T \geqslant 0.$$

Moreover, we are going to consider the renormalized SILT

$$\tilde{\gamma}_{X,\varepsilon}(T) = \gamma_{X,\varepsilon}(T) - \mathrm{e}^{\lambda\varepsilon} \int_0^T \int_{\mathbb{R}^d} X_s\big(G^{\lambda,\varepsilon}(x - \cdot)\big) X_s(\mathrm{d}x)\, \mathrm{d}s,$$

where

$$G^{\lambda,\varepsilon}(x) = \int_\varepsilon^\infty \mathrm{e}^{-\lambda t} p_t(x)\, \mathrm{d}t, \quad \lambda, \varepsilon \geqslant 0.$$

Notice that $G^{\lambda,\varepsilon}(x) \uparrow G^{\lambda,0}(x) = G^\lambda(x)$ as $\varepsilon \downarrow 0$ for all $x \in \mathbb{R}^d$, and if $d > \alpha$ then

$$G(x) = G^{0,0}(x) = c(\alpha, d)|x|^{\alpha - d}, \tag{1.3}$$

where

$$c(\alpha, d) = \frac{\Gamma((d - \alpha)/2)}{2^{\alpha/2}\pi^{d/2}\Gamma(\alpha/2)},$$

and Γ is the usual Gamma function. G is called Green function of Δ_α. Also notice that, for $\lambda > 0$, we have

$$\Delta_\alpha G^{\lambda,\varepsilon}(x) = \lambda G^{\lambda,\varepsilon}(x) - \mathrm{e}^{-\lambda\varepsilon} p_\varepsilon(x), \quad x \in \mathbb{R}^d. \tag{1.4}$$

Now we are ready to present our main result.

Theorem 1. *Let X be the (α, d, β)-superprocess with initial measure $X_0(\mathrm{d}x) = \mu(\mathrm{d}x) = h(x)\,\mathrm{d}x$, where h is bounded and integrable with respect to Lebesgue measure on $\mathbb{R}^d$. Let M be the martingale measure which appears in the martingale problem* (1.1) *for X.*

(a) *Let $d/2 < \alpha$. Then there exists a process $\gamma_X = \{\gamma_X(T)\colon T \geqslant 0\}$ such that for every $T > 0$, $\delta > 0$*

$$P\Big(\sup_{t \leqslant T}\big|\gamma_{X,\varepsilon}(t) - \gamma_X(t)\big| > \delta\Big) \to 0, \quad \textit{as } \varepsilon \downarrow 0.$$

Moreover, for any $\lambda > 0$,

$$\begin{aligned}\gamma_X(T) = {} & \lambda \int_0^T \int_0^t \int_{\mathbb{R}^{2d}} G^\lambda(x - y) X_s(\mathrm{d}x) X_t(\mathrm{d}y)\, \mathrm{d}t\, \mathrm{d}s - \int_0^T \int_{\mathbb{R}^d} X_T\big(G^\lambda(x - \cdot)\big) X_s(\mathrm{d}x)\, \mathrm{d}s \\ & + \int_0^T \int_{\mathbb{R}^d} X_s\big(G^\lambda(x - \cdot)\big) X_s(\mathrm{d}x)\, \mathrm{d}s + \int_0^T \int_{\mathbb{R}^d} \int_0^t \int_{\mathbb{R}^d} G^\lambda(x - y) M(\mathrm{d}s, \mathrm{d}y) X_t(\mathrm{d}x)\, \mathrm{d}t, \quad \textit{a.s.}\end{aligned} \tag{1.5}$$

(b) *Let $d/(2 + (1 + \beta)^{-1}) < \alpha \leqslant d/2$. Then there exists a process $\tilde{\gamma}_X = \{\tilde{\gamma}_X(T)\colon T \geqslant 0\}$ such that for every $T > 0$, $\delta > 0$*

$$P\Big(\sup_{t \leqslant T}\big|\tilde{\gamma}_{X,\varepsilon}(t) - \tilde{\gamma}_X(t)\big| > \delta\Big) \to 0, \quad \textit{as } \varepsilon \downarrow 0.$$

Moreover, for any $\lambda > 0$,

$$\tilde{\gamma}_X(T)=\lambda\int_0^T\int_0^t\int_{\mathbb{R}^{2d}} G^\lambda(x-y)X_s(\mathrm{d}x)X_t(\mathrm{d}y)\,\mathrm{d}t\,\mathrm{d}s-\int_0^T\int_{\mathbb{R}^d} X_T\big(G^\lambda(x-\cdot)\big)X_s(\mathrm{d}x)\,\mathrm{d}s$$
$$+\int_0^T\int_{\mathbb{R}^d}\int_0^t\int_{\mathbb{R}^d} G^\lambda(x-y)M(\mathrm{d}s,\mathrm{d}y)X_t(\mathrm{d}x)\,\mathrm{d}t,\quad a.s. \tag{1.6}$$

The processes γ_X and $\tilde{\gamma}_X$ are called SILT and renormalized SILT of X, respectively, and (1.5) *and* (1.6) *are called Tanaka-like formula for SILT.*

Remark 1. It is interesting to note that our bound on dimensions

$$d<\big(2+(1+\beta)^{-1}\big)\alpha,$$

for renormalized SILT does not converge, as $\beta\uparrow 1$, to the bound $d<3\alpha$ established by Rosen [15] for finite variance superprocess ($\beta=1$). For example, simple algebra shows that for $5/3<\alpha<2$, there is a SILT for finite variance superprocess in dimensions $d\leqslant 5$. However for any $\beta\in((3\alpha-5)/(5-2\alpha),1)$ we get the existence of SILT only in dimensions $d\leqslant 4$ and this bound not improve to $d\leqslant 5$ if $\beta\uparrow 1$. So, our bound is more restrictive, and we believe that it is related to the fact that (α,d,β)-superprocess (with $\beta<1$) has jumps. Our conjecture is that for $\beta<1$, the renormalized SILT defined by (1.6) does not exist in dimensions greater than $(2+(1+\beta)^{-1})\alpha$.

The common ways to prove the existence of SILT for the finite variance superprocesses (see e.g. [2,15]) do not work here. The reason for this is that such proofs strongly rely on the existence of high moments of X (at least of order four), and (α,d,β)-superprocess X has moments of order less than $1+\beta$. To overcome this difficulty let us consider the path properties of X more carefully. It is well known (see Theorem 6.1.3 of [4]) that, for $0<\beta<1$, the (α,d,β)-superprocess X is a.s. discontinuous and has jumps of the form $\Delta X_t=r\delta_x$, for some $r>0$, $x\in\mathbb{R}^d$. Here δ_x denotes the Dirac measure concentrated at x. Let

$$N_X(\mathrm{d}x,\,\mathrm{d}r,\,\mathrm{d}s)=\sum_{\{(x,r,s):\ \Delta X_s=r\delta_x\}}\delta_{(x,r,s)}, \tag{1.7}$$

be a random point measure on $\mathbb{R}^d\times\mathbb{R}_+\times\mathbb{R}_+$ with compensator measure $\widehat{N}_X$ given by

$$\widehat{N}_X(\mathrm{d}x,\,\mathrm{d}r,\,\mathrm{d}s)=\eta r^{-2-\beta}\,\mathrm{d}r\,X_s(\mathrm{d}x)\,\mathrm{d}s, \tag{1.8}$$

where

$$\eta=\frac{\beta(\beta+1)}{\Gamma(1-\beta)}.$$

Let $K>0$ fix. From [7] and [4] we have that the (α,d,β)-superprocess X has the following decomposition: Let $\varphi\in D(\Delta_\alpha)$, $t\geqslant 0$,

$$X_t(\varphi)=\mu(\varphi)+\int_0^t X_s(\Delta_\alpha\varphi)\,\mathrm{d}s-C_\beta(K)\int_0^t X_s(\varphi)\,\mathrm{d}s+\int_0^t\int_0^K\int_{\mathbb{R}^d} r\varphi(x)\widetilde{N}_X(\mathrm{d}x,\,\mathrm{d}r,\,\mathrm{d}s)$$
$$+\int_0^t\int_K^\infty\int_{\mathbb{R}^d} r\varphi(x)N_X(\mathrm{d}x,\,\mathrm{d}r,\,\mathrm{d}s), \tag{1.9}$$

where $\widetilde{N}_X=N_X-\widehat{N}_X$ is a martingale measure and

$$C_\beta(K)=\frac{\eta}{\beta K^\beta}.$$

As we have mentioned already, one of the problems of working with the (α,d,β)-superprocess X is dealing with "big" jumps. In fact, the "big" jumps produce the infinite variance of the process and they appear in the term corresponding to the integral with respect to N_X on (1.9). So, the first step in the establishing the existence of SILT for (α, d, β)-superprocess X is to "eliminate" those jumps. This is achieved via introducing the following auxiliary process.

Let us considerer the canonical space, $\Omega^\circ = D([0,\infty), M_F(\mathbb{R}^d))$, $\mathcal{F}^\circ = \mathcal{B}(\Omega^\circ)$ and $\mathcal{F}_t^\circ = \sigma\{Y_r^K\colon\ 0 \leqslant r \leqslant t\}$, where $Y_r^K(\omega^\circ) = \omega^\circ(r)$. For any $\mu \in M_F(\mathbb{R}^d)$ there exists (see [4]) a measure Q_μ on $(\Omega^\circ, \mathcal{F}^\circ)$, such that for any non-negative function $\varphi \in \mathcal{B}_b(\mathbb{R}^d)$

$$E_\mu\big[\exp\big(-Y_t^K(\varphi)\big)\big|\mathcal{F}_s^\circ\big] = \exp\big(-Y_s^K\big(V_{t-s}^K(\varphi)\big)\big), \quad \forall 0 < s \leqslant t, \tag{1.10}$$

and

$$E_\mu\big[\exp\big(-Y_t^K(\varphi)\big)\big] = \exp\big(-\mu\big(V_t^K(\varphi)\big)\big), \quad \forall t > 0 \tag{1.11}$$

(notice that the expectation is taken here with respect to the measure Q_μ). V_t^K is the unique non-negative solution for the non-linear equation

$$\begin{cases} \partial v_t^K/\partial t = (\Delta_\alpha - C_\beta(K))v_t^K - \Phi^K(v_t^K), \\ v_0^K = \varphi, \end{cases} \tag{1.12}$$

where

$$\Phi^K(x) = \eta \int_0^K \big(\mathrm{e}^{-ux} - 1 + ux\big) u^{-\beta-2}\,\mathrm{d}u. \tag{1.13}$$

Note that when $K = \infty$ the resulting process Y^∞ and the regular (α, d, β)-superprocess X have the same distribution. Now, for any $K > 0$, define the stopping time

$$\tau_K = \inf\big\{t > 0\colon\ |\Delta X_t| > K\big\}. \tag{1.14}$$

In Section 2 we will show that if we define the process which evolves as X up to time τ_K and then continues to evolve as Y^K starting at $X_{\tau_K -}$, then this process has the same law as Y^K. This together with the fact that $\tau_K \uparrow \infty$ as $K \to \infty$ (see Lemma 2) implies that it is enough to show existence of the SILT for the process Y^K. This task will be accomplished in Section 3, modulo some technical moment estimates that will be derived in Section 4. The main steps leading to the proof of Theorem 1 will be described in the next section.

2. Proof of Theorem 1

The process Y^K whose Laplace transform is given by (1.10), (1.11) has the following decomposition:

$$Y_t^K(\varphi) = \mu(\varphi) + \int_0^t Y_s^K(\Delta_\alpha\varphi)\,\mathrm{d}s - C_\beta(K)\int_0^t Y_s^K(\varphi)\,\mathrm{d}s + \int_0^t\int_0^K\int_{\mathbb{R}^d} r\varphi(x)\widetilde{N}_{Y^K}(\mathrm{d}x, \mathrm{d}r, \mathrm{d}s), \quad \forall t \geqslant 0, \tag{2.1}$$

where $\widetilde{N}_{Y^K} = N_{Y^K} - \widehat{N}_{Y^K}$, and

$$N_{Y^K}(\mathrm{d}x, \mathrm{d}r, \mathrm{d}s) = \sum_{\{(x,r,s)\colon\ \Delta Y_s^K = r\delta_x\}} \delta_{(x,r,s)},$$

$$\widehat{N}_{Y^K}(\mathrm{d}x, \mathrm{d}r, \mathrm{d}s) = \eta 1_{(0,K]}(r) r^{-2-\beta}\,\mathrm{d}r\, Y_s^K(\mathrm{d}x)\,\mathrm{d}s.$$

Note that N_{Y^K} is defined in a way analogous to (1.7), however it does not have jumps "greater" than K.

In the following lemma we are going to construct the probability space where Y^K coincides with X up to the stopping time τ_K.

Lemma 1. *There exists a probability space on which a pair of processes $(\ddot{Y}^K, X)$ is defined and possesses the following properties*:

(a) *$\ddot{Y}^K$ coincides in law with Y^K,*
(b) *$\ddot{Y}_t^K = X_t$, $\forall t < \tau_K$.*

Proof. Define

$$\Omega \equiv \Omega' \times \Omega^\circ, \quad \mathcal{F} \equiv \mathcal{F}' \times \mathcal{F}^\circ, \quad \mathcal{F}_t \equiv \mathcal{F}'_t \times \mathcal{F}^\circ_t,$$

and let

$$\ddot{Y}^K_t(w', w^\circ) \equiv \begin{cases} X_t(w'), & t < \tau_K(w'), \\ w^\circ(t - \tau_K(w')), & t \geqslant \tau_K(w'). \end{cases}$$

Define the measure P on $(\Omega, \mathcal{F})$:

$$P(B \times C) = \int_{\Omega'} 1_B(w') P^{\tau_K(w')}(C) P'(\mathrm{d}w'),$$

where

$$P^{\tau_K(w')}(C) = Q_{X_{\tau_K(\omega')-}}\big(\big\{w^\circ \in \Omega^\circ \colon \ddot{Y}^K(w', w^\circ) \in C\big\}\big). \tag{2.2}$$

Let $\varphi \in \mathrm{Dom}(\Delta_\alpha)$ and $t > 0$. From the definition of $\ddot{Y}^K$ we have

$$\ddot{Y}^K_t(\varphi) = \mu(\varphi) + \int_0^t \ddot{Y}^K_s(\Delta_\alpha \varphi)\,\mathrm{d}s - C_\beta(K) \int_0^t \ddot{Y}^K_s(\varphi)\,\mathrm{d}s + \int_0^t \int_0^K \int_{\mathbb{R}^d} r\varphi(x) \widetilde{N}_{\ddot{Y}^K}(\mathrm{d}x, \mathrm{d}r, \mathrm{d}s), \tag{2.3}$$

where $N_{\ddot{Y}^K}$ is defined by (1.7) for $t < \tau_K$ and

$$N_{\ddot{Y}^K}(\mathrm{d}x, \mathrm{d}r, \mathrm{d}s) = \sum_{\{(x,r,s)\colon \Delta \ddot{Y}^K_s = r\delta_x\}} \delta_{(x,r,s)},$$

for $t \geqslant \tau_K$. Let us check that $\int_0^t \int_0^K \int_{\mathbb{R}^d} r\varphi(x) \widetilde{N}_{\ddot{Y}^K}(\mathrm{d}x, \mathrm{d}r, \mathrm{d}s)$ is an $\mathcal{F}_t$-martingale: For any $t > u$, $B \in \mathcal{F}'_u$, $C \in \mathcal{F}^\circ_u$, we obtain by using the definition (2.2) of $P^{\tau_K(w')}$

$$P\left(1_{B\times C}\left(\int_0^t \int_0^K \int_{\mathbb{R}^d} r\varphi(x) \widetilde{N}_{\ddot{Y}^K}(\mathrm{d}x, \mathrm{d}r, \mathrm{d}s) - \int_0^u \int_0^K \int_{\mathbb{R}^d} r\varphi(x) \widetilde{N}_{\ddot{Y}^K}(\mathrm{d}x, \mathrm{d}r, \mathrm{d}s)\right)\right)$$

$$= \int_B P^{\tau_K(w')}\left(1_C\left(\int_0^t \int_0^K \int_{\mathbb{R}^d} r\varphi(x) \widetilde{N}_{\ddot{Y}^K}(\mathrm{d}x, \mathrm{d}r, \mathrm{d}s) - \int_0^u \int_0^K \int_{\mathbb{R}^d} r\varphi(x) \widetilde{N}_{\ddot{Y}^K}(\mathrm{d}x, \mathrm{d}r, \mathrm{d}s)\right)\right) P'(\mathrm{d}w')$$

$$= \int_B P^{\tau_K(w')}\left(1_C\left(\int_0^t \int_0^K \int_{\mathbb{R}^d} r\varphi(x) \widetilde{N}_{\ddot{Y}^K}(\mathrm{d}x, \mathrm{d}r, \mathrm{d}s) - \int_0^{(t\wedge\tau_K(w'))\vee u} \int_0^K \int_{\mathbb{R}^d} r\varphi(x) \widetilde{N}_{\ddot{Y}^K}(\mathrm{d}x, \mathrm{d}r, \mathrm{d}s)\right)\right) P'(\mathrm{d}w')$$

$$+ \int_B P^{\tau_K(w')}\left(1_C\left(\int_0^{(t\wedge\tau_K(w'))\vee u} \int_0^K \int_{\mathbb{R}^d} r\varphi(x) \widetilde{N}_{\ddot{Y}^K}(\mathrm{d}x, \mathrm{d}r, \mathrm{d}s) - \int_0^u \int_0^K \int_{\mathbb{R}^d} r\varphi(x) \widetilde{N}_{\ddot{Y}^K}(\mathrm{d}x, \mathrm{d}r, \mathrm{d}s)\right)\right) P'(\mathrm{d}w')$$

$$= \int_B P^{\tau_K(w')}\left(Q_{X_{\tau_K(w')-}}\left[1_C \int_{(t\wedge\tau_K(w'))\vee u}^t \int_0^K \int_{\mathbb{R}^d} r\varphi(x) \widetilde{N}_{\ddot{Y}^K}(\mathrm{d}x, \mathrm{d}r, \mathrm{d}s) \,\Big|\, \mathcal{F}^\circ_{t\wedge\tau_K(w'))\vee u}\right]\right) P'(\mathrm{d}w')$$

$$+ \int_B P^{\tau_K(w')}\left(1_C \int_u^{(t\wedge\tau_K(w'))\vee u} \int_0^{K-} \int_{\mathbb{R}^d} r\varphi(x) \widetilde{N}_X(\mathrm{d}x, \mathrm{d}r, \mathrm{d}s)\right) P'(\mathrm{d}w')$$

$$= \int_B P^{\tau_K(w')}\left(1_C Q_{X_{\tau_K(w')-}}\left(\int_{(t\wedge\tau_K(w'))\vee u}^t \int_0^K \int_{\mathbb{R}^d} r\varphi(x) \widetilde{N}_{\ddot{Y}^K}(\mathrm{d}x, \mathrm{d}r, \mathrm{d}s) \,\Big|\, \mathcal{F}^\circ_{(t\wedge\tau_K(w'))\vee u}\right)\right) P'(\mathrm{d}w')$$

$$+\int_B P^{\tau_K(w')}(C)\int_u^{(t\wedge\tau_K(w'))\vee u}\int_0^{K-}\int_{\mathbb{R}^d} r\varphi(x)\widetilde{N}_X(\mathrm{d}x,\,\mathrm{d}r,\,\mathrm{d}s)P'(\mathrm{d}w')$$

$$=\int_B 1_C\big(X_{\cdot\wedge u}(\omega')\big)1_{\{u<\tau_K\}}P'\left(\int_u^{(t\wedge\tau_K(w'))\vee u}\int_0^{K-}\int_{\mathbb{R}^d} r\varphi(x)\widetilde{N}_X(\mathrm{d}x,\,\mathrm{d}r,\,\mathrm{d}s)\middle|\mathcal{F}'_u\right)P'(\mathrm{d}w'),$$

where in the last equality for the first term we have used the fact that for P'-a.s. ω', $\widetilde{N}_{\ddot{Y}^K}$ is a $(Q_{X_{\tau_K(w')-}},\mathcal{F}_t)$-martingale measure on $\mathbb{R}^d\times\mathbb{R}_+\times[\tau_K\wedge t,t]$. As for the second term we have used the simple identity

$$P^{\tau_K(w')}(C)1_{\{u<\tau_K\}}=1_C\big(X_{\cdot\wedge u}(\omega')\big)1_{\{u<\tau_K\}}$$

for any $C\in\mathcal{F}^\circ_u$. Now use the fact that $\widetilde{N}_X$ is a $(P',\mathcal{F}'_t)$-martingale measure to get that

$$P'\left(\int_u^{(t\wedge\tau_K(w'))\vee u}\int_0^{K-}\int_{\mathbb{R}^d} r\varphi(x)\widetilde{N}_X(\mathrm{d}x,\,\mathrm{d}r,\,\mathrm{d}s)\middle|\mathcal{F}^\circ_u\right)=0,$$

and the proof that $\int_0^t\int_0^K\int_{\mathbb{R}^d} r\varphi(x)\widetilde{N}_{\ddot{Y}^K}(\mathrm{d}x,\,\mathrm{d}r,\,\mathrm{d}s)$ is a martingale is complete.

Then, due to the uniqueness of the decomposition ([7], Theorem 7) we conclude that $\ddot{Y}^K$ has the same distribution as Y^K. □

Convention. Based on the above lemma, from now on we will assume that Y^K, X are defined on the same probability space and $Y^K_t=X_t$, $\forall t<\tau_K$.

Now we are going to show that time τ_K can be made greater than any constant T with probability arbitrary close to 1 by taking K sufficiently large.

Lemma 2. *For every $T>0$ and $\varepsilon>0$, there exists $K>0$ such that $P(\tau_K\leqslant T)\leqslant\varepsilon$.*

Proof. Let Z^K_T the number of jumps of height greater than K in $[0,T]\times\mathbb{R}^d$, that is $Z^K_T=N([K,+\infty)\times[0,T]\times\mathbb{R}^d)$. Then there exists (see [13], page 1430) a standard Poisson process A^K_t such that

$$Z^K_T=A^K_{c_\beta K^{-1-\beta}\int_0^T X_s(\mathbb{R}^d)\,\mathrm{d}s},$$

for some positive constant c_β. Then from the Markov inequality we have,

$$\begin{aligned}
P(\tau_K\leqslant T)&=P\big(Z^K_T\geqslant 1\big)\\
&=P\left(A^K_{c_\beta K^{-1-\beta}\int_0^T X_s(\mathbb{R}^d)\,\mathrm{d}s}\geqslant 1\right)\\
&=P\left(A^K_{c_\beta K^{-1-\beta}\int_0^T X_s(\mathbb{R}^d)\,\mathrm{d}s}\geqslant 1,\ \frac{c_\beta}{K^{1+\beta}}\int_0^T X_s\big(\mathbb{R}^d\big)\,\mathrm{d}s\geqslant K^{-1}\right)\\
&\quad+P\left(A^K_{c_\beta K^{-1-\beta}\int_0^T X_s(\mathbb{R}^d)\,\mathrm{d}s}\geqslant 1,\ \frac{c_\beta}{K^{1+\beta}}\int_0^T X_s\big(\mathbb{R}^d\big)\,\mathrm{d}s< K^{-1}\right)\\
&\leqslant P\left(\int_0^T X_s\big(\mathbb{R}^d\big)\geqslant c_\beta^{-1}K^\beta\right)+P\big(A^K_{K^{-1}}\geqslant 1\big)
\end{aligned}$$

$$\leqslant c_\beta K^{-\beta} E_\mu\left[\int_0^T X_s(\mathbb{R}^d)\right] + \left(1 - P\left(A^K_{K^{-1}} = 0\right)\right)$$
$$= c_\beta T \mu(\mathbb{R}^d) K^{-\beta} + \left(1 - \exp(-K^{-1})\right).$$

The result follows, since the right-hand side goes to 0 as $K \to \infty$. □

Now the proof of Theorem 1 relies on the following proposition.

Proposition 1. *Let $K > 0$ and Y^K be the truncated (α, d, β)-superprocess with initial measure $Y_0^K(\mathrm{d}x) = \mu(\mathrm{d}x) = h(x)\,\mathrm{d}x$, where h is bounded and integrable with respect to Lebesgue measure $\mathrm{d}x$ on $\mathbb{R}^d$.*

(a) *For $d/2 < \alpha$ there exists a process $\gamma^K_{Y^K} = \{\gamma^K_{Y^K}(T)\colon T \geqslant 0\}$ such that*

$$\lim_{\varepsilon\downarrow 0} E\Big[\sup_{t<T}\big|\gamma^K_{Y^K,\varepsilon}(t) - \gamma^K_{Y^K}(t)\big|\Big] = 0, \quad \forall T > 0,$$

and for any $\lambda > 0$,

$$\gamma^K_{Y^K}(T) = \left(\lambda - C_\beta(K)\right)\int_0^T\int_0^t\int_{\mathbb{R}^{2d}} G^\lambda(x-y) Y_s^K(\mathrm{d}x) Y_t^K(\mathrm{d}y)\,\mathrm{d}t\,\mathrm{d}s - \int_0^T\int_{\mathbb{R}^d} Y_T^K\left(G^\lambda(x-\cdot)\right) Y_s^K(\mathrm{d}x)$$
$$+ \int_0^T\int_{\mathbb{R}^d} Y_s^K\left(G^\lambda(x-\cdot)\right) Y_s^K(\mathrm{d}x)\,\mathrm{d}s + \int_0^T\int_{\mathbb{R}^d}\int_0^t\int_{\mathbb{R}^d} G^\lambda(x-y) M^K(\mathrm{d}s,\mathrm{d}y) Y_t^K(\mathrm{d}x)\,\mathrm{d}t, \quad \text{a.s.}$$

(b) *For $d/(2 + (1+\beta)^{-1}) < \alpha \leqslant d/2$ there exists a process $\tilde\gamma^K_{Y^K} = \{\tilde\gamma^K_{Y^K}(T)\colon T \geqslant 0\}$ such that*

$$\lim_{\varepsilon\downarrow 0} E\Big[\sup_{t<T}\big|\tilde\gamma^K_{Y^K,\varepsilon}(t) - \tilde\gamma^K_{Y^K}(t)\big|\Big] = 0, \quad \forall T > 0,$$

and for any $\lambda > 0$,

$$\tilde\gamma^K_{Y^K}(T) = \left(\lambda - C_\beta(K)\right)\int_0^T\int_0^t\int_{\mathbb{R}^{2d}} G^\lambda(x-y) Y_s^K(\mathrm{d}x) Y_t^K(\mathrm{d}y)\,\mathrm{d}t\,\mathrm{d}s - \int_0^T\int_{\mathbb{R}^d} Y_T^K\left(G^\lambda(x-\cdot)\right) Y_s^K(\mathrm{d}x)\,\mathrm{d}s$$
$$+ \int_0^T\int_{\mathbb{R}^d}\int_0^t\int_{\mathbb{R}^d} G^\lambda(x-y) M^K(\mathrm{d}s,\mathrm{d}y) Y_t^K(\mathrm{d}x)\,\mathrm{d}t, \quad \text{a.s.}$$

Proof is postponed.
The above proposition immediately yields:

Proof of Theorem 1. Fix arbitrary $\varepsilon, \delta > 0$ and let $d/2 < \alpha$. Since $X_t = Y_t^K$ for any $t < \tau_K$, we immediately get that

$$\gamma_X(t) = \gamma_{Y^K}(t), \quad \forall t < \tau_K,$$

and $\gamma_X(t)$ satisfies Tanaka formula (1.5) for $t < \tau_K$. Moreover, since by Lemma 2, $\tau_K \uparrow \infty$, as $K \to \infty$, there is no problem to define $\gamma_X(t)$ satisfying (1.5) for any $t > 0$.

Now let us check the convergence part of the theorem. For any $T > 0$, by Lemma 2, we can fix $K > 0$ such that $P(\tau_K \leqslant T) \leqslant \delta$. Then

$$\lim_{\varepsilon_1\downarrow 0} P\Big(\sup_{t\leqslant T}\big|\gamma_{X,\varepsilon_1}(t)-\gamma_X(t)\big|>\varepsilon\Big)\leqslant \lim_{\varepsilon_1\downarrow 0} P\Big(\sup_{t\leqslant T}\big|\gamma_{X,\varepsilon_1}(t)-\gamma_X(t)\big|>\varepsilon,\tau_K>T\Big)+P(\tau_K\leqslant T)$$
$$\leqslant \lim_{\varepsilon_1\downarrow 0} P\Big(\sup_{t\leqslant T}\big|\gamma_{Y^K,\varepsilon_1}(t)-\gamma_{Y^K}(t)\big|>\varepsilon,\tau_K>T\Big)+\delta$$
$$=\delta,$$

and since $\delta,\varepsilon>0$ were arbitrary the proof of convergence is complete.

The proof of part (b) of the theorem goes along the same lines. □

3. Existence of SILT for Y^K — proof of Proposition 1

Fix arbitrary $K>0$. First, we derive very useful moment estimates for Y^K. Let $\{S_t^K\colon t\geqslant 0\}$ denote the solution of the partial differential equation

$$\frac{\partial v_t^K}{\partial t}=\big(\Delta_\alpha - C_\beta(K)\big)v_t^K.$$

That is, $\{S_t^K\colon t\geqslant 0\}$ is the semigroup defined as

$$S_t^K=\mathrm{e}^{-C_\beta(K)t}S_t. \tag{3.1}$$

Notice that $S_t^K\varphi\leqslant S_t\varphi$, for all non-negative bounded measurable functions φ.

Following Theorem 3.1 of [9] we have that for $\varphi,\psi\in\mathcal{B}_b(\mathbb{R}^d)$,

$$E\left[\exp\left(-Y_t^K(\varphi)-\int_0^t Y_s^K(\psi)\,\mathrm{d}s\right)\middle| Y_0^K=\mu\right]=\exp\big(-\mu\big(V_t^K(\varphi,\psi)\big)\big), \tag{3.2}$$

where $\mu\in\mathcal{M}_F(\mathbb{R}^d)$ and $V_t^K(\varphi,\psi)$ denotes the unique non-negative solution to the following evolution equation

$$v_t^K=S_t^K\varphi+\int_0^t S_s^K(\psi)\,\mathrm{d}s-\int_0^t S_{t-s}^K\big(\Phi^K\big(v_s^K\big)\big)\,\mathrm{d}s,\quad t\geqslant 0, \tag{3.3}$$

where

$$\Phi^K(x)=\sum_{m=2}^{\infty}\frac{(-1)^m}{m!}\chi(m)x^m \tag{3.4}$$

and

$$\chi(m)=\frac{\eta K^{m-1-\beta}}{m-1-\beta}. \tag{3.5}$$

Now we are going to calculate the first two moments of Y^K.

Lemma 3. *Let φ be a non-negative function on $\mathcal{B}_b(\mathbb{R}^d)$ and $t>0$. Then*

$$E_\mu\big[Y_t^K(\varphi)\big]=\mu\big(S_t^K\varphi\big), \tag{3.6}$$
$$E_\mu\big[\big(Y_t^K(\varphi)\big)^2\big]=\big(\mu\big(S_t^K\varphi\big)\big)^2+\chi(2)\mu\left(\int_0^t S_{t-s}^K\big(\big(S_s^K\varphi\big)^2\big)\,\mathrm{d}s\right).$$

Proof. From (3.2) we have

$$E_\mu\big[\mathrm{e}^{-\lambda Y_t^K(\varphi)}\big]=\mathrm{e}^{-\mu(v_t^K(\lambda))} \tag{3.7}$$

where

$$v_t^K(\lambda) = \lambda S_t^K \varphi - \int_0^t S_{t-s}^K\big(\Phi^K\big(v_s^K(\lambda)\big)\big)\,\mathrm{d}s. \tag{3.8}$$

Using the elementary inequality $\mathrm{e}^{-x} - 1 + x \leqslant x^2/2$, $x \geqslant 0$, and (1.13) we have

$$\Phi^K(x) \leqslant \frac{\chi(2)}{2}x^2, \quad x \geqslant 0.$$

Let $\|\cdot\|_\infty$ be the supremum norm, then $0 \leqslant v_t^K(\lambda) \leqslant \lambda S_t^K\varphi \leqslant \lambda\|\varphi\|_\infty$ and the previous inequality implies

$$\int_0^t S_{t-s}^K\big(\Phi^K\big(v_s^K(\lambda)\big)\big)\,\mathrm{d}s \leqslant \frac{\chi(2)\|\varphi\|_\infty^2 t}{2} \times \lambda^2.$$

Further from (3.8) we get

$$\lim_{\lambda\downarrow 0} \frac{\lambda S_t^K\varphi - v_t^K(\lambda)}{\lambda} = \lim_{\lambda\downarrow 0}\frac{1}{\lambda}\int_0^t S_{t-s}^K\big(\Phi^K\big(v_s^K(\lambda)\big)\big)\,\mathrm{d}s$$
$$\leqslant \lim_{\lambda\downarrow 0}\frac{\chi(2)\|\varphi\|_\infty^2 t}{2}\times\lambda = 0,$$

and we write this like

$$v_t^K(\lambda) = \lambda S_t^K\varphi - \mathrm{o}(\lambda). \tag{3.9}$$

This implies

$$\begin{aligned} E_\mu\big[Y_t^K(\varphi)\big] &= \lim_{\lambda\downarrow 0}\frac{1 - E_\mu[\mathrm{e}^{-\lambda Y_t^K(\varphi)}]}{\lambda} \\ &= \lim_{\lambda\downarrow 0}\frac{1 - \mathrm{e}^{-\lambda\mu(S_t^K\varphi)+\mathrm{o}(\lambda)}}{\lambda} \\ &= \lim_{\lambda\downarrow 0}\frac{1 - \mathrm{e}^{-\lambda\mu(S_t^K\varphi)+\mathrm{o}(\lambda)}}{\lambda\mu(S_t^K\varphi) - \mathrm{o}(\lambda)}\lim_{\lambda\downarrow 0}\frac{\lambda\mu(S_t^K\varphi) - \mathrm{o}(\lambda)}{\lambda} = \mu\big(S_t^K\varphi\big). \end{aligned}$$

Now, to calculate the second moment we follow the ideas used in the proof of Proposition 11 of Chapter II from [10]:

$$\begin{aligned} E_\mu\big[\big(Y_t^K(\varphi)\big)^2\big] &= \lim_{\lambda\downarrow 0}\frac{2}{\lambda^2}E_\mu\big[\mathrm{e}^{-\lambda Y_t^K(\varphi)} - 1 + \lambda Y_t^K(\varphi)\big] \\ &= \lim_{\lambda\downarrow 0}\frac{2}{\lambda^2}\big(\mathrm{e}^{-\mu(v_t^K(\lambda))} - 1 + \lambda\mu\big(S_t^K\varphi\big)\big) \\ &= \lim_{\lambda\downarrow 0}\frac{2}{\lambda^2}\big(\mathrm{e}^{-\lambda\mu(S_t^K\varphi)+\mu(\int_0^t S_{t-s}^K(\Phi^K(v_s^K(\lambda)))\,\mathrm{d}s)} - 1 + \lambda\mu\big(S_t^K\varphi\big)\big) \\ &= \lim_{\lambda\downarrow 0}\frac{2}{\lambda^2}\left(\sum_{n=0}^\infty\frac{1}{n!}\left(\mu\left(\int_0^t S_{t-s}^K\big(\Phi^K\big(v_s^K(\lambda)\big)\big)\,\mathrm{d}s\right) - \lambda\mu\big(S_t^K\varphi\big)\right)^n - 1 + \lambda\mu\big(S_t^K\varphi\big)\right). \end{aligned}$$

Using the series expansion (3.4) for Φ^K and (3.9) we obtain

$$\int_0^t S_{t-s}^K\big(\Phi^K\big(v_s^K(\lambda)\big)\big)\,\mathrm{d}s = \int_0^t S_{t-s}^K\big(\Phi^K\big(\lambda S_s^K\varphi - \mathrm{o}(\lambda)\big)\big)\,\mathrm{d}s$$

$$= \int_0^t S_{t-s}^K \left(\frac{\chi(2)}{2!} \lambda^2 \left(S_s^K \varphi\right)^2 + \mathrm{o}\left(\lambda^2\right) \right) \mathrm{d}s$$

$$= \frac{\chi(2)}{2} \lambda^2 \int_0^t S_{t-s}^K \left(\left(S_s^K \varphi\right)^2 \right) \mathrm{d}s + \mathrm{o}\left(\lambda^2\right).$$

Then

$$E_\mu\left[\left(Y_t^K(\varphi)\right)^2\right] = \lim_{\lambda \downarrow 0} \frac{2}{\lambda^2} \left(\frac{\chi(2)}{2} \lambda^2 \mu\left(\int_0^t S_{t-s}^K\left(\left(S_s^K \varphi\right)^2\right) \mathrm{d}s \right) + \mathrm{o}\left(\lambda^2\right) \right.$$

$$\left. + \frac{1}{2!} \left(\frac{\chi(2)}{2} \lambda^2 \mu\left(\int_0^t S_{t-s}^K\left(\left(S_s^K \varphi\right)^2\right) \mathrm{d}s \right) - \lambda \mu\left(S_t^K \varphi\right) + \mathrm{o}\left(\lambda^2\right) \right)^2 \right)$$

$$= \lim_{\lambda \downarrow 0} \frac{2}{\lambda^2} \left(\frac{\chi(2)}{2} \lambda^2 \mu\left(\int_0^t S_{t-s}^K\left(\left(S_s^K \varphi\right)^2\right) \mathrm{d}s \right) + \frac{1}{2} \lambda^2 \left(\mu\left(S_t^K \varphi\right)\right)^2 + \mathrm{o}\left(\lambda^2\right) \right)$$

$$= \chi(2) \mu\left(\int_0^t S_{t-s}^K\left(\left(S_s^K \varphi\right)^2\right) \mathrm{d}s \right) + \left(\mu\left(S_t^K \varphi\right)\right)^2,$$

and we are done. □

Remark 2. Using binary directed graphs, Dynkin in [6] gives a formula for the moments of superprocesses, where the Laplace functional (3.2) has an evolution equation (3.3) with only one term $m = 2$ in (3.4). For the Y^K superprocess it is also possible, but here the main difference is that the directed graphs are not necessarily binary.

Corollary 1. *Let φ, ψ be non-negative functions on $\mathcal{B}_b(\mathbb{R}^d)$ and $t \geqslant s > 0$. Then*

$$E_\mu\left[Y_t^K(\varphi) Y_s^K(\psi)\right] = \mu\left(S_t^K \varphi\right) \mu\left(S_s^K \psi\right) + \chi(2) \mu\left(\int_0^s S_r^K\left(S_{t-r}^K \varphi S_{s-r}^K \psi\right) \mathrm{d}r \right).$$

Proof. First, use the Markov property for Y^K to get

$$E_\mu\left[Y_t^K(\varphi) | \mathcal{F}_s\right] = Y_s^K\left(S_{t-s}^K \varphi\right).$$

Therefore

$$E_\mu\left[Y_t^K(\varphi) Y_s^K(\psi)\right] = E_\mu\left[Y_s^K\left(S_{t-s}^K \varphi\right) Y_s^K(\psi)\right]$$

$$= \frac{1}{4}\left(\left(E_\mu\left[Y_s^K\left(S_{t-s}^K \varphi\right) + Y_s^K(\psi)\right]\right)^2 - \left(E_\mu\left[Y_s^K\left(S_{t-s}^K \varphi\right) - Y_s^K(\psi)\right]\right)^2 \right),$$

and we are done by Lemma 3. □

Corollary 2. *Let φ be non-negative functions on $\mathcal{B}_b(\mathbb{R}^d \times \mathbb{R}^d)$ and $t \geqslant s > 0$. Then*

$$E_\mu\left[\int_{\mathbb{R}^d \times \mathbb{R}^d} \varphi(x, y) Y_t^K(\mathrm{d}x) Y_s^K(\mathrm{d}y) \right]$$

$$= \int_{\mathbb{R}^{4d}} \mu(\mathrm{d}x_1) \mu(\mathrm{d}x_2) p_t(z_1 - x_1) p_s(z_2 - x_2) \varphi(z_1, z_2) \, \mathrm{d}z_1 \, \mathrm{d}z_2$$

$$+\chi(2)\int_0^s\int_{\mathbb{R}^{4d}}\mu(\mathrm{d}x)p_r(y-x)\,\mathrm{d}y\,p_{t-r}(z_1-y)p_{s-r}(z_2-y)\varphi(z_1,z_2)\,\mathrm{d}z_1\,\mathrm{d}z_2\,\mathrm{d}r.$$

Proof. Use Corollary 1 and approximation of the $\psi(x,y)$ by functions in the form $\sum_i \varphi_i(x)\phi_i(y)$ to derive the result. We leave the details to the reader. □

Next proposition gives bounds on some fractional moments of Y^K and requires much more work than we have done in Lemma 3. Hence its proof will be postponed till Section 4.

Proposition 2. *Let* $1+\beta<p<2$ *and* $0<\varepsilon\leqslant 1$. *If*

$$d<\alpha\left(2+\frac{1}{p}\right),$$

then there exists a constant $c=c(K,p,d,\alpha,\beta)$ *such that*

$$E_\mu\left[\int_{\mathbb{R}^d}Y_t^K\big(p_\varepsilon(\cdot-x)\big)\left(\int_0^t Y_s^K\big(G^\lambda(\cdot\,\cdot-x)\big)\,\mathrm{d}s\right)^p\mathrm{d}x\right]<c(K,p,d,\alpha,\beta).$$

Moreover

$$E\left[\int_{\mathbb{R}^d}\left(\int_0^t Y_s^K\big(G^\lambda(\cdot\,\cdot-x)\big)\,\mathrm{d}s\right)^p Y_t^K(\mathrm{d}x)\right]<\infty. \tag{3.10}$$

Remark 3. When K goes to infinity, then $\chi(2)$ goes to infinity, hence $c(K,p,d,\alpha,\beta)$ goes to infinity and this is because $\chi(2)$ is part of $c(K,p,d,\alpha,\beta)$. The moment in (3.10) is infinity when $K=+\infty$, because the (α,d,β)-superprocess has moments of order less than $1+\beta$ and $p>1+\beta$.

Proof is postponed.

Now we can write the Tanaka-like formula for the approximating SILT of the truncated superprocess Y^K. From Fubini theorem, (1.4), (1.2) and (1.1) (the martingale problem for the truncated superprocess Y^K, [7]) we have

$$\begin{aligned}
\gamma^K_{Y^K,\varepsilon}(T)&=\int_0^T\int_0^t\int_{\mathbb{R}^{2d}}p_\varepsilon(x-y)Y_s^K(\mathrm{d}x)Y_t^K(\mathrm{d}y)\,\mathrm{d}s\,\mathrm{d}t\\
&=\lambda\mathrm{e}^{\lambda\varepsilon}\int_0^T\int_0^t\int_{\mathbb{R}^{2d}}G^{\lambda,\varepsilon}(x-y)Y_s^K(\mathrm{d}x)Y_t^K(\mathrm{d}y)\,\mathrm{d}t\,\mathrm{d}s\\
&\quad-\mathrm{e}^{\lambda\varepsilon}\int_0^T\int_{\mathbb{R}^d}\int_s^T\int_{\mathbb{R}^d}\Delta_\alpha G^{\lambda,\varepsilon}(x-y)Y_t^K(\mathrm{d}y)\,\mathrm{d}t\,Y_s^K(\mathrm{d}x)\,\mathrm{d}s\\
&=\big(\lambda-C_\beta(K)\big)\mathrm{e}^{\lambda\varepsilon}\int_0^T\int_0^t\int_{\mathbb{R}^{2d}}G^{\lambda,\varepsilon}(x-y)Y_s^K(\mathrm{d}x)Y_t^K(\mathrm{d}y)\,\mathrm{d}s\,\mathrm{d}t\\
&\quad-\mathrm{e}^{\lambda\varepsilon}\int_0^T\int_{\mathbb{R}^d}Y_T^K\big(G^{\lambda,\varepsilon}(x-\cdot)\big)Y_s^K(\mathrm{d}x)\,\mathrm{d}s+\mathrm{e}^{\lambda\varepsilon}\int_0^T\int_{\mathbb{R}^d}Y_s^K\big(G^{\lambda,\varepsilon}(x-\cdot)\big)Y_s^K(\mathrm{d}x)\,\mathrm{d}s\\
&\quad+\mathrm{e}^{\lambda\varepsilon}\int_0^T\int_{\mathbb{R}^d}\int_s^T\int_{\mathbb{R}^d}G^{\lambda,\varepsilon}(x-y)M^K(\mathrm{d}t,\mathrm{d}y)Y_s^K(\mathrm{d}x)\,\mathrm{d}s,
\end{aligned}\tag{3.11}$$

and

$$\begin{aligned}
\tilde{\gamma}^K_{Y^K,\varepsilon}(T) &= \gamma^K_{Y^K,\varepsilon}(T) - \mathrm{e}^{\lambda\varepsilon}\int_0^T\int_{\mathbb{R}^d} Y^K_s\big(G^{\lambda,\varepsilon}(x-\cdot)\big)Y^K_s(\mathrm{d}x)\,\mathrm{d}s \\
&= \big(\lambda - C_\beta(K)\big)\mathrm{e}^{\lambda\varepsilon}\int_0^T\int_0^t\int_{\mathbb{R}^{2d}} G^{\lambda,\varepsilon}(x-y)Y^K_s(\mathrm{d}x)Y^K_t(\mathrm{d}y)\,\mathrm{d}s\,\mathrm{d}t - \mathrm{e}^{\lambda\varepsilon}\int_0^T\int_{\mathbb{R}^d} Y^K_T\big(G^{\lambda,\varepsilon}(x-\cdot)\big)Y^K_s(\mathrm{d}x)\,\mathrm{d}s \\
&\quad + \mathrm{e}^{\lambda\varepsilon}\int_0^T\int_{\mathbb{R}^d}\int_s^T\int_{\mathbb{R}^d} G^{\lambda,\varepsilon}(x-y)M^K(\mathrm{d}t,\mathrm{d}y)Y^K_s(\mathrm{d}x)\,\mathrm{d}s. \qquad (3.12)
\end{aligned}$$

Note that stochastic integrals in (3.11) and (3.12) are well defined due to the moment bound given by Proposition 2.

Proof of Proposition 1. We are going to prove Proposition 1 via letting $\varepsilon \to 0$ in (3.11), and checking convergence of all the terms. By Corollary 2 and simple estimates we get

$$\begin{aligned}
&E\left[\int_0^T\int Y^K_s\big(G^{\lambda,\varepsilon}(x-\cdot)\big)Y^K_s(\mathrm{d}x)\,\mathrm{d}s\right] \\
&\leqslant \int_0^T\int_{\mathbb{R}^{4d}} \mu(\mathrm{d}x_1)\mu(\mathrm{d}x_2)p_s(z_1-x_1)p_s(z_2-x_2)G^\lambda(z_1-z_2)\,\mathrm{d}z_1\,\mathrm{d}z_2\,\mathrm{d}s \\
&\quad + \chi(2)\int_0^T\int_0^s\int_{\mathbb{R}^{4d}} \mu(\mathrm{d}x)p_r(y-x)\,\mathrm{d}y\,p_{s-r}(z_1-y)p_{s-r}(z_2-y)G^\lambda(z_1-z_2)\,\mathrm{d}z_1\,\mathrm{d}z_2\,\mathrm{d}r\,\mathrm{d}s \\
&\leqslant \int_0^T\int_{\mathbb{R}^{4d}} \mu(\mathrm{d}x_1)\mu(\mathrm{d}x_2)\int_0^\infty \mathrm{e}^{-\lambda u}p_u(z_1-z_2)p_s(z_1-x_1)p_s(z_2-x_2)\,\mathrm{d}z_1\,\mathrm{d}z_2\,\mathrm{d}u\,\mathrm{d}s \\
&\quad + \chi(2)\int_0^T\int_0^s\int_{\mathbb{R}^{4d}} \mu(\mathrm{d}x)p_r(y-x)\,\mathrm{d}y\int_0^\infty \mathrm{e}^{-\lambda u}p_u(z_1-z_2)p_{s-r}(z_1-y)\,\mathrm{d}z_1\,p_{s-r}(z_2-y)\,\mathrm{d}z_2\,\mathrm{d}u\,\mathrm{d}r\,\mathrm{d}s \\
&\leqslant \int_0^T\int_0^\infty \mathrm{e}^{-\lambda u}\int p_{u+2s}(x_1-x_2)\mu(\mathrm{d}x_1)\mu(\mathrm{d}x_2)\,\mathrm{d}u\,\mathrm{d}s \\
&\quad + \chi(2)\int_0^T\int_0^s\int \mu(\mathrm{d}x)p_r(y-x)\,\mathrm{d}y\int_0^\infty \mathrm{e}^{-\lambda u}p_{u+2s-2r}(0)\,\mathrm{d}u\,\mathrm{d}r\,\mathrm{d}s \\
&\leqslant \|h\|_\infty\mu(1)T\lambda^{-1} + \mu(1)\chi(2)\int_0^T\int_0^s\int_0^\infty \mathrm{e}^{-\lambda u}(u+2s-2r)^{-d/\alpha}\,\mathrm{d}u\,\mathrm{d}r\,\mathrm{d}s, \quad \forall T>0, \qquad (3.13)
\end{aligned}$$

where the last integral is convergent if $d < 2\alpha$. Using (3.13), the bound $G^\lambda \geqslant G^{\lambda,\varepsilon}$ and the monotone convergence theorem to get

$$\lim_{\varepsilon\downarrow 0} E\left[\sup_{t<T}\left|\int_0^t\int Y^K_s\big(G^\lambda(x-\cdot) - G^{\lambda,\varepsilon}(x-\cdot)\big)Y^K_s(\mathrm{d}x)\,\mathrm{d}s\right|\right]$$

$$\leqslant \lim_{\varepsilon\downarrow 0} E\left[\int_0^T\int\int Y_s^K\big(G^\lambda(x-\cdot)-G^{\lambda,\varepsilon}(x-\cdot)\big)Y_s^K(\mathrm{d}x)\,\mathrm{d}s\right]$$
$$=0,\quad \forall T>0. \tag{3.14}$$

In a similar way we can prove that

$$\lim_{\varepsilon\downarrow 0} E\left[\sup_{T<L}\left|\int_0^T\int_0^t\int_{\mathbb{R}^{2d}}\big(G^\lambda(x-y)-G^{\lambda,\varepsilon}(x-y)\big)Y_s^K(\mathrm{d}x)Y_t^K(\mathrm{d}y)\,\mathrm{d}s\,\mathrm{d}t\right|\right]=0, \tag{3.15}$$

and

$$\lim_{\varepsilon\downarrow 0} E\left[\sup_{T<L}\left|\int_0^T\int_{\mathbb{R}^d} Y_T^K\big(G^\lambda(x-\cdot)-G^{\lambda,\varepsilon}(x-\cdot)\big)Y_s^K(\mathrm{d}x)\,\mathrm{d}s\right|\right]=0, \tag{3.16}$$

for all $L>0$ and $d<3\alpha$.

Now let us deal with the stochastic integral

$$\int_0^T\int F^\varepsilon(t,x)M^K(\mathrm{d}t,\,\mathrm{d}x),$$

where

$$F^\varepsilon(t,x)=\int_0^t\int_{\mathbb{R}^d} G^{\lambda,\varepsilon}(x-y)Y_s^K(\mathrm{d}y)\,\mathrm{d}s.$$

This integral is well defined if (see [11])

$$E\left[\left(\sum_{t\in J\cap[0,T]} F^\varepsilon\big(t,\Delta Y_t^K\big)^2\right)^{1/2}\right]<+\infty, \tag{3.17}$$

where J denotes the set of all jump times of X. Let

$$d<\alpha\left(2+\frac{1}{1+\beta}\right), \tag{3.18}$$

hence we can choose $p\in(1+\beta,2)$ such that

$$d<\alpha\left(2+\frac{1}{p}\right). \tag{3.19}$$

Since $p\in(1+\beta,2)$ we can use the Jensen inequality to get

$$\left(\sum_{i\in I}a_i\right)^{p/2}\leqslant\sum_{i\in I}a_i^{p/2},$$

if $a_i\geqslant 0$ for all $i\in I$. This yields

$$E\left[\left(\sum_{t\in J\cap[0,T]} F^\varepsilon\big(t,\Delta Y_t^K\big)^2\right)^{1/2}\right]\leqslant\left(E\left[\sum_{t\in J\cap[0,T]} F^\varepsilon\big(t,\Delta Y_t^K\big)^p\right]\right)^{1/p}$$
$$=\left(E\left[\eta\int_0^T\int\int_0^K u^{-\beta-2}\big(F^\varepsilon(t,u\delta_x)\big)^p\,\mathrm{d}u\,Y_t^K(\mathrm{d}x)\,\mathrm{d}t\right]\right)^{1/p}$$
$$=c\left(\int_0^T E\left[\int\left(\int_0^t\int G^{\lambda,\varepsilon}(x-y)Y_s^K(\mathrm{d}y)\,\mathrm{d}s\right)^p Y_t^K(\mathrm{d}x)\right]\mathrm{d}t\right)^{1/p}.$$

Since p satisfies (3.19), the condition (3.17) follows from Proposition 2.

Let $F = F^0$. By Burkholder–Davis–Gundy inequality (see [11]) and the previous argument we get

$$
\begin{aligned}
&E\left[\sup_{t\leqslant T}\left|\int_0^t\int \big(F(s,x)-F^\varepsilon(s,x)\big)M^K(\mathrm{d}s,\,\mathrm{d}x)\right|\right]\\
&\leqslant E\left[\sup_{t\leqslant T}\int_0^t\int \big(\big|F(s,x)-F^\varepsilon(s,x)\big|\big)M^K(\mathrm{d}s,\,\mathrm{d}x)\right]\\
&\leqslant cE\left[\left(\sum_{t\in J\cap[0,T]}\big((F-F^\varepsilon)\big(t,\Delta Y_t^K\big)\big)^2\right)^{1/2}\right]\\
&\leqslant c\left(\int_0^T E\left[\int\left(\int_0^t\int\big|\big(G^\lambda-G^{\lambda,\varepsilon}\big)(x-y)\big|Y_s^K(\mathrm{d}y)\,\mathrm{d}s\right)^p Y_t^K(\mathrm{d}x)\right]\mathrm{d}t\right)^{1/p}\\
&\to 0,\quad \text{as } \varepsilon\downarrow 0,\ \forall T>0,
\end{aligned}
\tag{3.20}
$$

where the last convergence follows by Proposition 2 and the monotone convergence theorem. Now combine (3.14)–(3.16), (3.18) and (3.20) to get that all the terms in (3.11) converge and the proof of part (a) is complete. By (3.15), (3.16), (3.18) and (3.20) we get that all the terms in (3.12) converge and hence the part (b) of the proposition follows. □

4. Proof of Proposition 2: estimation of fractional moments

In what follows we will use the following well known equalities. For $p \in (1,2)$

$$
z^{p-1} = \eta_p \int_0^\infty \big(1-\mathrm{e}^{-\lambda z}\big)\lambda^{-p}\,\mathrm{d}\lambda \tag{4.1}
$$

and

$$
z^{p} = p\eta_p \int_0^\infty \big(\mathrm{e}^{-\lambda z}-1+\lambda z\big)\lambda^{-p-1}\,\mathrm{d}\lambda \tag{4.2}
$$

where

$$
\eta_p = \frac{p-1}{\Gamma(2-p)}.
$$

Proposition 3. *Let $1 < 1+\beta < p < p' < 2$. Then there exists a constant $c = c(K,\beta,p,p')$ such that for any non-negative functions $\varphi, \psi \in \mathcal{B}_b(\mathbb{R}^d)$,*

$$
\begin{aligned}
&E\left[Y_t^K(\varphi)\left(\int_0^t Y_s^K(\psi)\,\mathrm{d}s\right)^p\middle|\,Y_0^K=\mu\right]\\
&\leqslant c\left\{\mu\big(S_t^K\varphi\big)+\mu\big(S_t^K\varphi\big)\left(\mu\left(\int_0^t S_s^K\psi\,\mathrm{d}s\right)\right)^p+\mu\big(S_t^K\varphi\big)\mu\left(\int_0^t S_{t-s}^K\left(\left(\int_0^s S_r^K\psi\,\mathrm{d}r\right)^{p'}\right)\mathrm{d}s\right)\right.\\
&\quad+\mu\left(\int_0^t S_{t-s}^K\left(S_s^K\varphi\int_0^s S_r^K\psi\,\mathrm{d}r\right)\mathrm{d}s\right)+\mu\left(\int_0^t S_{t-s}^K\left(S_s^K\varphi\int_0^s S_r^K\psi\,\mathrm{d}r\right)\mathrm{d}s\right)\left(\mu\left(\int_0^t S_s^K\psi\,\mathrm{d}s\right)\right)^{p-1}
\end{aligned}
$$

$$+\mu\left(\int_0^t S_{t-s}^K\left(\int_0^s S_{s-r}^K\left(S_r^K\varphi\int_0^r S_u^K\psi\,\mathrm{d}u\right)\mathrm{d}r\left(\int_0^s S_r^K\psi\,\mathrm{d}r\right)^{p-1}\right)\mathrm{d}s\right)$$
$$+\mu\left(\int_0^t S_{t-s}^K\left(S_s^K\varphi\left(\int_0^s S_r^K\psi\,\mathrm{d}r\right)^p\right)\mathrm{d}s\right)$$
$$+\mu\left(\int_0^t S_{t-s}^K\left(S_s^K\varphi\int_0^s S_{s-r}^K\left(\left(\int_0^r S_u^K\psi du\right)^{p'}\right)\mathrm{d}r\right)\mathrm{d}s\right)\Bigg\},\quad \forall t>0. \tag{4.3}$$

Proof. Fix an arbitrary $t>0$. By (4.2) we obtain

$$E_\mu\left[Y_t^K(\varphi)\left(\int_0^t Y_s^K(\psi)\,\mathrm{d}s\right)^p\right]=p\eta_p\int_0^\infty\lambda^{-p-1}\Bigg(E\left[Y_t^K(\varphi)\,\mathrm{e}^{-\lambda\int_0^t Y_s^K(\psi)\,\mathrm{d}s}\right]$$
$$-E\left[Y_t^K(\varphi)\right]+\lambda E\left[Y_t^K(\varphi)\int_0^t Y_s^K(\psi)\,\mathrm{d}s\right]\Bigg)\mathrm{d}\lambda. \tag{4.4}$$

Now we will bound the moments on the right-hand side of the above expression. First of all, by Corollary 1, we have

$$E_\mu\left[Y_t^K(\varphi)\int_0^t Y_s^K(\psi)\,\mathrm{d}s\right]=\mu\left(S_t^K\varphi\right)\mu\left(\int_0^t S_s^K\psi\,\mathrm{d}s\right)+\chi(2)\mu\left(\int_0^t\int_0^s S_r^K\left(S_{t-r}^K\varphi S_{s-r}^K\psi\right)\mathrm{d}r\,\mathrm{d}s\right). \tag{4.5}$$

Moreover, from Fubini theorem we get the following useful equality

$$\int_0^t\int_0^s S_r^K\left(S_{t-r}^K\varphi S_{s-r}^K\psi\right)\mathrm{d}r\,\mathrm{d}s=\int_0^t S_{t-r}^K\left(S_r^K\varphi\int_0^r S_s^K\psi\,\mathrm{d}s\right)\mathrm{d}r. \tag{4.6}$$

Now let us estimate the remaining moment. Use the Laplace transform (3.2) and the dominated convergence theorem to obtain

$$E_\mu\left[Y_t^K(\varphi)\,\mathrm{e}^{-\lambda\int_0^t Y_s^K(\psi)\,\mathrm{d}s}\right]=-\lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}E_\mu\left[\mathrm{e}^{-\lambda\int_0^t Y_s^K(\psi)\,\mathrm{d}s-\varepsilon Y_t^K(\varphi)}-\mathrm{e}^{-\lambda\int_0^t Y_s^K(\psi)\,\mathrm{d}s}\right]$$
$$=-\lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\left(\mathrm{e}^{-\mu(V_t^K(\varepsilon\varphi,\lambda\psi))}-\mathrm{e}^{-\mu(V_t^K(0,\lambda\psi))}\right).$$

From (3.2), we can easily derive that

$$V_s^N(\varphi,\psi)\geqslant V_s^N(0,\psi)\geqslant 0,\quad \forall s\geqslant 0, \tag{4.7}$$

and hence by the dominated convergence theorem, we get

$$V_t^K(\varepsilon\varphi,\lambda\psi)\to V_t^K(0,\lambda\psi),\quad \text{as } \varepsilon\downarrow 0.$$

Using the same argument we get

$$E\left[Y_t^K(\varphi)\,\mathrm{e}^{-\lambda\int_0^t Y_s^K(\psi)\,\mathrm{d}s}\right]=\mathrm{e}^{-\mu(V_t^K(0,\lambda\psi))}\mu\left(\lim_{\varepsilon\downarrow 0}\frac{V_t^K(\varepsilon\varphi,\lambda\psi)-V_t^K(0,\lambda\psi)}{\varepsilon}\right).$$

Following the argument in Section 6.3 of [5] we can show that $U_t^K(\varphi,\lambda\psi)$ defined by

$$U_t^K(\varphi,\lambda\psi)=\lim_{\varepsilon\downarrow 0}\frac{V_t^K(\varepsilon\varphi,\lambda\psi)-V_t^K(0,\lambda\psi)}{\varepsilon}$$

satisfies the following equation,

$$U_t^K(\varphi,\lambda\psi)=S_t^K\varphi-\int_0^t S_{t-s}^K\big(U_s^K(\varphi,\lambda\psi)\big(\Phi^K\big)'\big(V_s^K(0,\lambda\psi)\big)\big)\,\mathrm{d}s, \tag{4.8}$$

with $\Phi'(x)=\mathrm{d}\Phi(x)/\mathrm{d}x$.

Use (3.6), (4.4)–(4.6) and (4.8) to get

$$\begin{aligned}
&E\left[Y_t^K(\varphi)\left(\int_0^t Y_s^K(\psi)\,\mathrm{d}s\right)^p\right]\\
&=p\eta_p\int_0^\infty\left(\mathrm{e}^{-\mu(V_t^K(0,\lambda\psi))}\mu\big(U_t^K(\varphi,\lambda\psi)\big)+\lambda\mu\big(S_t^K\varphi\big)\mu\left(\int_0^t S_s^K\psi\,\mathrm{d}s\right)\right.\\
&\quad\left.-\mu\big(S_t^K\varphi\big)+\lambda\chi(2)\mu\left(\int_0^t S_{t-s}^K\left(S_s^K\varphi\int_0^s S_r^K\psi\,\mathrm{d}r\right)\mathrm{d}s\right)\right)\lambda^{-p-1}\,\mathrm{d}\lambda\\
&=p\eta_p\mu\left(\int_0^\infty\left(S_t^K\varphi\mathrm{e}^{-\lambda\mu(\int_0^t S_s^K\psi\,\mathrm{d}s)}-S_t^K\varphi+\lambda S_t^K\varphi\mu\left(\int_0^t S_s^K\psi\,\mathrm{d}s\right)\right.\right.\\
&\quad+S_t^K\varphi\mathrm{e}^{-\mu(V_t^K(0,\lambda\psi))}-S_t^K\varphi\,\mathrm{e}^{-\lambda\mu(\int_0^t S_s^K\psi\,\mathrm{d}s)}+\lambda\chi(2)\int_0^t S_{t-s}^K\left(S_s^K\varphi\int_0^s S_r^K\psi\,\mathrm{d}r\right)\mathrm{d}s\\
&\quad\left.\left.-\mathrm{e}^{-\mu(V_t^K(0,\lambda\psi))}\int_0^t S_{t-s}^K\big(U_s^K(\varphi,\lambda\psi)\big(\Phi^K\big)'\big(V_s^K(0,\lambda\psi)\big)\big)\,\mathrm{d}s\right)\lambda^{-p-1}\,\mathrm{d}\lambda\right)\\
&=\mu\big(S_t^K\varphi\big)\left(\mu\left(\int_0^t S_s^K\psi\,\mathrm{d}s\right)\right)^p+I_1+I_2+I_3
\end{aligned}$$

where

$$\begin{aligned}
I_1&=p\eta_p\mu\left(S_t^K\varphi\int_0^\infty\big(\mathrm{e}^{-\mu(V_t^K(0,\lambda\psi))}-\mathrm{e}^{-\lambda\mu(\int_0^t S_s^K\psi\,\mathrm{d}s)}\big)\lambda^{-p-1}\,\mathrm{d}\lambda\right),\\
I_2&=p\eta_p\mu\left(\int_0^\infty\left(\lambda\chi(2)\int_0^t S_{t-s}^K\left(S_s^K\varphi\int_0^s S_r^K\psi\,\mathrm{d}r\right)\mathrm{d}s\right.\right.\\
&\quad\left.\left.-\int_0^t S_{t-s}^K\big(U_s^K(\varphi,\lambda\psi)\big(\Phi^K\big)'\big(V_s^K(0,\lambda\psi)\big)\big)\,\mathrm{d}s\right)\lambda^{-p-1}\,\mathrm{d}\lambda\right),\\
I_3&=p\eta_p\mu\left(\int_0^\infty\big(1-\mathrm{e}^{-\mu(V_t^K(0,\lambda\psi))}\big)\int_0^t S_{t-s}^K\big(U_s^K(\varphi,\lambda\psi)\big(\Phi^K\big)'\big(V_s^K(0,\lambda\psi)\big)\big)\,\mathrm{d}s\,\lambda^{-p-1}\,\mathrm{d}\lambda\right).
\end{aligned}$$

By the elementary inequality $1-\mathrm{e}^{-x}\leqslant x$, for $x\geqslant 0$, and (3.5) we have

$$\big(\Phi^K\big)'(x)\leqslant\eta\frac{K^{1-\beta}}{1-\beta}x=\chi(2)x. \tag{4.9}$$

Using (3.3) and (4.7) it is easy to derive that $U_t^K(\varphi, \lambda\psi) \geqslant 0$ and

$$U_t^K(\varphi, \lambda\psi) \leqslant S_t^K \varphi, \tag{4.10}$$

$$V_t^K(0, \lambda\psi) \leqslant \lambda \int_0^t S_s^K \psi \, \mathrm{d}s. \tag{4.11}$$

The above inequalities and (4.8) yield the following bound on I_3:

$$\begin{aligned} I_3 &\leqslant c\mu\left(\int_0^\infty \int_0^t S_{t-s}^K \big(S_s^K \varphi V_s^K(0, \lambda\psi)\big) \, \mathrm{d}s \big(1 - \mathrm{e}^{-\mu(V_t^K(0,\lambda\psi))}\big) \lambda^{-p-1} \, \mathrm{d}\lambda \right) \\ &\leqslant c\mu\left(\int_0^t S_{t-s}^K \left(S_s^K \varphi \int_0^s S_r^K \psi \, \mathrm{d}r \right) \mathrm{d}s \right) \int_0^\infty \big(1 - \mathrm{e}^{-\lambda\mu(\int_0^t S_l^K \psi \, \mathrm{d}l)}\big) \lambda^{-p} \, \mathrm{d}\lambda \\ &\leqslant c\left(\mu\left(\int_0^t S_s^K \psi \, \mathrm{d}s \right) \right)^{p-1} \mu\left(\int_0^t S_{t-s}^K \left(S_s^K \varphi \int_0^s S_r^K \psi \, \mathrm{d}r \right) \mathrm{d}s \right), \end{aligned} \tag{4.12}$$

where the last inequality follows by (4.1).

Let us take care of I_2. By (4.8) we get

$$\begin{aligned} I_2 &= p\eta_p\mu\left(\int_0^\infty \left(\lambda\chi(2) \int_0^t S_{t-s}^K \left(S_s^K \varphi \int_0^s S_r^K \psi \, \mathrm{d}r \right) \mathrm{d}s - \int_0^t S_{t-s}^K \big(S_s^K \varphi \big(\Phi^K\big)'\big(V_s^K(0, \lambda\psi)\big)\big) \, \mathrm{d}s \right.\right. \\ &\quad \left.\left. + \int_0^t S_{t-s}^K \left(\big(\Phi^K\big)'\big(V_s^K(0, \lambda\psi)\big) \int_0^s S_{s-r}^K \big(U_r^K(\varphi, \lambda\psi)\big(\Phi^K\big)'\big(V_r^K(0, \lambda\psi)\big)\big) \, \mathrm{d}r \right) \mathrm{d}s \right) \lambda^{-p-1} \, \mathrm{d}\lambda \right) \\ &= J_1 + J_2, \end{aligned} \tag{4.13}$$

where

$$J_1 = p\eta_p\mu\left(\int_0^\infty \left(\lambda\chi(2) \int_0^t S_{t-s}^K \left(S_s^K \varphi \int_0^s S_r^K \psi \, \mathrm{d}r \right) \mathrm{d}s - \int_0^t S_{t-s}^K \big(S_s^K \varphi \big(\Phi^K\big)'\big(V_s^K(0, \lambda\psi)\big)\big) \, \mathrm{d}s \right) \lambda^{-p-1} \, \mathrm{d}\lambda \right),$$

$$J_2 = p\eta_p\mu\left(\int_0^\infty \int_0^t S_{t-s}^K \left(\big(\Phi^K\big)'\big(V_s^K(0, \lambda\psi)\big) \int_0^s S_{s-r}^K \big(U_r^K(\varphi, \lambda\psi)\big(\Phi^K\big)'\big(V_r^K(0, \lambda\psi)\big)\big) \, \mathrm{d}r \right) \mathrm{d}s \, \lambda^{-p-1} \, \mathrm{d}\lambda \right).$$

Let us estimate J_2. First, by (1.13), (4.1) and (4.2) we obtain

$$\begin{aligned} \int_0^\infty \big(\Phi^K\big)'\big(V_s^K(0, \lambda\psi)\big) \lambda^{-p} \, \mathrm{d}\lambda &= \eta \int_0^\infty \int_0^K \big(1 - \mathrm{e}^{-w V_s^K(0,\lambda\psi)}\big) w^{-\beta-1} \, \mathrm{d}w \, \lambda^{-p} \, \mathrm{d}\lambda \\ &\leqslant \eta \int_0^K \int_0^\infty \big(1 - \mathrm{e}^{-\lambda w \int_0^s S_r \psi \, \mathrm{d}r}\big) \lambda^{-p} \, \mathrm{d}\lambda \, w^{-\beta-1} \, \mathrm{d}w \\ &= \eta\eta_p^{-1} \int_0^K \left(w \int_0^s S_r^K \psi \, \mathrm{d}r \right)^{p-1} w^{-\beta-1} \, \mathrm{d}w \\ &= c\left(\int_0^s S_r^K \psi \, \mathrm{d}r \right)^{p-1}. \end{aligned}$$

Use this and (4.9)–(4.11) to get

$$
\begin{aligned}
J_2 &\leqslant c\mu\left(\int_0^\infty\int_0^t S_{t-s}^K\left(\left(\Phi^K\right)'\left(V_s^K(0,\lambda\psi)\right)\int_0^s S_{s-r}^K\left(S_r^K\varphi V_r^K(0,\lambda\psi)\right)\mathrm{d}r\right)\mathrm{d}s\lambda^{-p-1}\,\mathrm{d}\lambda\right)\\
&\leqslant c\mu\left(\int_0^t S_{t-s}^K\left(\int_0^s S_{s-r}^K\left(S_r^K\varphi\int_0^r S_u^K\psi\,\mathrm{d}u\right)\mathrm{d}r\int_0^\infty\left(\Phi^K\right)'\left(V_s^K(0,\lambda\psi)\right)\lambda^{-p}\,\mathrm{d}\lambda\right)\mathrm{d}s\right)\\
&\leqslant c\mu\left(\int_0^t S_{t-s}^K\left(\int_0^s S_{s-r}^K\left(S_r^K\varphi\int_0^r S_u^K\psi\,\mathrm{d}u\right)\mathrm{d}r\left(\int_0^s S_l^K\psi\,\mathrm{d}l\right)^{p-1}\right)\mathrm{d}s\right).
\end{aligned}
\tag{4.14}
$$

Now let us estimate J_1:

$$
\begin{aligned}
J_1 &= p\eta_p\mu\left(\int_0^\infty\left(\lambda\chi(2)\int_0^t S_{t-s}^K\left(S_s^K\varphi\int_0^s S_r^K\psi\,\mathrm{d}r\right)\right.\right.\\
&\qquad\left.\left.-\int_0^t S_{t-s}^K\left(S_s^K\varphi\eta\int_0^K\left(1-\mathrm{e}^{-wV_s^K(0,\lambda\psi)}\right)w^{-\beta-1}\,\mathrm{d}w\right)\mathrm{d}s\right)\lambda^{-p-1}\,\mathrm{d}\lambda\right)\\
&= p\eta_p\mu\left(\int_0^t S_{t-s}^K\left(S_s^K\varphi\left(\int_0^\infty\left[\lambda\chi(2)\int_0^s S_r^K\psi\,\mathrm{d}r+\eta\int_0^K\left(\mathrm{e}^{-wV_s^K(0,\lambda\psi)}-1\right)w^{-\beta-1}\,\mathrm{d}w\right]\lambda^{-p-1}\,\mathrm{d}\lambda\right)\right)\mathrm{d}s\right).
\end{aligned}
$$

Using the identity

$$
\chi(2)=\eta\int_0^K w^{-\beta}\,\mathrm{d}w,
$$

we obtain

$$
\begin{aligned}
J_1 &= p\eta_p\mu\left(\int_0^t S_{t-s}^K\left(S_s^K\varphi\eta\int_0^\infty\int_0^K\left(\mathrm{e}^{-wV_s^K(0,\lambda\psi)}-1+\lambda w\int_0^s S_r^K\psi\,\mathrm{d}r\right)w^{-\beta-1}\,\mathrm{d}w\,\lambda^{-p-1}\,\mathrm{d}\lambda\right)\mathrm{d}s\right)\\
&= p\eta\eta_p\mu\left(\int_0^t S_{t-s}^K\left(S_s^K\varphi\left[\int_0^K\int_0^\infty\left(\mathrm{e}^{-\lambda w\int_0^s S_r\psi\,\mathrm{d}r}-1+\lambda w\int_0^s S_r^K\psi\,\mathrm{d}r\right)\lambda^{-p-1}\,\mathrm{d}\lambda\,w^{-\beta-1}\,\mathrm{d}w\right.\right.\right.\\
&\qquad\left.\left.\left.+\int_0^K\int_0^\infty\left(\mathrm{e}^{-wV_s^K(0,\lambda\psi)}-\mathrm{e}^{-\lambda w\int_0^s S_r^K\psi\,\mathrm{d}r}\right)\lambda^{-p-1}\,\mathrm{d}\lambda\,w^{-\beta-1}\,\mathrm{d}w\right]\right)\mathrm{d}s\right)\\
&= \mu\left(\int_0^t S_{t-s}^K\left(S_s^K\varphi\eta\int_0^K\left(w\int_0^s S_r^K\psi\,\mathrm{d}r\right)^p w^{-\beta-1}\,\mathrm{d}w\right)\right)\\
&\qquad+p\eta\eta_p\mu\left(\int_0^t S_{t-s}^K\left(S_s^K\varphi\int_0^K\int_0^\infty\left(\mathrm{e}^{-wV_s^K(0,\lambda\psi)}-\mathrm{e}^{-\lambda w\int_0^s S_r^K\psi\,\mathrm{d}r}\right)\lambda^{-p-1}\,\mathrm{d}\lambda\,w^{-\beta-1}\,\mathrm{d}w\right)\mathrm{d}s\right)\\
&= c\mu\left(\int_0^t S_{t-s}^K\left(S_s^K\varphi\left(\int_0^s S_r^K\psi\,\mathrm{d}r\right)^p\right)\right)\mathrm{d}s+p\eta\eta_p\mu\left(\int_0^t S_{t-s}^K\left(S_s^K\varphi Q(s)\right)\mathrm{d}s\right),
\end{aligned}
\tag{4.15}
$$

where

$$\begin{aligned}Q(s)&=\int_0^K\int_0^\infty\left(\mathrm{e}^{-wV_s^K(0,\lambda\psi)}-\mathrm{e}^{\lambda w\int_0^s S_r^K\psi\,\mathrm{d}r}\right)\lambda^{-p-1}\,\mathrm{d}\lambda\,w^{-\beta-1}\,\mathrm{d}w\\&\leqslant\int_0^K\int_0^\infty\left|wV_s^K(0,\lambda\psi)-\lambda w\int_0^s S_r^K\psi\,\mathrm{d}r\right|\lambda^{-p-1}\,\mathrm{d}\lambda\,w^{-\beta-1}\,\mathrm{d}w\\&=c\left(\int_0^K+\int_K^\infty\right)\left|V_s^K(0,\lambda\psi)-\int_0^s S_r^K(\lambda\psi)\,\mathrm{d}r\right|\lambda^{-p-1}\,\mathrm{d}\lambda\\&=(Q_1+Q_2)(s).\end{aligned}$$

By (3.3)

$$Q_1(s)=c\int_0^K\int_0^s S_{s-r}^K\Phi^K\left(V_r^K(0,\lambda\psi)\right)\mathrm{d}r\,\lambda^{-p-1}\,\mathrm{d}\lambda.$$

Due to $1<1+\beta<p<p'<2$ we have, from the elementary inequality $0\leqslant\mathrm{e}^{-x}-1-x\leqslant cx^{p'}$, for $x\geqslant 0$, that

$$\Phi^K(x)\leqslant c\eta\frac{K^{p'-\beta-1}}{p'-\beta-1}x^{p'}.$$

Using this we obtain

$$\begin{aligned}Q_1(s)&\leqslant c\int_0^K\int_0^s S_{s-r}^K\left(\left(V_r^K(0,\lambda\psi)\right)^{p'}\right)\mathrm{d}r\,\lambda^{-p-1}\,\mathrm{d}\lambda\\&\leqslant c\int_0^K\int_0^s S_{s-r}^K\left(\left(\int_0^r S_u^K(\lambda\psi)\,\mathrm{d}u\right)^{p'}\right)\mathrm{d}r\,\lambda^{-p-1}\,\mathrm{d}\lambda\\&=c\int_0^s S_{s-r}^K\left(\left(\int_0^r S_u^K\psi\,\mathrm{d}u\right)^{p'}\right)\mathrm{d}r.\end{aligned}\tag{4.16}$$

Apply triangle inequality and (4.11) to bound Q_2:

$$\begin{aligned}Q_2(s)&=c\int_K^\infty\left|V_s^K(0,\lambda\psi)-\int_0^s S_r^K(\lambda\psi)\,\mathrm{d}r\right|\lambda^{-p-1}\,\mathrm{d}\lambda\\&\leqslant 2c\int_K^\infty\int_0^s S_r^K(\lambda\psi)\,\mathrm{d}r\,\lambda^{-p-1}d\lambda\\&=c\int_0^s S_r^K\psi\,\mathrm{d}r.\end{aligned}\tag{4.17}$$

Finally, let us estimate I_1. Proceeding as before we have

$$I_1=p\eta_p\mu\left(S_t^K\varphi\left(\int_0^K+\int_K^\infty\right)\left(\mathrm{e}^{-\mu(V_t^K(0,\lambda\psi))}-\mathrm{e}^{-\lambda\mu(\int_0^t S_s^K\psi\,\mathrm{d}s)}\right)\lambda^{-p-1}\mathrm{d}\lambda\right)$$

$$\leqslant p\eta_p\mu\left(S_t^K\varphi\int_0^K\left|\mu\left(V_t^K(0,\lambda\psi)\right)-\mu\left(\int_0^t S_s^K\psi\,\mathrm{d}s\right)\right|\lambda^{-p-1}\,\mathrm{d}\lambda\right)+p\eta_p\mu\left(S_t^K\varphi\int_K^\infty 2\lambda^{-p-1}\,\mathrm{d}\lambda\right)$$

$$\leqslant c\mu\left(S_t^K\varphi\right)\int_0^K\mu\left(\int_0^t S_{t-s}^K\left(V_s^K(0,\lambda\psi)\right)^{p'}\,\mathrm{d}s\right)\lambda^{-p-1}\,\mathrm{d}\lambda+c\mu\left(S_t^K\varphi\right)$$

$$=c\mu\left(S_t^K\varphi\right)\mu\left(\int_0^t S_{t-s}^K\left(\int_0^s S_r^K\psi\,\mathrm{d}r\right)^{p'}\mathrm{d}s\right)+c\mu\left(S_t^K\varphi\right). \tag{4.18}$$

Combining (4.12)–(4.18) and (3.6) we obtain (4.3). □

Now, the proof of Proposition 2 is based on the bounds that we will get on all the terms on the right-hand side of (4.3).

Lemma 4. *Let $\mu(\mathrm{d}x)=h(x)\,\mathrm{d}x$, where h is bounded and integrable. Then*

$$\sup_{x\in\mathbb{R}^d}\mu\left(S_sG^\lambda(\cdot\cdot-x)\right)\leqslant c_{4.19}(h,\lambda)<\infty. \tag{4.19}$$

Proof. Using the explicit expression for μ we have,

$$\begin{aligned}\mu\left(S_sG^\lambda(\cdot\cdot-x)\right)&=\iint p_s(y-z)G^\lambda(z-x)\,\mathrm{d}z\,\mu(\mathrm{d}y)\\&\leqslant\iint p_s(y-z)G^\lambda(z-x)\,\mathrm{d}z\|h\|_\infty\,\mathrm{d}y\\&=\|h\|_\infty\left\|G^\lambda\right\|_1=\|h\|_\infty\lambda^{-1},\end{aligned}$$

recall that $\|\cdot\|_\infty$ is the supremum norm. □

In the next two lemmas we are going to use the following basic inequalities: For $d>\alpha$ and $\delta\in(0,1)$, we have ([12], Lemma 4)

$$p_t(x)\leqslant ct^{\delta-1}|x|^{\alpha-d-\alpha\delta},\quad t>0,x\in\mathbb{R}^d\backslash\{0\}, \tag{4.20}$$

and the Riesz convolution formula

$$\int_{\mathbb{R}^d}|x-z|^{a-d}|z-y|^{b-d}\,\mathrm{d}z=c|x-y|^{a+b-d}, \tag{4.21}$$

whenever $a,b>0$, $a+b<d$ and $x,y\in\mathbb{R}^d$.

Also define the indicator function:

$$\kappa(x)=\mathbf{1}\left(|x|\leqslant 1\right).$$

Lemma 5. *Let $\alpha<d$. Then, for any $\delta\in(0,1)$ and $a\in[0,d)$, we have*

$$\int_{\mathbb{R}^d}p_s(y-z)|z-x|^{-a}\,\mathrm{d}z\leqslant c_1+c_2s^{\delta-1}\left(|y-x|^{\alpha-a-\delta\alpha}\kappa(y-x)+1\right),\quad\forall y,x\in\mathbb{R}^d, \tag{4.22}$$

where $c_1\geqslant 1$, $c_2>0$ are constants.

Proof. First let us prove (4.22) for the case $\alpha-\delta\alpha<a$. Use (4.20) and (4.21) to get

$$\begin{aligned}\int_{\mathbb{R}^d} p_s(y-z)|z-x|^{-a}\,\mathrm{d}z &= \int_{\mathbb{R}^d} p_s(y-x-z)|z|^{-a}\,\mathrm{d}z\\ &= \Bigl(\int_{|z|>1} + \int_{|z|\leqslant 1}\Bigr) p_s(y-x-z)|z|^{-a}\,\mathrm{d}z\\ &\leqslant 1 + \int_{|z|\leqslant 1} s^{\delta-1}|y-x-z|^{\alpha-d-\delta\alpha}|z|^{-a}\,\mathrm{d}z\\ &\leqslant 1 + cs^{\delta-1}|y-x|^{\alpha-a-\delta\alpha}\\ &= 1 + cs^{\delta-1}\bigl(|y-x|^{\alpha-a-\delta\alpha}\mathbf{1}\bigl(|y-x|\leqslant 1\bigr) + |y-x|^{\alpha-a-\delta\alpha}\mathbf{1}\bigl(|y-x|>1\bigr)\bigr)\\ &\leqslant 1 + cs^{\delta-1}\bigl(|y-x|^{\alpha-a-\delta\alpha}\kappa(y-x)+1\bigr).\end{aligned}$$

Now, suppose $\alpha - \delta\alpha \geqslant a$. Using a simple coupling argument, as in Lemma 5.1 of [14], we have

$$\int_{\mathbb{R}^d} p_s(y-x-z)|z|^{-a}\,\mathrm{d}z \leqslant \int_{\mathbb{R}^d} p_s(z)|z|^{-a}\,\mathrm{d}z.$$

By the scaling relationship

$$p_t(x) = t^{-d/\alpha} p_1\bigl(t^{-1/\alpha}x\bigr), \quad t>0,\ x\in\mathbb{R}^d, \tag{4.23}$$

we get

$$\begin{aligned}\int_{\mathbb{R}^d} p_s(z)|z|^{-a}\,\mathrm{d}z &= s^{-a/\alpha}\int_{\mathbb{R}^d} p_1\bigl(s^{-1/\alpha}z\bigr)\bigl|s^{-1/\alpha}z\bigr|^{-a} s^{-d/\alpha}\,\mathrm{d}z\\ &= s^{-a/\alpha}\int_{\mathbb{R}^d} p_1(z)|z|^{-a}\,\mathrm{d}z.\end{aligned}$$

Therefore,

$$\begin{aligned}\int_{\mathbb{R}^d} p_s(y-z)|z-x|^{-a}\,\mathrm{d}z &\leqslant cs^{-a/\alpha}\bigl(\mathbf{1}\bigl(|s|\leqslant 1\bigr)+\mathbf{1}\bigl(|s|>1\bigr)\bigr)\\ &\leqslant cs^{\delta-1}+c,\end{aligned}$$

and we are done. □

Lemma 6. *For any $\delta\in(0,1)$ there exists $c(t)$ such that for any $T>0$, $\sup_{t<T} c(t)<\infty$ and*

$$\int_0^t S_s G^\lambda(\cdot - x)(y)\,\mathrm{d}s \leqslant c(t)\bigl(|y-x|^{2\alpha-d-\delta\alpha}\kappa(y-x)+1\bigr), \quad \forall y,x\in\mathbb{R}^d. \tag{4.24}$$

Proof. Let $\alpha<d$. Since $G^\lambda(x)\leqslant c|x|^{\alpha-d}$, take $a=d-\alpha$, apply Lemma 5 and make additional integration with respect to time. If $\alpha\geqslant d$, then by the unimodality of p_1,

$$\begin{aligned}\int_0^t S_s G^\lambda(\cdot - x)(y)\,\mathrm{d}s &= \int_0^t\int_0^\infty \mathrm{e}^{-\lambda r} p_{r+s}(x-y)\,\mathrm{d}r\,\mathrm{d}s\\ &\leqslant c\int_0^t\int_0^\infty \mathrm{e}^{-\lambda r}(r+s)^{-d/\alpha}\,\mathrm{d}r\,\mathrm{d}s\\ &\leqslant c(t), \forall t\geqslant 0,\end{aligned}$$

and we are done. □

Lemma 7. *Let $1 < q < 2$, and $d < \alpha(2 + 1/q)$. Then there exists $c(t)$ such that for any $T > 0$, $\sup_{t<T} c(t) < \infty$ and for any $y, x \in \mathbb{R}^d$,*

$$\int_0^t S_{t-s}\left(\left(\int_0^s S_r G^\lambda(\cdot - x)\,\mathrm{d}r\right)^q\right)(y)\,\mathrm{d}s \leqslant c(t), \tag{4.25}$$

$$\int_0^t S_{s+\varepsilon}\left(\left(\int_0^s S_r G^\lambda(\cdot - x)\,\mathrm{d}r\right)^q\right)(y)\,\mathrm{d}s \leqslant c(t), \quad \forall \varepsilon \in [0,1]. \tag{4.26}$$

Proof. Since $d < \alpha(2 + 1/q)$ it is easy to check that we can fix $\delta \in (0,1)$ sufficiently small such that,

$$q(2\alpha - d - \delta\alpha) + \alpha - \delta\alpha > 0. \tag{4.27}$$

By Lemma 6,

$$\left(\int_0^s S_r G^\lambda(\cdot - x)\,\mathrm{d}r\right)^q(y) \leqslant \sup_{s\leqslant t} c(s)\big(|y-x|^{q(2\alpha-d-\delta\alpha)}\kappa(y-x) + 1\big). \tag{4.28}$$

Now take $a = -q(2\alpha - d - \delta\alpha)$. If $a < 0$ then the result follows trivially, due to the fact that then the right-hand side of (4.28) is uniformly bounded for any $y, x \in \mathbb{R}^d$.

If $a \geqslant 0$, we apply again Lemma 5 to conclude that the result follows if $q(2\alpha - d - \delta\alpha) + \alpha - \delta\alpha > 0$. But this is exactly the condition (4.27) which is satisfied due to the choice of δ. □

Proof of Proposition 2. From (3.1) we see that $S_t^K \leqslant S_t$, hence Proposition 3 implies

$$\begin{aligned}
&E_\mu\left[\int_{\mathbb{R}^d} Y_t^K\big(p_\varepsilon(\cdot - x)\big)\left(\int_0^t Y_s^K\big(G^\lambda(\cdot\,\cdot - x)\big)\,\mathrm{d}s\right)^p \mathrm{d}x\right] \\
&\quad\leqslant c\Bigg\{\int_{\mathbb{R}^d} \mu\big(S_t p_\varepsilon(\cdot - x)\big)\,\mathrm{d}x + \int_{\mathbb{R}^d} \mu\big(S_t p_\varepsilon(\cdot - x)\big)\left(\mu\left(\int_0^t S_s G^\lambda(\cdot\,\cdot - x)\,\mathrm{d}s\right)\right)^p \mathrm{d}x \\
&\qquad + \int_{\mathbb{R}^d} \mu\big(S_t p_\varepsilon(\cdot - x)\big)\mu\left(\int_0^t S_{t-s}\left(\int_0^s S_r G^\lambda(\cdot\,\cdot - x)\,\mathrm{d}r\right)^{p'} \mathrm{d}s\right)\mathrm{d}x \\
&\qquad + \int_{\mathbb{R}^d} \mu\left(\int_0^t S_{t-s}\left(S_s p_\varepsilon(\cdot - x)\int_0^s S_r G^\lambda(\cdot\,\cdot - x)\,\mathrm{d}r\right)\mathrm{d}s\right)\mathrm{d}x \\
&\qquad + \int_{\mathbb{R}^d} \mu\left(\int_0^t S_{t-s}\left(S_s p_\varepsilon(\cdot - x)\int_0^s S_r G^\lambda(\cdot\,\cdot - x)\,\mathrm{d}r\right)\mathrm{d}s\right)\left(\mu\left(\int_0^t S_s G^\lambda(\cdot\,\cdot - x)\,\mathrm{d}s\right)\right)^{p-1} \mathrm{d}x \\
&\qquad + \int_{\mathbb{R}^d} \mu\left(\int_0^t S_{t-s}\left(\int_0^s S_{s-r}\left(S_r p_\varepsilon(\cdot - x)\int_0^r S_u G^\lambda(\cdot\,\cdot - x)\,\mathrm{d}u\right)\mathrm{d}r\left(\int_0^s S_r G^\lambda(\cdot\,\cdot - x)\,\mathrm{d}r\right)^{p-1}\right)\mathrm{d}s\right)\mathrm{d}x \\
&\qquad + \int_{\mathbb{R}^d} \mu\left(\int_0^t S_{t-s}\left(S_s p_\varepsilon(\cdot - x)\left(\int_0^s S_r G^\lambda(\cdot\,\cdot - x)\,\mathrm{d}r\right)^p\right)\mathrm{d}s\right)\mathrm{d}x \\
&\qquad + \int_{\mathbb{R}^d} \mu\left(\int_0^t S_{t-s}\left(S_s p_\varepsilon(\cdot - x)\int_0^s S_{s-r}\left(\int_0^r S_u G^\lambda(\cdot\,\cdot - x)\,\mathrm{d}u\right)^{p'} \mathrm{d}r\right)\mathrm{d}s\right)\mathrm{d}x\Bigg\} = c\sum_{i=1}^8 I_i(\varepsilon).
\end{aligned}$$

We will check the boundedness of all the terms $I_i(\varepsilon), i=1,\ldots,8$. First note, that for $d\leqslant\alpha$ all the terms $I_i(\varepsilon), i=1,\ldots,8$, can be bounded very easily, and we leave it to check to the reader. We will consider the case $\alpha<d$. The first two terms, $I_1(\varepsilon)$ and $I_2(\varepsilon)$ are easy to handle. By the Fubini theorem and Lemma 4 we get

$$I_1(\varepsilon)+I_2(\varepsilon)\leqslant\mu(1)\big(1+(c_{4.19}t)^p\big).$$

By Lemma 7 we easily get

$$I_3(\varepsilon)+I_8(\varepsilon)\leqslant\mu(1)c(t).$$

For $I_7(\varepsilon)$ we get the following

$$\begin{aligned}
I_7(\varepsilon)&=\int_0^t\int_{\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}^d} p_{t-s}(y-z)p_{s+\varepsilon}(z-x)\left(\int_0^s S_rG^\lambda(z-x)\,\mathrm{d}r\right)^p \mathrm{d}s\,\mathrm{d}x\,\mathrm{d}z\,\mu(\mathrm{d}y)\\
&=\int_0^t\int_{\mathbb{R}^d\times\mathbb{R}^d} p_{t-s}(y-z)\,\mathrm{d}z\,S_{s+\varepsilon}\left(\int_0^s S_rG^\lambda(\cdot)\,\mathrm{d}r\right)^p(0)\,\mathrm{d}s\,\mu(\mathrm{d}y)\\
&=\mu(1)\int_0^t S_{s+\varepsilon}\left(\int_0^s S_rG^\lambda(\cdot)\,\mathrm{d}r\right)^p(0)\,\mathrm{d}s\\
&\leqslant\mu(1)c(t),
\end{aligned}\tag{4.29}$$

where the last inequality follows by Lemma 7. It is also easy to check that

$$\begin{aligned}
I_4(\varepsilon)&\leqslant\int_{\mathbb{R}^d}\mu\left(\int_0^t S_{t-s}\left(S_sp_\varepsilon(\cdot-x)\left(\left(\int_0^s S_rG^\lambda(\cdot\cdot-x)\,\mathrm{d}r\right)^p+1\right)\right)\mathrm{d}s\right)\mathrm{d}x\\
&\leqslant I_7(\varepsilon)+\mu(1)t,
\end{aligned}\tag{4.30}$$

and

$$I_5(\varepsilon)\leqslant c_{4.19}^{p-1}I_4(\varepsilon).$$

The last term we have to handle is $I_6(\varepsilon)$:

$$\begin{aligned}
I_6(\varepsilon)&\leqslant\int_{\mathbb{R}^{3d}}\left(\int_{x:\,|y_1-x|\leqslant1}+\int_{x:\,|y_1-x|>1}\right)\left(\int_0^t p_{t-s}(y-y_1)\int_0^s p_{s-r}(y_1-z)p_{r+\varepsilon}(z-x)\int_0^r S_uG^\lambda(z-x)\,\mathrm{d}u\,\mathrm{d}r\right.\\
&\quad\left.\times\left(\int_0^s S_rG^\lambda(y_1-x)\,\mathrm{d}r\right)^{p-1}\mathrm{d}s\right)\mathrm{d}y_1\,\mathrm{d}z\,\mathrm{d}x\,\mu(\mathrm{d}y)\\
&=I_{6,1}(\varepsilon)+I_{6,2}(\varepsilon).
\end{aligned}$$

Our condition on d implies that we can choose $\delta\in(0,1/3)$ sufficiently small such that

$$(2\alpha-d)(p+1)-\delta\alpha(p+2)>-d.\tag{4.31}$$

By (4.20), Lemmas 5 and 6 we get

$$\begin{aligned}
I_{6,1}(\varepsilon)&\leqslant c(t)\int_{\mathbb{R}^{3d}}\int_{x:\,|y_1-x|\leqslant1}\left(\int_0^t p_{t-s}(y-y_1)\int_0^s p_{s-r}(y_1-z)(r+\varepsilon)^{\delta-1}\right.\\
&\quad\left.\times\big(|z-x|^{3\alpha-2d-2\delta\alpha}\kappa(z-x)+|z-x|^{\alpha-d-\delta\alpha}\big)\,\mathrm{d}r\big(|y_1-x|^{(2\alpha-d-\delta\alpha)(p-1)}+1\big)\,\mathrm{d}s\right)\mathrm{d}y_1\,\mathrm{d}z\,\mathrm{d}x\,\mu(\mathrm{d}y)
\end{aligned}$$

$$
\begin{aligned}
&= c(t) \int_{\mathbb{R}^{2d}} \int_{x:\,|y_1-x|\leqslant 1} \Bigg(\int_0^t p_{t-s}(y-y_1) \int_0^s (r+\varepsilon)^{\delta-1} \\
&\quad \times \int_{\mathbb{R}^d} p_{s-r}(y_1-z)\big(|z-x|^{3\alpha-2d-2\delta\alpha}\kappa(z-x) + |z-x|^{\alpha-d-\delta\alpha}\big)\,\mathrm{d}z\,\mathrm{d}r \\
&\quad \times \big(|y_1-x|^{(2\alpha-d-\delta\alpha)(p-1)}+1\big)\,\mathrm{d}s \Bigg)\,\mathrm{d}y_1\,\mathrm{d}x\,\mu(\mathrm{d}y) \\
&\leqslant c(t) \int_{\mathbb{R}^{2d}} \int_{x:\,|y_1-x|\leqslant 1} \int_0^t p_{t-s}(y-y_1) \int_0^s (r+\varepsilon)^{\delta-1} \\
&\quad \times \big[1 + (s-r)^{\delta-1}\big(|y_1-x|^{4\alpha-2d-3\delta\alpha} + |y_1-x|^{2\alpha-d-2\delta\alpha}+1\big)\big]\,\mathrm{d}r \\
&\quad \times \big(|y_1-x|^{(2\alpha-d-\delta\alpha)(p-1)}+1\big)\,\mathrm{d}s\,\mathrm{d}y_1\,\mathrm{d}x\,\mu(\mathrm{d}y) \\
&= c(t) \int_{\mathbb{R}^{2d}} \int_0^t p_{t-s}(y-y_1) \Bigg[\int_0^s (r+\varepsilon)^{\delta-1}\,\mathrm{d}r \\
&\quad \times \int_{x:\,|y_1-x|\leqslant 1} \big(1+|y_1-x|^{(2\alpha-d-\delta\alpha)(p-1)}\big)\,\mathrm{d}x + \int_0^s (r+\varepsilon)^{\delta-1}(s-r)^{\delta-1}\,\mathrm{d}r \\
&\quad \times \int_{x:\,|y_1-x|\leqslant 1} \big(1+|y_1-x|^{(2\alpha-d-\delta\alpha)(p-1)+4\alpha-2d-3\alpha\delta} \\
&\quad + |y_1-x|^{(2\alpha-d-\delta\alpha)(p-1)+2\alpha-d-2\alpha\delta} + |y_1-x|^{(2\alpha-d-\delta\alpha)(p-1)} \\
&\quad + |y_1-x|^{4\alpha-2d-3\alpha\delta} + |y_1-x|^{2\alpha-d-2\alpha\delta}\big)\,\mathrm{d}x \Bigg]\,\mathrm{d}y_1\,\mu(\mathrm{d}y) \\
&\leqslant c(t)\mu(1),
\end{aligned}
$$

where the last inequality follows by (4.31). As for the $I_{6,2}(\varepsilon)$, by Lemma 6 we get

$$
\begin{aligned}
I_{6,2}(\varepsilon) &\leqslant \sup_{x,y_1:\,|y_1-x|>1} \left(\int_0^t S_r G^\lambda(y_1-x)\,\mathrm{d}r \right)^{p-1} \int_{\mathbb{R}^{3d}} \int_0^t p_{t-s}(y-y_1) \\
&\quad \times \int_0^s \int_0^r \int_0^\infty p_{s-r}(y_1-z)\,p_{r+\varepsilon+u+v}(0)\,\mathrm{e}^{-\lambda v}\,\mathrm{d}v\,\mathrm{d}u\,\mathrm{d}r\,\mathrm{d}s\,\mathrm{d}y_1\,\mathrm{d}z\,\mu(\mathrm{d}y) \\
&\leqslant c(t)\mu(1) \int_0^r \int_0^s \int_0^r \int_0^\infty (r+\varepsilon+u+v)^{-d/\alpha}\,\mathrm{e}^{-\lambda v}\,\mathrm{d}v\,\mathrm{d}u\,\mathrm{d}r\,\mathrm{d}s
\end{aligned}
$$

and the last integral is bounded if $d < 3\alpha$. By combining all the above estimates we are done with the first part of the proposition.

Now we are going to prove the second part of the proposition. Take

$$
\varphi(y) = \left(\int_0^t \int G^{\lambda,\varepsilon}(y-x) Y_s^K(\mathrm{d}x)\,\mathrm{d}s \right)^p, \quad y \in \mathbb{R}^d.
$$

For each $n \in \mathbb{N}$ define the truncations functions, $\varphi_n = \varphi \wedge n$. Then $0 \leqslant \varphi_n \uparrow \varphi$, as $n \to 0$. Since φ_n is bounded and $p_1(z)\,\mathrm{d}z Y_t^K(\mathrm{d}y)$ is a finite measure, we have by the dominated convergence theorem and the scaling relationship (4.23) the following estimation

$$\begin{aligned}\int \varphi_n(x) Y_t^K(\mathrm{d}x) &= \lim_{\delta\downarrow 0} \iint \varphi_n\big(\delta^{1/\alpha} z + x\big) p_1(z)\,\mathrm{d}z\, Y_t^K(\mathrm{d}x)\\ &\leqslant \liminf_{\delta\downarrow 0} \iint \varphi\big(\delta^{1/\alpha} z + x\big) p_1(z)\,\mathrm{d}z\, Y_t^K(\mathrm{d}x)\\ &= \liminf_{\delta\downarrow 0} \iint p_\delta(x-y) Y_t^K(\mathrm{d}x)\varphi(y)\,\mathrm{d}y.\end{aligned}$$

Letting $n \to \infty$, by the monotone convergence theorem, we have

$$\int \left(\int_0^t Y_s^K\big(G^{\lambda,\varepsilon}(\cdot\cdot - x)\big)\,\mathrm{d}s\right)^p Y_t^K(\mathrm{d}x) \leqslant \liminf_{\delta\downarrow 0} \int \left(\int_0^t Y_s^K\big(G^{\lambda,\varepsilon}(\cdot\cdot - x)\big)\,\mathrm{d}s\right)^p Y_t^K\big(p_\delta(\cdot - x)\big)\,\mathrm{d}x.$$

From the Fatou lemma we get

$$E\left[\int \left(\int_0^t Y_s^K\big(G^{\lambda,\varepsilon}(\cdot\cdot - x)\big)\,\mathrm{d}s\right)^p Y_t^K(\mathrm{d}x)\right] \leqslant \liminf_{\delta\downarrow 0} E\left[\int \left(\int_0^t Y_s^K\big(G^{\lambda,\varepsilon}(\cdot\cdot - x)\big)\,\mathrm{d}s\right)^p Y_t^K\big(p_\delta(\cdot - x)\big)\,\mathrm{d}x\right].$$

Since $G^{\lambda,\varepsilon} \leqslant G^\lambda$ we have by the already proven part of Proposition 2,

$$\begin{aligned}&E\left[\int \left(\int_0^t Y_s^K\big(G^{\lambda,\varepsilon}(\cdot\cdot - x)\big)\,\mathrm{d}s\right)^p Y_t^K(\mathrm{d}x)\right] \leqslant \liminf_{\delta\downarrow 0} E\left[\int \left(\int_0^t Y_s^K\big(G^{\lambda}(\cdot\cdot - x)\big)\,\mathrm{d}s\right)^p Y_t^K\big(p_\delta(\cdot - x)\big)\,\mathrm{d}x\right]\\ &\quad\leqslant c(K,p,d,\alpha,\beta).\end{aligned}$$

Using once again the monotone convergence theorem, as $\varepsilon \to 0$, we arrive at

$$E\left[\int \left(\int_0^t Y_s^K\big(G^{\lambda}(\cdot\cdot - x)\big)\,\mathrm{d}s\right)^p Y_t^K(\mathrm{d}x)\right] < c(K,p,d,\alpha,\beta),$$

and we are done. □

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