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Mixing rank-one actions of locally compact Abelian groups ☆

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Abstract

Using techniques related to the (C, F)-actions we construct explicitly mixing rank-one (by cubes) actions T of $G = \mathbb{R}^{d_1} \times \mathbb{Z}^{d_2}$ for any pair of non-negative integers d_1, d_2. It is also shown that $h(T_g) = 0$ for each $g \in G$.

Résumé

En utilisant des techniques liées aux action (C, F), nous construisons explicitement des actions mélangeantes de rang un (par cubes), T de $G = \mathbb{R}^{d_1} \times \mathbb{Z}^{d_2}$ pour toute paire de nombres entiers $d_1, d_2 \geqq 0$. On prouve aussi que $h(T_g) = 0$ pour chaque $g \in G$.

MSC: 37A25; 37A15

Keywords: Ergodic action; Mixing; Rank-one action; Entropy

0. Introduction

Mixing rank-one transformations (and actions of more general groups) have been of interest in ergodic theory since 1970 when Ornstein constructed an example of mixing transformation without square root [18]. His method was used later as the core of a number of other remarkable constructions (see [20,21,13,10,17,19,8], etc.) Since then the dynamical properties of mixing rank-one transformations have been deeply investigated. It is now well known that such transformations are mixing of all orders [14,22] and have minimal self-joinings of all orders [15,22]. This implies in turn that they are prime and have trivial centralizer [21]. The results on multiple mixing were extended to rank-one mixing actions of $\mathbb{R}^d$ and $\mathbb{Z}^d$ [22–24] and to rank-one mixing actions of a wide class of discrete countable Abelian groups having an element of infinite order [11].

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Despite this progress, there are not many concrete examples of rank-one mixing actions that are known. Most of them were obtained via stochastic cutting-and-stacking techniques using "random spacers". Ornstein initiated this technique in [18], and more recent generalizations include the constructions of $\mathbb{R}$-actions in [19] and actions of infinite sums of finite groups in [8], as well as the del Junco–Madore actions of Abelian extensions of $\mathbb{Z}^d$ by locally finite groups [10,17]. The latter actions were only shown to be weakly mixing but conjectured to be mixing in [17]. While demonstrating the existence of mixing rank-one actions (which is a non-trivial problem!), these works do not exhibit a specific such transformation or action. In 1992 Adams and Friedman [2] gave a non-random algorithm that leads to a mixing rank-one construction. Using the ideas from that manuscript Adams [1] proved in 1998 the old conjecture that the classical staircase is mixing. That gave the first explicit example of mixing cutting-and-stacking transformation. Higher dimensional mixing staircase $\mathbb{Z}^d$-actions were later constructed in [3]. We note that the complete proof of the fact that they are mixing was given there only in dimension $d = 2$. As one of the consequences of our work, we complete the proof for all $d > 2$ (see Remark 4.12 below). Recently, a more general family of mixing "polynomial" staircase $\mathbb{Z}$-actions was constructed in [5]. Another interesting non-random construction appears in a recent work [12] devoted to smooth realizations of mixing rank-one flows on the 3-torus.

Our main purpose here is *to construct explicitly a family of mixing rank-one actions of* $\mathbb{R}^{d_1} \times \mathbb{Z}^{d_2}$ *for all non-negative* d_1 *and* d_2. It seems plausible that Orntein's stochastic method also can be adapted to produce mixing rank-one actions of these groups. We note however that our construction is more general and the 'randomness' can be incorporated into it (see [5] and [8] for a detailed discussion on that for $\mathbb{Z}$-actions and actions of infinite sums of finite groups respectively). Moreover, the main advantage of our approach is that the examples in our family are 'absolutely concrete', i.e., the parameters in the construction are all explicitly specified—the 'spacer mappings' are polynomials with known coefficients.

As a corollary we show that this family includes all the examples of mixing rank-one $\mathbb{Z}^d$-actions constructed previously in [1,3] and [5]. Our approach is based on ideas that first appeared in those three works. However, in this paper we proceed entirely in the framework of (C, F)-actions for locally compact second countable (l.c.s.c.) Abelian groups, and in fact we develop a large part of the theory in the more general context of these actions. In particular, we encounter here some new problems that are specific to higher dimensions and the continuity of the groups. Recall that the (C, F)-construction of finite measure-preserving actions of discrete countable amenable groups appeared in [10] as an algebraic counterpart of the "geometrical" cutting-and-stacking method developed for $\mathbb{Z}$-actions. Later it was used (in a modified form) by the authors in the framework of infinite measure-preserving and non-singular countable Abelian group actions, as a convenient tool for modeling examples and counterexamples with various properties of weak mixing and multiple recurrence (see [6,7,9]).

Let G be a non-compact l.c.s.c. Abelian group and $T = (T_g)_{g\in G}$ a measurable action of G on a standard probability space $(X, \mathfrak{B}, \mu)$.

Definition 0.1. T is said to be *mixing* if for all subsets $A, B \in \mathfrak{B}$ we have

$$\lim_{g\to\infty} \mu(T_g A \cap B) = \mu(A)\mu(B). \tag{0.1}$$

A sequence $g_n \to \infty$ in G is called *mixing* if (0.1) holds along g_n as $n \to \infty$.

Notice that an action is mixing whenever each sequence converging to infinity in G contains a mixing subsequence.

Definition 0.2.

(i) A *Rokhlin tower or column* for T is a triple (Y, f, F), where $Y \in \mathfrak{B}$, F is a relatively compact subset of G and $f : Y \to F$ is a measurable mapping such that for any Borel subset $H \subset F$ and an element $g \in G$ with $g + H \subset F$, one has $f^{-1}(g + H) = T_g f^{-1}(H)$.

(ii) We say that T is of *funny rank-one* if there exists a sequence of Rokhlin towers (Y_n, f_n, F_n) such that $\lim_{n\to\infty} \mu(Y_n) = 1$ and for any subset $B \in \mathfrak{B}$, there is a sequence of Borel subsets $H_n \subset F_n$ such that

$$\lim_{n\to\infty} \mu\big(B \Delta f_n^{-1}(H_n)\big) = 0.$$

(iii) If $G = \mathbb{R}^{d_1} \times \mathbb{Z}^{d_2}$, T is of funny rank-one and, in addition, the subsets F_n from (ii) are as follows

$$F_n = \big\{(t_1, \ldots, t_{d_1+t_2}) \in G \mid 0 \leqslant t_i < a_n \text{ for all } i = 1, \ldots, d_1 + d_2\big\}$$

for some $a_n \in \mathbb{R}$, $n = 1, 2, \ldots$, then we say that T is of *rank-one* (or *rank-one by cubes*).

It is easy to see that any funny rank-one action is ergodic.

Note that what we call funny rank-one is called rank-one by del Junco and Yassawi in case G is discrete and countable and $G \neq \mathbb{Z}$ [11]; also del Junco and Yassawi require in addition that the sequence of sets $(F_n)_{n=1}^{\infty}$ is a Følner sequence.

The paper is organized as follows. In Section 1 we extend the concept of (C, F)-action introduced for countable discrete groups (see [10,6]) to the class of l.c.s.c. Abelian ones. A special family of actions, whose mixing properties will be under investigation in subsequent sections, is defined. In Section 2 we introduce a concept of uniformly mixing sequence and prove a fundamental lemma (Lemma 2.2) linking the uniform mixing along some special sequences with Cesàro means for the 'spacer mappings'. Then we find a sufficient condition for the total ergodicity of the actions under considerations. We also start to check the uniform mixing property for some special sequences. In particular, we show that if a sequence is of 'moderate growth' relative to a fixed Følner sequence in G then (under some extra conditions on G and the action) it is uniformly mixing (Lemma 2.9). Section 3 is devoted to the actions with restricted growth—the property which was phrased explicitly in [5] for $G = \mathbb{Z}$ but used already in [1] and [3] in an implicit form. We note that our definition of restricted growth differs from that introduced in [5] (the latter does not extend from $\mathbb{Z}$ to arbitrary l.c.s.c. Abelian group actions). However, they are equivalent for polynomial staircase actions. Theorem 3.5 provides a sufficient condition for the (C, F)-actions with restricted growth to be mixing. We also include here a couple of statements (Lemmas 3.9–3.11) facilitating verification of this condition for the $\mathbb{R}^{d_1} \times \mathbb{Z}^{d_2}$-actions to be constructed in the next section. Section 4 contains the main results of the paper: Theorems 4.9–4.11 and 4.13 which provide families of mixing rank-one actions of $\mathbb{R}^d$ with $d > 1$, $\mathbb{R}$, $\mathbb{Z}^d$ and $\mathbb{R}^{d_1} \times \mathbb{Z}^{d_2}$ respectively. Every such action is determined completely by a sequence of positive integers $(r_n)_{n=1}^{\infty}$ (corresponding to the sequence of 'cuts' in the cutting-and-stacking construction) and a sequence $(s_n)_{n=1}^{\infty}$ of 'monotonic' polynomials of $d_1 + d_2$ variables (corresponding to the sequence of 'spacer's maps' on the n-th step). The sequences are chosen in the following way: $(r_n)_{n=1}^{\infty}$ is any sequence of sub-exponential growth with $\lim_{n\to\infty} r_n = \infty$ and $(s_n)_{n=1}^{\infty}$ consists of some specially selected quadratic polynomials from Example 4.2. Moreover, if $d_1 \neq 1$ (and only in this case) then $(s_n)_{n=1}^{\infty}$ can be chosen constant. If $d_1 = 1$ then $(s_n)_{n=1}^{\infty}$ can be chosen consisting of two alternating polynomials. Furthermore, using our techniques plus the Hilbertian van der Corput trick we can also treat a more complicated case where $(s_n)_{n=1}^{\infty}$ consists of polynomials of degree > 2 (see Proposition 4.14). Example 4.15 provides a family of rank-one mixing transformations including the polynomial staircases from [5]. In the final section (Section 5) we show that the actions constructed in Section 4 have 'very weak' stochastic properties—the entropy of any individual transformation from such actions is zero. This fact holds for any rank-one (by cubes) action. However, it is no longer true for a more general class of actions of rank-one 'by rectangles' (see [20] for a counterexample).

1. (C, F)-actions of locally compact Abelian groups

In this section we introduce the (C, F)-actions of l.c.s.c. Abelian groups and specify a subclass of them (see Definitions 1.2 and 1.4). We explain how the classical cutting-and-stacking transformations are included into this subclass (Remark 1.6). The aim of the paper is to show that this subclass contains mixing actions.

Let G be a l.c.s.c. Abelian group. Denote by λ_G a (σ-finite) Haar measure on it. Given two subsets $E, F \subset G$, by $E + F$ we mean their algebraic sum, i.e., $E + F = \{e + f \mid e \in E, f \in F\}$. The algebraic difference $E - F$ is defined in a similar way. We hope that the reader will not confuse it with the set theoretical difference $E \setminus F$. If E is a singleton, say $E = \{e\}$, then we will write $e + F$ for $E + F$. If $(E - E) \cap (F - F) = \{0\}$ then E and F are called *independent*. For an element $g \in G$ and a subset $E \subset G$, we set $E(g) = E \cap (E - g)$.

To define a (C, F)-action of G we need two sequences $(F_n)_{n\geqslant 0}$ and $(C_n)_{n>0}$ of subsets in G such that the following hold

$$F_0 \subset F_1 \subset F_2 \subset \cdots \text{ is a Følner sequence in } G, \tag{1.1}$$

$$C_n \text{ is finite and } \#C_n > 1, \tag{1.2}$$

$$F_n + C_{n+1} \subset F_{n+1}, \tag{1.3}$$

$$F_n \text{ and } C_{n+1} \text{ are independent.} \tag{1.4}$$

We put $X_n := F_n \times \prod_{k>n} C_k$, endow X_n with the standard product Borel σ-algebra and define a Borel embedding $X_n \to X_{n+1}$ by setting

$$(f_n, c_{n+1}, c_{n+2}, \ldots) \mapsto (f_n + c_{n+1}, c_{n+2}, \ldots). \tag{1.5}$$

Then we have $X_1 \subset X_2 \subset \cdots$. Hence $X := \bigcup_n X_n$ endowed with the natural Borel σ-algebra, say $\mathfrak{B}$, is a standard Borel space. Given a Borel subset $A \subset F_n$, we denote the set

$$\left\{x \in X \mid x = (f_n, c_{n+1}, c_{n+2} \ldots) \in X_n \text{ and } f_n \in A\right\}$$

by $[A]_n$ and call it an *n-cylinder*. It is clear that the σ-algebra $\mathfrak{B}$ is generated by the family of all cylinders.

Now we are going to define a measure on $(X, \mathfrak{B})$. Let κ_n stand for the equidistribution on C_n and $\nu_n := (\#C_1 \cdots \#C_n)^{-1}\lambda_G \restriction F_n$ on F_n. We define a product measure μ_n on X_n by setting

$$\mu_n = \nu_n \times \kappa_{n+1} \times \kappa_{n+2} \times \cdots,$$

$n \in \mathfrak{N}$. Then the embeddings (1.5) are all measure preserving. Hence a σ-finite measure μ on X is well defined by the restrictions $\mu \restriction X_n = \mu_n$, $n \in \mathbb{N}$. To put it in another way, $(X, \mu) = \operatorname{inj\,lim}_n (X_n, \mu_n)$. Since

$$\mu_{n+1}(X_{n+1}) = \frac{\nu_{n+1}(F_{n+1})}{\nu_{n+1}(F_n + C_{n+1})}\mu_n(X_n) = \frac{\lambda_G(F_{n+1})}{\lambda_G(F_n)\#C_{n+1}}\mu_n(X_n),$$

it follows that μ is finite if and only if

$$\prod_{n=0}^{\infty} \frac{\lambda_G(F_{n+1})}{\lambda_G(F_n)\#C_{n+1}} < \infty, \quad \text{i.e.,} \quad \sum_{n=0}^{\infty} \frac{\lambda_G(F_{n+1} \setminus (F_n + C_{n+1}))}{\lambda_G(F_n)\#C_{n+1}} < \infty. \tag{1.6}$$

For the rest of the paper we will assume that (1.6) is satisfied. Moreover, we choose (i.e., normalize) λ_G in such a way that $\mu(X) = 1$.

To construct a measure-preserving action of G on (X, μ), we fix a filtration $K_1 \subset K_2 \subset \cdots$ of G by compact subsets. Thus $\bigcup_{m=1}^{\infty} K_m = G$. Given $n, m \in \mathbb{N}$, we set

$$D_m^{(n)} := \left(\bigcap_{k \in K_m} (F_n - k) \cap F_n \right) \times \prod_{k>n} C_k \subset X_n$$

and

$$R_m^{(n)} := \left(\bigcap_{k \in K_m} (F_n + k) \cap F_n \right) \times \prod_{k>n} C_k \subset X_n.$$

It is easy to verify that

$$D_{m+1}^{(n)} \subset D_m^{(n)} \subset D_m^{(n+1)} \quad \text{and} \quad R_{m+1}^{(n)} \subset R_m^{(n)} \subset R_m^{(n+1)}.$$

We define a Borel mapping

$$K_m \times D_m^{(n)} \ni (g, x) \mapsto T_{m,g}^{(n)} x \in R_m^{(n)}$$

by setting for $x = (f_n, c_{n+1}, c_{n+2}, \ldots)$,

$$T_{m,g}^{(n)}(f_n, c_{n+1}, c_{n+2}, \ldots) := (g + f_n, c_{n+1}, c_{n+2}, \ldots).$$

Now let $D_m := \bigcup_{n=1}^{\infty} D_m^{(n)}$ and $R_m := \bigcup_{n=1}^{\infty} R_m^{(n)}$. Then a Borel mapping

$$K_m \times D_m \ni (g, x) \mapsto T_{m,g} x \in R_m$$

is well defined by the restrictions $T_{m,g} \restriction D_m^{(n)} = T_{m,g}^{(n)}$ for $g \in K_m$ and $n \geqslant 1$. It is easy to see that $D_m \supset D_{m+1}$, $R_m \supset R_{m+1}$ and $T_{m,g} \restriction D_{m+1} = T_{m+1,g}$ for all m. It follows from (1.1) that $\mu_n(D_m^{(n)}) \to 1$ and $\mu_n(R_m^{(n)}) \to 1$ as $n \to \infty$. Hence $\mu(D_m) = \mu(R_m) = 1$ for all $m \in \mathbb{N}$. Finally we set $\widehat{X} := \bigcap_{m=1}^{\infty} D_m \cap \bigcap_{m=1}^{\infty} R_m$ and define a Borel mapping

$$T : G \times \widehat{X} \ni (g, x) \to T_g x \in \widehat{X}$$

by setting $T_g x := T_{m,g} x$ for some (and hence any) m such that $g \in K_m$. It is clear that $\mu(\widehat{X}) = 1$.

Proposition 1.1. *$T = (T_g)_{g\in G}$ is a free Borel measure preserving action of G on a conull subset of the standard probability space $(X, \mathfrak{B}, \mu)$. It is of funny rank-one.*

Proof. It suffices to verify only the latter claim. According to Definition 0.2 we have to find a sequence of Rokhlin towers 'appoximating' the dynamical system. Let s_n denote the projection of $X_n = F_n \times C_{n+1} \times \cdots$ onto the first coordinate. It is easy to see that the sequence (X_n, s_n, F_n) is as desired. □

Throughout the paper we will not distinguish between two measurable sets (or mappings) which agree almost everywhere. It is easy to see that T does not depend on the choice of filtration $(K_m)_{m=1}^{\infty}$.

Definition 1.2. T is called the *(C, F)-action of G associated with $(C_n, F_n)_n$*.

We will often use the following simple properties of (X, μ, T): for Borel subsets $A, B \subset F_n$,

$$[A \cap B]_n = [A]_n \cap [B]_n, \tag{1.7}$$

$$[A]_n = [A + C_{n+1}]_{n+1} = \bigsqcup_{c\in C_{n+1}} [A + c]_{n+1}, \tag{1.8}$$

$$T_g[A]_n = [A + g]_n \quad \text{if } A + g \subset F_n, \tag{1.9}$$

$$\mu\big([A]_n\big) = \#C_{n+1} \cdot \mu\big([A + c]_{n+1}\big) \quad \text{for every } c \in C_{n+1}, \tag{1.10}$$

$$\mu\big([A]_n\big) \leqslant \frac{\lambda_G(A)}{\lambda_G(F_n)}, \tag{1.11}$$

where the sign $\sqcup$ means the union of mutually disjoint sets.

Recall that an action T of G on $(X, \mathfrak{B}, \mu)$ is *partially rigid* if there exists $\delta > 0$ with

$$\liminf_{g\to\infty} \mu(T_g B \cap B) \geqslant \delta\mu(B) \quad \text{for all } B \in \mathfrak{B}.$$

It is clear that partial rigidity is incompatible with the mixing. For the (C, F)-actions, there is a simple condition that implies the partial rigidity.

Proposition 1.3. *If $\liminf_{n\to\infty} \#C_n < \infty$ then T is partially rigid and hence is not mixing.*

Proof. Let $n_i < n_2 < \cdots$ be a sequence of indices with $\#C_{n_1} = \#C_{n_2} = \cdots$. Select $c_i \neq c_i'$ in C_i and set $g_i := c_i - c_i'$. Then $g_i \notin F_{n_i} - F_{n_i}$ by (1.4). On the other hand, it follows from (1.1) that $\bigcup_{i=1}^{\infty}(F_{n_i} - F_{n_i}) = G$. Hence $g_i \to \infty$ as $i \to \infty$. Take a cylinder $B \in \mathfrak{B}$. We can represent it eventually (i.e., for all large enough i) as $B = [B_i]_{n_i}$, where B_i is a Borel subset of F_{n_i}. If follows from (1.7)–(1.10) that

$$\begin{aligned}
\mu(T_{g_i} B \cap B) &= \mu\big(T_{g_i}[B_{i-1} + C_{n_i}]_{n_i} \cap [B_{i-1} + C_{n_i}]_{n_i}\big) \\
&\geqslant \mu\big(T_{g_i}[B_{i-1} + c_i']_{n_i} \cap [B_{i-1} + c_i]_{n_i}\big) \\
&= \mu\big([B_{i-1} + c_i]_{n_i}\big) = \frac{1}{\#C_{n_i}}\mu\big([B_{i-1}]_{n_i-1}\big) \\
&= \frac{1}{\#C_{n_1}}\mu(B).
\end{aligned}$$

Since the cylinders generate a dense subalgebra in $\mathfrak{B}$, we are done. □

Now we isolate a special subfamily of (C, F)-actions to show in the sequel that it contains mixing actions.

Let H be a discrete countable group, and let ϕ_n, s_n and c_{n+1} be three mappings from H to G such that ϕ_n is a homomorphism, $s_n(0) = 0$ and $c_{n+1} := \phi_n + s_n$, $n \in \mathbb{N}$. Suppose that

$$(H_n)_{n\geqslant 0} \text{ is a Følner sequence in } H,\ 0 \in H_n \tag{1.12}$$

and

$$\phi_n(H) \text{ is a lattice in } G. \tag{1.13}$$

Now we define $F_n \subset G$ to be a Borel fundamental domain for $\phi_n(H)$ (i.e., a subset which meets every $\phi_n(H)$-coset exactly once) and put $C_{n+1} := c_{n+1}(H_n)$, $n \geqslant 0$. Assume that (1.1)–(1.4) are all satisfied.

Definition 1.4. We call the corresponding (C, F)-action T of G on the probability space $(X, \mathfrak{B}, \mu)$ *the action associated with* $(H_n, \phi_n, s_n, F_n)_n$.

In view of Proposition 1.3, we will always assume that $\lim_{n\to\infty} \# H_n = \infty$.

Notice also that if s_n are all trivial, i.e. $s_n(h) = 0$ for all $h \in H_n$ then the action of G associated with (H_n, ϕ_n, s_n, F_n) has pure point spectrum with rational eigenvalues only. This simple fact will not be used in this paper. We leave its proof to the reader.

In the statements of our main results here it will be assumed that the mappings s_n are polynomials of degree > 1.

Definition 1.5. [16] For any $h \in H$, the *h-derivative* of s is a mapping $\partial_h s : H \to G$ given by $\partial_h s(k) = s(k+h) - s(k)$. Let d be a non-negative integer. Then s is called a *polynomial of degree* $\leqslant d$ if for any $h_1, \ldots, h_{d+1} \in H \setminus \{0\}$, we have $\partial_{h_1} \cdots \partial_{h_{d+1}} s = 0$. The minimal d with this property is called *the degree* of s.

It is easy to see that every polynomial of degree 0 is constant. As was shown in [16], a polynomial of degree one is a non-constant affine mapping (i.e., a homomorphism plus a constant). A polynomial from $\mathbb{Z}^d$ to $\mathbb{R}^l$ is an l-tuple of usual polynomials in d variables with real coefficients. A polynomial from $\mathbb{Z}^d$ to $\mathbb{Z}^l$ is an l-tuple $(p_1, \ldots, p_l)$ of usual polynomials in d variables with rational coefficients such that $p_i(\mathbb{Z}^d) \subset \mathbb{Z}$ for all $i = 1, \ldots, l$.

Remark 1.6. Here we are going to explain how the (C, F)-construction for $\mathbb{Z}$-actions is related to the classical cutting-and-stacking construction. Recall that the latter one defines ergodic measure-preserving transformations on intervals in $\mathbb{R}$ (or on $[a, +\infty)$) furnished with Lebesgue measure via an inductive procedure. A *column* is an ordered collection of intervals, called *levels*, of the same length. The number of levels is called the *height* of the column. The associated *column mapping* is defined by translation of each level to the level above it (i.e., next in the order). Hence the column mapping is defined from all but the top level onto all but the bottom level. Suppose now that we are given a sequence $(r_n)_{n=1}^{\infty}$ of positive integers and a sequence of arrays of non-negative integers $(\sigma_n(j),\ j = 0, 1, \ldots, r_n - 1)_{n=1}^{\infty}$. Then we define inductively a sequence of columns as follows. Let the initial column Y_0 consists of one level of length 1. Suppose that on the n-th step we have a column Y_n consisting of levels $I(i,n)$, $0 \leqslant i < a_n$. Cut every $I(i,n)$ into r_n sublevels $I_k(i,n)$, $0 \leqslant k < r_n$, numbered from left to right. Then we obtain r_n subcolumns $Y_{n,k} := \{I_k(i,n) \mid i = 0, \ldots, a_n - 1\}$, $0 \leqslant k < r_n$, of Y_n of the same height. Now place $\sigma_n(k)$ *spacers* (i.e., the intervals of the same length as $I_k(i,n)$) above $Y_{n,k}$ and stack the resulting subcolumns with spacers right to the top of left. This yields a new column Y_{n+1} of height $a_{n+1} = a_n r_n + \sum_{k=0}^{r_n-1} \sigma_n(k)$ and a natural inclusion of Y_n into Y_{n+1}. Notice that the associated $(n+1)$-column mapping restricted to Y_n coincides with the n-th column mapping. Hence the associated sequence of column mappings approaches a transformation defined on all but a measure zero subset of the union of the initial level and the spacers added at each column. It is easy to see that this transformation corresponds exactly to the (C, F)-action of $\mathbb{Z}$ associated with $(H_n, \phi_n, s_n, F_n)_n$ if we put $H_n := \{0, 1, \ldots, r_n - 1\}$, $\phi_n(t) := a_n t$, $s_n(t) := \sum_{k=0}^{t} \sigma_n(k)$ and $F_n = \{0, 1, \ldots, a_n - 1\}$. If we set $\sigma_n(k) = k$ for all $0 \leqslant k < r_n$ and $n \in \mathbb{N}$ then the corresponding cutting-and-stacking transformation is called a *staircase*. If, moreover, $r_n = n$ for all $n \in \mathbb{N}$, we obtain the *classical staircase* which is finite measure-preserving. In case the sequence $(\sigma_n)_{n=1}^{\infty}$ consists of polynomials, the corresponding transformations are called *polynomial staircases* [5].

2. Uniformly mixing sequences

For the remaining of the paper $(X, \mathfrak{B}, \mu, T)$ will stand for the (C, F)-action of G associated to a sequence $(H_n, \phi_n, s_n, F_n)_n$.

In this section we prove a fundamental Lemma 2.2 and use it to show that some special sequences in G are uniformly mixing. As an auxiliary result for that we exhibit a sufficient condition for the total ergodicity of T. A connection between the uniform mixing and total ergodicity is established in Corollary 2.6.

Definition 2.1. A sequence $(g_n)_{n=1}^{\infty}$ of elements from G is called *uniformly mixing* for T if

$$\sup_{A\subset F_n} \bigl|\mu\bigl(T_{g_n}[A]_n \cap B\bigr) - \mu\bigl([A]_n\bigr)\mu(B)\bigr| \to 0 \quad \text{as } n\to\infty \tag{2.1}$$

for every subset $B \in \mathfrak{B}$.

It is easy to see that if a sequence is uniformly mixing then it is mixing.

Notice that a somewhat different definition for uniform mixing was given in [5] in case $G=\mathbb{Z}$. To state it precisely we assume that the sequence $(H_n, \phi_n, s_n, F_n)_n$ is chosen as described in Remark 1.6. Then a sequence of positive integers $(g_n)_{n=1}^{\infty}$ was called uniformly mixing in [5] if

$$\sum_{f\in F_{p_n}} \bigl|\mu\bigl(T_{g_n}[f]_{p_n} \cap B\bigr) - \mu\bigl([f]_{p_n}\bigr)\mu(B)\bigr| \to 0$$

for every subset $B \in \mathfrak{B}$, where p_n is the unique positive integer such that $a_{p_n} \leqslant g_n < a_{p_n+1}$. We only observe that this implies the uniform mixing in the sense of Definition 2.1 if $p_n \geqslant n$ for all n.

Let L be a finite set and $a: L \to G$ a mapping. We define a linear operator $\mathcal{M}_{a,L}$ in $L^2(X,\mu)$ by setting

$$\mathcal{M}_{a,L}(f) := \frac{1}{\#L}\sum_{l\in L} f\circ T_{a(l)}.$$

Let P_0 stand for the projection onto the subspace of constant functions, i.e. $P_0(f) = \int_X f\,\mathrm{d}\mu$. The inner product in $L^2(X,\mu)$ will be denoted by $\langle\cdot,\cdot\rangle$.

Fix a sequence $(h_n)_{n\geqslant 1}$ of elements from H. For brevity, we will denote $\partial_{h_n} s_n$ by s_n', $n\in\mathbb{N}$.

Lemma 2.2. *If the following conditions are satisfied*

$$\frac{\#H_n(h_n)}{\#H_n} \to 1, \tag{2.2}$$

$$\frac{1}{\#H_n}\sum_{h\in H_n(h_n)} \frac{\lambda_G(F_n \setminus F_n(s_n'(h)))}{\lambda_G(F_n)} \to 0 \tag{2.3}$$

then for all n-cylinders $A, B\subset X$, we have

$$\begin{aligned}\mu(T_{\phi_n(h_n)}A\cap B) &= \frac{1}{\#H_n}\sum_{h\in H_n(h_n)} \mu(A\cap T_{s_n'(h)}B) + \bar{o}(1)\\ &= \bigl\langle \chi_A, \mathcal{M}_{s_n', H_n(h_n)}(\chi_B)\bigr\rangle + \bar{o}(1),\end{aligned}$$

where χ_A and χ_B are the indicators of A and B respectively and $\bar{o}(1)$ denotes a sequence that tends to 0 and that does not depend on A and B. The same formula holds as well for

(i) *an arbitrary subset $B\in\mathfrak{B}$ and A as above with $\bar{o}(1)$ depending on B only and*
(ii) *arbitrary subsets $A, B\in\mathfrak{B}$ with $\bar{o}(1)$ depending on both A and B.*

Proof. Let A_n and B_n be the Borel subsets of F_n such that $A=[A_n]_n$ and $B=[B_n]_n$. For $h\in H_n(h_n)$, we put $A_{n,h} := A_n \cap F_n(-s_n'(h))$. Then

$$A_{n,h} - s_n'(h) \subset F_n. \tag{2.4}$$

We also make a simple but important observation that

$$\begin{aligned}\phi_n(h_n) + c_{n+1}(h) &= \phi_n(h_n) + \phi_n(h) + s_n(h)\\ &= \phi_n(h+h_n) + s_n(h+h_n) - s_n'(h)\\ &= c_{n+1}(h+h_n) - s_n'(h).\end{aligned} \tag{2.5}$$

It follows from (1.8), (1.9), (2.4), (1.3), (2.5) and (1.10) that

$$\begin{aligned}\mu(T_{\phi_n(h_n)}A\cap B) &= \sum_{h\in H_n}\mu\big(T_{\phi_n(h_n)}\big[A_n+c_{n+1}(h)\big]_{n+1}\cap[B_n]_n\big)\\ &= \sum_{h\in H_n(h_n)}\big(\mu\big(T_{\phi_n(h_n)}\big[A_{n,h}+c_{n+1}(h)\big]_{n+1}\cap[B_n]_n\big)\\ &\quad\pm\mu\big(\big[(A_n\setminus A_{n,h})+c_{n+1}(h)\big]_{n+1}\big)\big)\pm\sum_{h\in H_n\setminus H_n(h_n)}\mu\big(\big[F_n+c_{n+1}(h)\big]_{n+1}\big)\\ &= \sum_{h\in H_n(h_n)}\Big(\mu\big(\big[A_{n,h}-s'_n(h)+c_{n+1}(h+h_n)\big]_{n+1}\cap[B_n]_n\big)\pm\frac{1}{\#H_n}\mu\big([A_n\setminus A_{n,h}]_n\big)\Big)\\ &\quad\pm\Big(1-\frac{\#H_n(h_n)}{\#H_n}r\Big).\end{aligned}$$

Notice that for all $c\in C_{n+1}$ and $h\in H_n(h_n)$, we have by (1.7), (1.8) and (1.11),

$$[A_{n,h}-s'_n(h)+c]_{n+1}\cap[B_n]_n=\big[\big(\big(A_{n,h}-s'_n(h)\big)\cap B_n\big)+c\big]_{n+1} \tag{2.6}$$

and

$$\mu\big(\big[(A_n\setminus A_{n,h})\big]_n\big)\leqslant\frac{\lambda_G(A_n\setminus A_{n,h})}{\lambda_G(F_n)}\leqslant\frac{\lambda_G(F_n\setminus F_n(s'_n(h)))}{\lambda_G(F_n)}. \tag{2.7}$$

Hence it follows from (1.10), (2.2), (2.3), (2.6) and (2.7) that

$$\mu(T_{\phi_n(h_n)}A\cap B)=\frac{1}{\#H_n}\sum_{h\in H_n(h_n)}\mu\big(\big[\big(A_{n,h}-s'_n(h)\big)\cap B_n\big]_n\big)+\bar{o}(1).$$

Applying (1.7), (1.9) and (2.4) we obtain

$$\begin{aligned}\mu(T_{\phi_n(h_n)}A\cap B) &= \frac{1}{\#H_n}\sum_{h\in H_n(h_n)}\mu\big(T_{-s'_n(h)}[A_{n,h}]_n\cap[B_n]_n\big)+\bar{o}(1)\\ &= \frac{1}{\#H_n}\sum_{h\in H_n(h_n)}\big(\mu(A\cap T_{s'_n(h)}B)\pm\mu\big([A_n\setminus A_{n,h}]_n\big)\big)+\bar{o}(1).\end{aligned}$$

It remains to make use of (2.7), (2.3) and (2.2).

The final claim of Lemma 2.2 follows from the fact that the cylinders generate a dense subalgebra in $\mathfrak{B}$. □

Corollary 2.3. *Let* (2.2) *and* (2.3) *hold. Then the following are satisfied*:

(i) *The sequence* $(\phi_n(h_n))_{n=1}^{\infty}$ *is mixing for* T *if and only if* $\mathcal{M}_{s'_n,H_n(h_n)}\to P_0$ *in the weak operator topology.*
(ii) *If* $\mathcal{M}_{s'_n,H_n(h_n)}\to P_0$ *in the strong operator topology then* $(\phi_n(h_n))_{n=1}^{\infty}$ *is uniformly mixing.*

We now examine when T is totally ergodic. Recall some standard definitions.

Definition 2.4.

(i) Given a subset $B\in\mathfrak{B}$, we denote by G_B the *stabilizer* of B, i.e., $G_B:=\{g\in G\mid T_gB=B\}$.
(ii) T is called *totally ergodic* if for any co-compact subgroup $K\subset G$, the action $(T_g)_{g\in K}$ is ergodic.
(iii) T is called *weakly mixing* if the diagonal action $(T_g\times T_g)_{g\in G}$ of G is ergodic. Equivalently, if there exist a function $f\in L^2(X,\mu)$ and a continuous character χ of G such that $f\circ T_g=\chi(g)f$ a.e. then f is constant.

It is easy to see that T is totally ergodic if and only if the stabilizer of any subset $B\in\mathfrak{B}$ with $0<\mu(B)<1$ is not co-compact. Moreover, if an action is weakly mixing then it is totally ergodic. The converse is true for $G=\mathbb{R}$ but it does not hold for general groups.

Proposition 2.5. *Let* (2.2) *and* (2.3) *hold. Let K be a co-compact subgroup of G and $\pi : G \to G/K$ stand for the corresponding quotient map. Denote by κ_n the image of the equidistributed probability on $H_n(h_n)$ under the mapping $(\pi \circ s'_n)_*$, $n \in \mathbb{N}$. If κ_n does not $*$-weakly converge to a Dirac δ-measure on G/K then K is not the stabilizer of any measurable subset $B \in \mathfrak{B}$ with $0 < \mu(B) < 1$.*

Proof. Suppose that the contrary holds, i.e., there exists $B \in \mathfrak{B}$ with $0 < \mu(B) < 1$ and $K = G_B$. Then the quotient compact group G/K acts naturally on the sub-σ-algebra $\mathcal{F}$ of $(T_g)_{g\in K}$-invariant subsets. Denote this action by $\widehat{T}$. Then $\widehat{T}_{\pi(g)}A := T_g A$ for all $g \in G$ and $A \in \mathfrak{F}$. It is clear that $\widehat{T}$ is free. We set $a_n := \pi(\phi_n(h_n))$. Passing to a subsequence, if necessary, we may assume without loss of generality that a_n converges to some $a \in G$. Then $\mu(\widehat{T}_{a_n} B \Delta \widehat{T}_a B) \to 0$ as $n \to \infty$. Hence

$$\mu(B) = \mu(T_{\phi_n(h_n)}B) = \mu\big(T_{\phi_n(h_n)}B \cap \widehat{T}_{a_n}B\big) = \mu\big(T_{\phi_n(h)}B \cap \widehat{T}_a B\big) + \bar{o}(1).$$

We then deduce from this formula and Lemma 2.2(ii) that

$$\begin{aligned}\mu(B) &= \frac{1}{\#H_n}\sum_{h\in H_n(h_n)} \mu\big(B \cap T_{s'_n(h)}\widehat{T}_a B\big) + \bar{o}(1)\\ &= \frac{1}{\#H_n(h_n)}\sum_{h\in H_n(h_n)} \mu\big(\widehat{T}_{\pi(s'_n(h))+a}B \cap B\big) + \bar{o}(1)\\ &= \int_{G/K} \mu\big(\widehat{T}_{a+b}B \cap B\big)\,\mathrm{d}\kappa_n(b) + \bar{o}(1). \end{aligned} \tag{2.8}$$

Since G/K is compact, we may assume (passing to a subsequence, if necessary) that κ_n converges $*$-weakly to a probability κ which is not a Dirac δ-measure by the condition of the proposition. Hence passing to a limit in (2.8) we obtain

$$\mu(B) = \int_{G/H} \mu\big(\widehat{T}_{a+b}B \cap B\big)\,\mathrm{d}\kappa(b)$$

Hence $\mu(B) = \mu(\widehat{T}_{a+b}B \cap B)$, i.e., $B = \widehat{T}_{a+b}B$, for κ-a.a. $b \in G/K$. Since $\widehat{T}$ is free, we deduce that $\operatorname{Supp}\kappa = \{-a\}$. Hence $\operatorname{Supp}\kappa$ is a singleton, a contradiction. □

Now we are interested in the following particular case. There exist a non-zero $k \in H$ and a polynomial $s : H \to G$ of degree 2 such that $h_n = k$ and $s_n = s$ for all $n \in \mathbb{N}$. Then, of course, (2.2) is satisfied. Moreover, for any non-zero $t \in H$, we have

$$\partial_t s(h) = \psi_t(h) + a_t, \quad \text{at all } h \in H$$

for some non-trivial homomorphism $\psi_t : H \to G$ and an element $a_t \in G$ (see the text following Definition 1.5). Hence $s'_n(h) = \psi_k(h) + a_k$ for all $h \in H$ and $n \in \mathbb{N}$.

Corollary 2.6. *Let the above assumptions and* (2.3) *hold. Then the following are satisfied*:

(i) *If the action $(T_{\psi_k(h)})_{h\in H}$ is ergodic then the sequence $(\phi_n(k))_{n=1}^{\infty}$ is uniformly mixing.*
(ii) *If the subgroup generated by $\bigcup_{t\in H}\psi_t(H)$ is dense in G then T is totally ergodic.*
(iii) *If the subgroup generated by $\bigcup_{t\in H}\psi_t(H)$ is dense in G and $\psi_k(H)$ is a lattice in G then the sequence $(\phi_n(k))_{n=1}^{\infty}$ is uniformly mixing.*

Proof. (i) follows from Corollary 2.3 and the von Neumann mean ergodic theorem for $(T_{\psi_k(h)})_{h\in H}$.

(ii) Suppose that the contrary holds, i.e., there exists a co-compact subgroup $K \subset G$ and a subset $B \subset X$ such that $0 < \mu(B) < 1$ and $K = G_B$. Fix $t \in H$, $t \neq 0$. Then κ_n is the translation of $(\pi \circ \psi_t)_*\nu_n$ by $\pi(a_t)$, where ν_n stands for the equidistribution on $H_n(t)$, $n \in \mathbb{N}$. Denote by G_t the closure of the subgroup $(\pi \circ \psi_t)(H)$ in G/K. Since $H_n(t)$ is a Følner sequence in H and $\pi \circ \psi_t : H \to G/K$ is a homomorphism, it is easy to verify that $(\pi \circ \psi_t)_*\nu_n$ converges

$*$-weakly to the Haar measure λ_{G_t} which is considered now as a measure on G/K supported on G_t. It follows from Proposition 2.5 that $\operatorname{Supp}\lambda_{G_t}$ is a singleton, i.e. $G_t = K$. However we deduce from the condition of the corollary that the subgroup generated by $\bigcup_{t\in H} G_t$ is dense in G/K. Hence $K = G$ which contradicts to the ergodicity of T.

(iii) follows directly from (i) and (ii). □

Remark 2.7. We will also need the following slight extension of Corollary 2.6 which is proved in a similar way. Assume that (2.3) holds for a constant sequence $h_n = k \neq 0$. Let S be a finite family of polynomials from H to G of degree 2. For all $s \in S$ and $t \in H$, we then have

$$\partial_t s(h) = \psi_t^s(h) + a_t^s, \quad h \in H,$$

where $\psi_t^s : H \to G$ is a homomorphism and a_t^s is an element of G. Let $(s_n)_{n=1}^{\infty}$ be a sequence of elements of S such that every element of S occurs in $(s_n)_{n=1}^{\infty}$ infinitely many times. Denote by T the (C, F)-action associated with $(H_n, \phi_n, s_n, F_n)_n$. Then the following are satisfied.

(i) If the action $(T_{\psi_k^s(h)})_{h\in H}$ is ergodic for any $s \in S$ then the sequence $(\phi_n(k))_{n=1}^{\infty}$ is uniformly mixing.
(ii) If the subgroup generated by $\bigcup_{s\in S}\bigcup_{t\in H}\psi_t^s(H)$ is dense in G then T is totally ergodic.
(iii) If the subgroup generated by $\bigcup_{s\in S}\bigcup_{t\in H}\psi_t^s(H)$ is dense in G and $\psi_k^s(H)$ is a lattice in G for all $s \in S$ then $(\phi_n(k))_{n=1}^{\infty}$ is uniformly mixing.

Now we return to the general case and prove two lemmas.

Lemma 2.8. *Suppose that there exists a finite subset $Q \subset H$ such that*

$$F_n + F_n \subset \phi_n(Q) + F_n \quad \text{for all } n \in \mathbb{N}. \tag{2.9}$$

Let $g_n = \phi_n(k) + f_n$ for some $k \in H$ and $f_n \in F_n$, $n \in \mathbb{N}$. If the sequence $(\phi_n(t))_{n=1}^{\infty}$ is uniformly mixing for every $t \in k + Q$ then so is the sequence $(g_n)_{n=1}^{\infty}$.

Proof. Since $f_n + F_n \subset \phi_n(Q) + F_n$ and F_n is a fundamental domain for $\phi_n(H)$, a finite partition of F_n is well defined as follows: $F_n = \bigsqcup_{q\in Q} F_n(f_n - \phi_n(q))$. Now let A be an n-cylinder in X and $A = [A_n]_n$ for some Borel subset $A_n \subset F_n$. Consider the induced partition $\bigsqcup_{q\in Q} A_{n,q}$ of A_n, where $A_{n,q} := A_n \cap F(f_n - \phi_n(q))$. Then by (1.9),

$$T_{g_n}[A_{n,q}]_n = T_{\phi_n(k+q)}T_{f_n-\phi_n(q)}[A_{n,q}]_n = T_{\phi_n(k+q)}\big[f_n - \phi_n(q) + A_q\big]_n.$$

Hence for any subset $B \in \mathfrak{B}$,

$$\mu(T_{g_n}A \cap B) - \mu(A)\mu(B) = \sum_{q\in Q}\big(\mu\big(T_{\phi_n(k+q)}[A_{n,q}]_n \cap B\big) - \mu\big([A_{n,q}]_n \cap B\big)\big).$$

It remains to use the uniform mixing of $(\phi_n(t))_{n=1}^{\infty}$, $t \in k + Q$. □

Suppose that—in addition to (2.9)—there is a finite subset $Q^\bullet \subset H$ such that

$$F_n - F_n \subset \phi_n\big(Q^\bullet\big) + F_n \quad \text{for all } n \in \mathbb{N}. \tag{2.10}$$

We will assume in the following that $Q^\bullet$ is the minimal subset with this property. It is clear that $0 \in Q^\bullet$. We set $F_n^\bullet := \phi_n(Q^\bullet) + F_n$. Then there exists a finite subset $Q_+ \subset H$ such that

$$\big(F_n^\bullet + F_n^\bullet\big) \setminus F_n^\bullet \subset \phi_n(Q_+) + F_n \quad \text{for all } n \in \mathbb{N}.$$

Again we will assume that Q_+ is the minimal subset with this property. It is clear that $Q^\bullet \cap Q_+ = \emptyset$.

Lemma 2.9. *Let* (2.9) *and* (2.10) *hold and let for any $t \in H \setminus \{0\}$, the sequence $(\phi_n(t))_{n=1}^{\infty}$ be uniformly mixing. Take $g_n \in (F_n^\bullet + F_n^\bullet) \setminus F_n^\bullet$ for all $n \in \mathbb{N}$. If for an integer $l > 0$, we have*

$$-lQ_+ \cap (\underbrace{Q + \cdots + Q}_{l \text{ times}}) = \emptyset$$

then the sequence $(lg_n)_{n=1}^{\infty}$ is uniformly mixing.

Proof. Since Q_+ is finite, we can partition the sequence g_n into finitely many subsequences of the form $\phi_n(q)+f_n$ for some $f_n\in F_n$, $n\in\mathbb{N}$, and a fixed $q\in Q_+$. Therefore it is enough to assume that g_n itself enjoys this property. Then condition (2.9) applied $l-1$ times yields

$$lg_n=\phi_n(lq)+\underbrace{f_n+\cdots+f_n}_{l\text{ times}}=\phi_n(lq)+\phi_n(q_1+\cdots+q_{l-1})+\tilde{f}_n$$

for some $q_1,\dots,q_{l-1}\in Q$ and $\tilde{f}_n\in F_n$. Since $0\notin lq+q_1+\cdots+q_{l-1}+Q$ by the condition of the lemma, we may apply Lemma 2.8 to complete the proof. □

3. Restricted growth condition

In this section we introduce a restricted growth condition for the (C,F)-actions specified in Section 2. It is an analogue of a concept considered in [5] for $G=\mathbb{Z}$. The concept, in turn, is a counterpart of a sufficient condition introduced in [1] for a staircase to be mixing. The main result here is Theorem 3.5, which provides a sufficient condition for mixing in the class of (C,F)-actions with restricted growth. We conclude the section with some technical statements needed to verify this condition for some special $\mathbb{R}^{d_1}\times\mathbb{Z}^{d_2}$-actions that will be constructed in the next section.

Definition 3.1. We say that T satisfies the *restricted growth* condition if (2.3) holds for any sequence $h_n\in H$ such that there exists $\lim_{n\to\infty}\#H_n(h_n)/\#H_n>0$.

Definition 3.2. Given two sequences $A_n, B_n\subset G$, we write $A_n\sim_n B_n$ if

$$\frac{\lambda_G(A_n\Delta B_n)}{\lambda_G(F_n)}\to 0\quad\text{as } n\to\infty.$$

It follows from (1.11) that $\mu([A_n]_n\Delta[B_n]_n)\to 0$ whenever $A_n\sim_n B_n$ and $A_n, B_n\subset F_n$.

Lemma 3.3. *Suppose that* (2.9) *and the following hold*

$$\frac{1}{\#H_n}\sum_{h\in H_n}\frac{\lambda_G(F_n\setminus F_n(s_n(h)))}{\lambda_G(F_n)}\to 0\quad \textit{as } n\to\infty. \tag{3.1}$$

Let $g_n:=\phi_n(h_n)+f_n$ for some $h_n\in H$ and $f_n\in F_n$, $n\in\mathbb{N}$. Then

$$F_{n+1}(g_n)\sim_{n+1}\bigsqcup_{h\in H_n(h_n)}\bigl(c_{n+1}(h)+F_n\bigr).$$

Proof. Since

$$\begin{aligned}\lambda_G\Bigl(F_{n+1}\Delta\bigsqcup_{h\in H_n}\bigl(\phi_n(h)+F_n\bigr)r\Bigr)&\leqslant\lambda_G\bigl(F_{n+1}\setminus(F_n+C_{n+1})\bigr)+\sum_{h\in H_n}\lambda_G\bigl(\bigl(c_{n+1}(h)+F_n\bigr)\Delta\bigl(\phi_n(h)+F_n\bigr)\bigr)\\&=\lambda_G\bigl(F_{n+1}\setminus(F_n+C_{n+1})\bigr)+\sum_{h\in H_n}\lambda_G\bigl(\bigl(s_n(h)+F_n\bigr)\Delta F_n\bigr),\end{aligned}$$

it follows from (3.1) and (1.6) that $F_{n+1}\sim_{n+1}(\phi_n(H_n)+F_n)$. For $q\in Q$, we set $F_{n,q}:=F_n(f_n-\phi_n(q))$ and $F'_{n,q}:=F_n(-f_n+\phi_n(q))$. Then (2.9) yields

$$F_n=\bigsqcup_{q\in Q}F_{n,q}=\bigsqcup_{q\in Q}F'_{n,q}. \tag{3.2}$$

It follows that

$$F_{n+1}-g_n\sim_{n+1}\bigsqcup_{h\in H_n}\bigsqcup_{q\in Q}\bigl(\phi_n(h)-\phi_n(h_n)-f_n+F'_{n,q}\bigr)=\bigsqcup_{q\in Q}\bigsqcup_{h\in H_n}\bigl(\phi_n(h-h_n-q)+F_{n,q}\bigr).$$

Therefore

$$F_{n+1}\cap(F_{n+1}-g_n)\sim_{n+1}\bigsqcup_{q\in Q}\bigsqcup_{h\in H_n(h_n+q)}\bigl(\phi_n(h)+F_{n,q}\bigr). \tag{3.3}$$

Next, since $\lambda_G(F_{n+1})\geqslant \#H_n\lambda_G(F_n)$ and

$$\begin{aligned}\frac{\lambda_G(\bigsqcup_{q\in Q}\bigsqcup_{h\in H_n(h_n+q)\Delta H_n(h_n)}(\phi_n(h)+F_{n,q}))}{\lambda_G(F_{n+1})}&\leqslant\frac{\sum_{q\in Q}\#(H_n(h_n+q)\Delta H_n(h_n))\lambda_G(F_{n,q})}{\#H_n\lambda_G(F_n)}\\&\leqslant\sum_{q\in Q}\frac{\#(H_n(h_n+q)\Delta H_n(h_n))}{\#H_n}\to 0,\end{aligned}$$

it follows from (3.3) and (3.2) that

$$F_{n+1}(g_n)\sim_{n+1}\bigsqcup_{q\in Q}\bigsqcup_{h\in H_n(h_n)}\bigl(\phi_n(h)+F_{n,q}\bigr)=\bigsqcup_{h\in H_n(h_n)}\bigl(\phi_n(h)+F_n\bigr).$$

It remains to use (3.1) once more to conclude that

$$\bigsqcup_{h\in H_n(h_n)}\bigl(\phi_n(h)+F_n\bigr)\sim_{n+1}\bigsqcup_{h\in H_n(h_n)}\bigl(c_{n+1}(h)+F_n\bigr).\qquad\square$$

Corollary 3.4. *If* (2.9) *and* (3.1) *are satisfied then*

$$(A+C_{n+1})\cap F_{n+1}(g_n)\sim_{n+1}A+c_{n+1}\bigl(H_n(h_n)\bigr)$$

uniformly in $A\subset F_n$. *Hence*

$$\sup_{A\subset F_n}\mu\Bigl(\bigl([A]_n\cap\bigl[F_{n+1}(g_n)\bigr]_{n+1}\bigr)\Delta\bigl[A+c_{n+1}\bigl(H_n(h_n)\bigr)\bigr]_{n+1}\Bigr)=\bar{o}(1).$$

Now we are going to prove the main result of the section.

Theorem 3.5. *Let* T *be a* (C,F)*-action associated with* $(H_n,\phi_n,s_n,F_n)_n$ *and satisfying* (2.9), (2.10), (3.1) *and the restricted growth condition. Let* $Q^\bullet\supset -Q$. *If* $\mathcal{M}_{s_n',H_n(h_n)}\to P_0$ *strongly for any sequence* $h_n\in H\setminus\{0\}$ *such that*

$$\liminf_{n\to\infty}\#H_n(h_n)/\#H_n>0$$

then T *is mixing.* (*Here* s_n' *stands for the* h_n*-derivative of* s_n.)

Proof. Take any sequence $g_n\to\infty$ in G. We are going to prove that it contains a mixing subsequence. Since $F_1^\bullet\subset F_2^\bullet\subset\cdots$ and $\bigcup_{n=1}^\infty F_n^\bullet=G$, we can find p_n such that $g_n\in F_{p_n+1}^\bullet\setminus F_{p_n}^\bullet$ for all $n\in\mathbb{N}$. It is clear that $p_n\to\infty$. Notice that

$$F_{n+1}^\bullet+F_{n+1}\subset\phi_{n+1}\bigl(Q^\bullet+Q\bigr)+F_{n+1}.$$

Then using a similar idea as in the proof of Lemma 2.8, we partition F_{p_n+1} as follows

$$F_{p_n+1}=\bigsqcup_{q\in Q^\bullet+Q}F_{p_n+1}\bigl(g_n-\phi_{p_n+1}(q)\bigr).$$

Take two p_n-cylinders $A,B\subset X$. Let A_m and B_m be the Borel subsets of F_m such that $A=[A_m]_m$ and $B=[B_m]_m$ for all $m\geqslant p_n$. For $q\in Q^\bullet+Q$, we put $A_{p_n+1,q}:=A_{p_n+1}\cap F_{p_n+1}(g_n-\phi_{p_n}(q))$. We have now

$$T_{g_n}[A_{p_n+1,q}]_{p_n+1}=T_{\phi_{p_n+1}(q)}\bigl[A_{p_n+1,q}+g_n-\phi_{p_n+1}(q)\bigr]_{p_n+1} \tag{3.4}$$

and $A_{p_n+1}=\bigsqcup_{q\in Q^\bullet+Q}A_{p_n+1,q}$. For any $q\in Q^\bullet+Q$, the constant sequence $h_n:=q$, $n\in\mathbb{N}$ satisfies (2.2). Moreover, it satisfies (2.3) since T has restricted growth (we need this to apply Corollary 2.3 later). By the condition of the

theorem, $\mathcal{M}_{\partial_q s_n, H_n(q)} \to P_0$ strongly whenever $q \neq 0$. Therefore by Corollary 2.3(ii), the sequence $(\phi_n(q))_{n=1}^{\infty}$ is uniformly mixing. Then it follows from (3.4) that

$$\sup_{A \text{ is a } p_n\text{-cylinder}} \left|\mu\big(T_{g_n}[A_{p_n+1,q}]_{p_n+1} \cap B\big) - \mu\big([A_{p_n+1,q}]_{p_n+1}\big)\mu(B)\right| \to 0 \tag{3.5}$$

for every $q \in (Q^\bullet + Q) \setminus \{0\}$.

It remains to consider the case $q = 0$ which is more involved. Let us represent g_n as $g_n = \phi_{p_n}(h_n) + f_n$ for some (uniquely determined) $h_n \in H \setminus Q^\bullet$ and $f_n \in F_{p_n}$. Since $A_{p_n+1,0} = A_{p_n+1} \cap F_{p_n+1}(g_n)$ and (3.1) holds, it follows from Corollary 3.4 that

$$\sup_{A \text{ in a } p_n\text{-cylinder}} \left|\mu\big([A_{p_n+1,0}]_{p_n+1}\big) - \mu(A)\frac{\#H_{p_n}(h_n)}{\#H_{p_n}}\right| = \bar{o}(1). \tag{3.6}$$

Hence, if $\liminf_{n\to\infty} \#H_{p_n}(h_n)/\#H_{p_n} = 0$,

$$\liminf_{n\to\infty} \sup_{A \text{ in a } p_n\text{-cylinder}} \mu\big([A_{p_n+1,0}]_{p_n+1}\big) = 0.$$

Therefore passing to a subsequence in $(p_n)_{n=1}^{\infty}$ we conclude that (3.5) is also true for $q = 0$. This implies that the corresponding subsequence of $(g_n)_{n=1}^{\infty}$ is mixing. Thus from now on we may assume that $\liminf_{n\to\infty} \#H_{p_n}(h_n)/\#H_{p_n} > 0$. It follows from Corollary 3.4 that

$$\mu\big(T_{\phi_{p_n}(h_n)}[A_{p_n+1,0}]_{p_n+1} \cap B\big) = \mu\big(T_{\phi_{p_n}(h_n)}\big[A_{p_n} + c_{p_n+1}\big(H_{p_n}(h_n)\big)\big]_{p_n+1} \cap B\big) + \bar{o}(1),$$

where $\bar{o}(1)$ does not depend on the p_n-cylinders A and B. Using that together with the restricted growth condition and arguing almost literally as in the proof of Lemma 2.2 we obtain

$$\begin{aligned}\mu\big(T_{\phi_{p_n}(h_n)}[A_{p_n+1,0}]_{p_n+1} \cap B\big) &= \frac{1}{\#H_{p_n}} \sum_{h \in H_{p_n}(h_n)} \mu\big(T_{-s'_n(h)}[A_{p_n}]_{p_n} \cap B\big) + \bar{o}(1)\\ &= \frac{\#H_{p_n}(h_n)}{\#H_{p_n}}\big\langle \chi_A, \mathcal{M}_{s'_n, H_{p_n}(h_n)}(\chi_B)\big\rangle + \bar{o}(1),\end{aligned}$$

where s'_n is the h_{p_n}-derivative of s_n and $\bar{o}(1)$ does not depend on A and B. It is easy to find a sequence $(\hat{h}_n)_{n=1}^{\infty}$ with $\liminf_{n\to\infty} \#H_n(\hat{h}_n)/\#H_n > 0$, $\hat{h}_n \neq 0$ and $\hat{h}_{p_n} = h_n$ for all $n \in \mathbb{N}$. Let $\hat{s}'_n$ denote the $\hat{h}_n$-derivative of s_n. Then $s'_n = \hat{s}'_{p_n}$. By the condition of the theorem, $\mathcal{M}_{\hat{s}'_n, H_n(\hat{h}_n)} \to P_0$ strongly. Hence $\mathcal{M}_{s'_n, H_{p_n}(h_n)} \to P_0$ strongly. Therefore we deduce from the above and (3.6) that

$$\begin{aligned}\mu\big(T_{\phi_{p_n}(h_n)}[A_{p_n+1,0}]_{p_n+1} \cap B\big) &= \frac{\#H_{p_n}(h_n)}{\#H_{p_n}} \cdot \mu(A)\mu(B) + \bar{o}(1)\\ &= \mu\big([A_{n+1,0}]_{p_n+1}\big)\mu(B) + \bar{o}(1),\end{aligned} \tag{3.7}$$

where $\bar{o}(1)$ does not depend on A. To complete the proof, we will use once more the 'partition trick'. Let $A_{p_n}^q := A_{p_n} \cap F_{p_n}(f_n - \phi_{p_n}(q))$ for $q \in Q$. Then in view of (2.9) $A_{p_n} = \bigsqcup_{q\in Q} A_{p_n}^q$ and $f_n - \phi_{p_n}(q) + A_{p_n}^q \subset F_{p_n}$ for all $q \in Q$. It is easy to see that (3.7) is also true if we replace the sequence $(h_n)_{n\geqslant 1}$ with the sequence $(h_n + q)_{n\geqslant 1}$ for any $q \in Q$. Notice that $h_n + q \neq 0$ since $Q^\bullet \supset -Q$. Hence applying Corollary 3.4 and (3.7) we obtain

$$\begin{aligned}\mu\big(T_{g_n}[A_{p_n+1,0}]_{p_n+1} \cap B\big) &= \mu\big(T_{g_n}\big[A_{p_n} + c_{p_n+1}\big(H_{p_n}(h_n)\big)\big]_{p_n+1} \cap B\big) + \bar{o}(1)\\ &= \sum_{q\in Q} \mu\big(T_{\phi_{p_n}(h_n+q)}\big[f_n - \phi_{p_n}(q) + A_{p_n}^q + c_{p_n+1}\big(H_{p_n}(h_n)\big)\big]_{p_n+1} \cap B\big) + \bar{o}(1)\\ &= \sum_{q\in Q} \mu\big(\big[f_n - \phi_{p_n}(q) + A_{p_n}^q + c_{p_n+1}\big(H_{p_n}(h_n)\big)\big]_{p_n+1}\big)\mu(B) + \bar{o}(1)\\ &= \sum_{q\in Q} \mu\big(\big[A_{p_n}^q + c_{p_n+1}\big(H_{p_n}(h_n)\big)\big]_{p_n+1}\big)\mu(B) + \bar{o}(1)\\ &= \mu\big(\big[A_{p_n} + c_{p_n+1}\big(H_{p_n}(h_n)\big)\big]_{p_n+1}\big)\mu(B) + \bar{o}(1)\\ &= \mu\big([A_{p_n+1,0}]_{p_n+1}\big)\mu(B) + \bar{o}(1).\end{aligned}$$

This plus (3.5) imply that the sequence $(g_n)_{n=1}^{\infty}$ is mixing. □

The following material will be used in the next section to check the condition on the strong convergence from Theorem 3.5.

Definition 3.6. Let I, J be finite subsets in H and let ϵ be a non-negative number. We say that I is ϵ-tiled by J if there exists a finite set $F \subset H$ such that the following are satisfied:

(i) $J + F \subset I$,
(ii) J and F are independent and
(iii) $\#(I \setminus (J + F)) \leqslant \epsilon \# I$.

Lemma 3.7. *Let I be ϵ-tiled by J. Then for any function $f \in L^2(X, \mu)$ and any homomorphism $\alpha : H \to G$,*

$$\|\mathcal{M}_{\alpha,I}(f)\|_2 \leqslant \|\mathcal{M}_{\alpha,J}(f)\|_2 + \epsilon \|f\|_2.$$

Proof. Since T preserves μ, it follows that

$$\|\mathcal{M}_{\alpha,h+J}(f)\|_2 = \|\mathcal{M}_{\alpha,J}(f) \circ T_{\alpha(h)}\|_2 = \|\mathcal{M}_{\alpha,J}(f)\|_2$$

for every $h \in H$. Let F be as in Definition 3.6. Then

$$\begin{aligned}\|\mathcal{M}_{\alpha,I}(f)\|_2 &\leqslant \frac{\#J}{\#I} \sum_{h\in F} \|\mathcal{M}_{\alpha,h+J}(f)\|_2 + \frac{\#(I \setminus (J+F))}{\#I} \|\mathcal{M}_{\alpha,I\setminus(J+F)}(f)\|_2 \\ &\leqslant \|\mathcal{M}_{\alpha,J}(f)\|_2 + \epsilon \|f\|_2. \qquad \square\end{aligned}$$

Let $h \in H$, $i, j \in \mathbb{Z}$ and $0 \leqslant i < j$. The subset $\{ih, (i+1)h, \ldots, jh\}$ is called an *h-interval*.

Remark 3.8. Let $V \subset H$ be a finite subset, I, J two h-intervals in H and $l \in \mathbb{N}$. If V is tiled (i.e., 0-tiled) by I then V is also $\frac{l\#J}{\#I}$-tiled by the lh-interval $l \cdot J$.

Lemma 3.9. *Let $\alpha : H \to G$ be a homomorphism. Let $(h_n)_{n\geqslant 1}$ be a sequence in H such that for some subset $A \subset X$ and every $l \in \mathbb{N}$, we have $\mu(T_{l\alpha(h_n)}A \cap A) \to \mu(A)^2$ as $n \to \infty$. Then for any sequence of h_n-intervals I_n whose cardinality is constant, say L, the following inequality holds eventually (i.e., for all large enough n)*

$$\|\mathcal{M}_{\alpha,I_n}(\chi_A) - \mu(A)\|_2^2 \leqslant \frac{2}{L}.$$

Proof. Since T preserves μ, without loss of generality we may assume that $I_n = \{0, h_n, \ldots, (L-1)h_n\}$. Now the inequality follows from the formula

$$\left\| \frac{1}{L} \sum_{i=0}^{L-1} \chi_A \circ T_{i\alpha(h_n)} - \mu(A) \right\|_2^2 = \frac{\mu(A)}{L} + \frac{1}{L^2} \sum_{i\neq j} \big(\mu(T_{|i-j|\alpha(h_n)}A \cap A) - \mu(A)^2\big),$$

which is established by a straightforward calculation. □

Lemma 3.10. *Let $\alpha : H \to G$ be a homomorphism, $(h_n)_{n\geqslant 1}$ a sequence in H and $(m_n)_{n\geqslant 1}$ a sequence in $\mathbb{N}$. Let V_n be a finite subset of H which is tiled by an h_n-interval I_n. If $m_n/\#I_n \to 0$ and $\mu(T_{lm_n\alpha(h_n)}A \cap A) \to \mu(A)^2$ as $n \to \infty$ for any integer $l > 0$ then $\mathcal{M}_{\alpha,V_n}(\chi_A) \to \mu(A)$ in $L^2(X, \mu)$.*

Proof. For an $\epsilon > 0$, fix an integer $r > \epsilon^{-1}$. Let J_n be a h_n-interval of cardinality r. By Lemma 3.9,

$$\|\mathcal{M}_{\alpha,m_nJ_n}(\chi_A) - \mu(A)\|_2^2 < 2\epsilon$$

eventually in n. It follows from Remark 3.8 that V_n is $\frac{m_n r}{\#I_n}$-tiled by the $m_n h_n$-interval $m_n J_n$. By Lemma 3.7,

$$\big\|\mathcal{M}_{\alpha,V_n}(\chi_A)-\mu(A)\big\|_2 \leqslant \big\|\mathcal{M}_{\alpha,m_n J_n}(\chi_A)-\mu(A)\big\|_2 + \frac{m_n r}{\#I_n} < 3\epsilon$$

eventually in n. □

The following results will be used while proving mixing of the (C,F)-actions whose 'spacer mappings' s_n are polynomials of degree > 2.

Lemma 3.11 *(Hilbertian van der Corput trick). Let $(v_h)_{h\in H}$ stand for a bounded family of vectors in a Hilbert space. If for any $k \in H \setminus \{0\}$, we have $\lim_{n\to\infty} \frac{1}{\#H_n}\sum_{h\in H_n}\langle v_{h+k}, v_h\rangle = 0$ then $\lim_{n\to\infty}\|\frac{1}{\#H_n}\sum_{h\in H_n} v_h\| = 0$.*

Proof. In case $H=\mathbb{Z}$, we refer the reader to [4] for a short proof. Only a slight and obvious modification of that proof is needed to adapt it to the general case. □

Corollary 3.12. *Given a map $t: H \to G$, let $\langle \mathcal{M}_{\partial_k t, H_n}\chi_B, \chi_B\rangle \to \mu(B)^2$ for every non-trivial $k \in H$ and all $B \in \mathfrak{B}$. Then $\mathcal{M}_{t,H_n} \to P_0$ strongly as operators in $L^2(X,\mu)$.*

Proof. Fix a subset $B \in \mathfrak{B}$. For any $h \in H$, we set $v_h := \chi_B \circ T_{t(h)} - \mu(B) \in L^2(X,\mu)$. Then $\langle v_{h+k}, v_h\rangle = \langle \chi_B \circ T_{\partial_k t(h)}, \chi_B\rangle - \mu(B)^2$. It remains to apply Lemma 3.11. □

4. Mixing rank-one actions of $\mathbb{R}^{d_1} \times \mathbb{Z}^{d_2}$

In this section we prove the main results of the paper—Theorems 4.9–4.11 and 4.12.

Let d_1 and d_2 be non-negative integers such that $d := d_1 + d_2 \neq 0$. We set $G = \mathbb{R}^{d_1} \times \mathbb{Z}^{d_2}$ and $H = \mathbb{Z}^d$. For $g = (g_1, \dots, g_d) \in G$, we let $\|g\|_\infty := \max_{1\leqslant i\leqslant d}|g_i|$. If $g_i \geqslant 0$ for all $i = 1, \dots, d$ we write $g \geqslant 0$. In a similar way we define $\|h\|_\infty$ and $h \geqslant 0$ for $h \in H$.

To define a mixing (C,F)-action of G we first fix a sequence of positive integers $r_n > 2$ which goes to infinity as $n \to \infty$. Some restrictions on its growth will be imposed later. Let $s_n^{(1)}: H \to \mathbb{R}^{d_1}$ be a usual polynomial with real coefficients and $s_n^{(2)}: H \to \mathbb{Z}^{d_2}$ a usual polynomial with rational coefficients such that $s_n^{(1)}(0) = 0$ and $s_n^{(2)}(0) = 0$. Then $s_n := (s_n^{(1)}, s_n^{(2)})$ is a polynomial mapping from H to G in the sense of Definition 1.5. Let $H_n := \{h \geqslant 0 \mid \|h\|_\infty < r_n\}$. We define a sequence of positive reals $(a_n)_{n\geqslant 0}$ recurrently by setting

$$a_{n+1} := \text{the integer part of } a_n r_n + \max_{h\in H_n}\big\|s_n(h)\big\|_\infty$$

and choosing a_0 arbitrarily. It is clear that $a_{n+1} \geqslant a_0 r_1 \cdots r_n$ for all $n \in \mathbb{N}$. Since $r_n \to \infty$, the sequence $(a_n)_{n=1}^\infty$ grows faster than any exponent. We finally let

$$F_n := \big\{g \geqslant 0 \mid \|g\|_\infty < a_n\big\},$$
$$\phi_n(h) := a_n h \quad \text{for } h \in H \text{ and}$$
$$C_{n+1} := (\phi_n + s_n)(H_n),$$

for $n = 0, 1, \dots$. It is easy to see that (1.1)–(1.3), (1.12) and (1.13) are satisfied. In the case $d = 1$, the condition (1.4) is satisfied too if s_n is non-negative and non-decreasing on $\mathbb{Z}_+$, for instance if the coefficients of s_n are all non-negative. The situation is more difficult when $d > 1$. That is why we introduce the following definition.

Definition 4.1. A mapping $s: H \to G$ is monotonic if for all non-negative $x, y \in H$ with $\|x-y\|_\infty = 1$,

$$\max\left\{\frac{s(x)_i - s(y)_i}{x_i - y_i}\;\middle|\; \text{for all } 1 \leqslant i \leqslant d \text{ such that } x_i \neq y_i\right\} \geqslant 0.$$

Example 4.2. Let $s(x)_i = (\alpha_i x_i + \gamma_i)(x_1 + \cdots + x_d) + \beta_i x_i^2 + \delta_i x_i$ for some reals $\alpha_i, \beta_i, \gamma_i$ and δ_i such that $\alpha_i > 0$, $\gamma_i \geqslant 0$, $\alpha_i + 2\beta_i \geqslant 0$ and $\alpha_i + \beta_i + \delta_i \geqslant 0$ for $i = 1, \ldots, d$. Suppose that the mapping

$$H \ni x \mapsto s(x) = \big(s(x)_1, \ldots, s(x)_d\big)$$

takes values in G. Then it is monotonic. To show this, consider two non-negative elements $x \neq y \in H$ such that $\|x - y\|_\infty = 1$ and set $z := y - x$. Without loss of generality we may assume that $z_1 + \cdots + z_d \geqslant 0$. Then there exists a coordinate j such that $z_j = 1$. By a straightforward calculation,

$$s(x+z)_j - s(x)_j = (\alpha_j x_j + \gamma_j) \sum_{i=1}^{d} z_i + \alpha_j \cdot \sum_{i \neq j} (x_i + z_i) + (\alpha_j + 2\beta_j) x_j + \alpha_j + \beta_j + \delta_j \geqslant 0.$$

We observe that in [3], the following monotonic polynomial $s : \mathbb{Z}^d \to \mathbb{Z}^d$ was used while constructing $\mathbb{Z}^d$-staircase actions:

$$s(x)_i = x_i(x_1 + \cdots + x_d) - \big(x_i^2 + x_i\big)/2, \quad i = 1, \ldots, d.$$

(By this formula we correct a misprint on page 849 of [3].) If, moreover $d = 1$, we get $s(x) = x(x-1)/2$, i.e., the polynomial corresponding to the classical staircase shown to be mixing in [1].

Lemma 4.3. *If s_n is monotonic and*

$$\max_{x \in H_n} \big\| s_n(x) \big\|_\infty < a_n/2 \tag{4.1}$$

then (1.4) *holds.*

Proof. Suppose that the contrary holds. Then there exist $x \neq y \in H_n$ and $f, f' \in F_n$ such that $a_n x + s_n(x) + f = a_n y + s_n(y) + f'$. Since

$$F_n - F_n = \big\{ g \in G \mid \|g\|_\infty < a_n \big\},$$

it follows that $\|a_n(x-y) + s_n(x) - s_n(y)\|_\infty < a_n$. If $\|x - y\|_\infty \geqslant 2$ then we get a contradiction with (4.1). If $\|x - y\|_\infty = 1$ then we get a contradiction with the fact that s_n is monotonic. □

Now suppose that $s_n = s$ for all $n \geqslant 0$, where s is a monotonic polynomial of degree $l > 1$. Let λ_G stand for the direct product of the Lebesgue measure on $\mathbb{R}^{d_1}$ and the 'counting' measure on $\mathbb{Z}^{d_2}$. Let us check (1.6). Since $\lambda_G(F_n) = a_n^d$ and

$$\lambda_G\big(F_{n+1} \setminus (F_n + C_{n+1})\big) = \Big(r_n a_n + \max_{h \in H_n} \big\| s(h) \big\|_\infty \pm 1\Big)^d - (r_n a_n)^d,$$

we see that (1.6) is satisfied if and only if

$$\sum_{n=1}^{\infty} \frac{\max_{h \in H_n} \|s(h)\|_\infty}{r_n a_n} < \infty.$$

Of course, there exists a strictly positive limit $\lim_{n\to\infty} \max_{h \in H_n} \|s(h)\|_\infty / r_n^l$. Thus (1.6) holds if and only if

$$\sum_{n=1}^{\infty} \frac{r_n^{l-1}}{a_n} < \infty. \tag{4.2}$$

It is easy to see that (2.9) and (2.10) are satisfied if we set

$$Q := \big\{ h \in H \mid h \geqslant 0 \text{ and } \|h\|_\infty \leqslant 1 \big\} \quad \text{and} \quad Q^\bullet := -Q. \tag{4.3}$$

Proposition 4.4. *Let* (4.2) *be satisfied. If*

$$r_n^l / a_n \to 0 \tag{4.4}$$

then the (C, F)-action T associated with $(H_n, \phi_n, s_n, F_n)_n$ is well defined and it satisfies the restricted growth condition. Also (3.1) *holds.*

Proof. Notice first that (4.4) implies (4.1) eventually. Hence (1.4) holds eventually by Lemma 4.3. Without loss of generality we may assume that it holds for all n. Since (1.1)–(1.3), (1.12) and (1.13) are also satisfied, it follows that the associated (C, F)-action T of G is well defined. Take a sequence $(h_n)_{n=1}^{\infty}$ in H with $\liminf_{n\to\infty} \#H_n(h_n)/\#H_n > 0$ (in fact, it suffices to have only $H_n \cap (H_n + h_n) \neq \emptyset$ eventually). It follows from (4.4) that

$$\max_{h\in H_n(h_n)} \frac{\|\partial_{h_n} s(h)\|_\infty}{a_n} \leqslant \max_{h,k\in H_n} \frac{\|s(h) - s(k)\|_\infty}{a_n} \to 0.$$

On the other hand, for any $b \in G$ with $\|b\|_\infty < a_n$, we have

$$\frac{\lambda_G(F_n \setminus (F_n + b))}{\lambda_G(F_n)} \leqslant \frac{a_n^d - (a_n - \|b\|_\infty)^d}{a_n^d} \leqslant \frac{d\|b\|_\infty}{a_n}.$$

Hence

$$\frac{1}{\#H_n} \sum_{h\in H_n(h_n)} \frac{\lambda_G(F_n \setminus F_n(\partial_{h_n} s(h)))}{\lambda_G(F_n)} \leqslant \max_{h\in H_n(h_n)} \frac{\lambda_G(F_n \setminus (F_n + \partial_{h_n} s(h)))}{\lambda_G(F_n)} \to 0,$$

i.e., (2.3) holds. (3.1) is verified in a similar way. □

We see, in particular, that if r_n is of sub-exponential growth, i.e., $r_n/\xi^n \to 0$ for some $\xi > 1$, then both (4.2) and (4.4) hold and hence T is well defined for any monotonic s.

Now consider in more detail the case where $s : H \to G$ is a polynomial of degree 2. Given $t \in H$, we have $\partial_t s(h) = \psi_t(h) + a_t$ for some homomorphism $\psi_t : H \to G$ and an element $a_t \in G$. It is easy to verify that $\psi_{t_1+t_2}(h) = \psi_{t_1}(h) + \psi_{t_2}(h)$ for all $t_1, t_2 \in H$. Hence the map $H \times H \ni (t, h) \mapsto \psi_t(h) \in G$ is a 'bihomomorphic' form. Then there is $\gamma > 0$ such that

$$\big\|\psi_t(h)\big\|_\infty \leqslant \gamma\|t\|_\infty\|h\|_\infty \quad \text{for all } t, h \in H. \tag{4.5}$$

Fix a standard 'basis' in G (and H):

$$e_1 := (1, 0, \ldots, 0),\ e_2 := (0, 1, 0, \ldots, 0), \ldots, e_d := (0, \ldots, 0, 1).$$

Then we can identify ψ_t with the corresponding $(d \times d)$-matrix with real entries for any $t \in H$.

The following statement based essentially on Theorem 3.5 provides convenient sufficient conditions for the (C, F)-actions under consideration to be mixing. Combined with Corollary 2.6 it will turn proofs of the main results into verifications of almost purely algebraic nature.

Proposition 4.5. *Let the polynomial s be monotonic and* (4.2) *and* (4.4) *hold with $l = 2$. Moreover, suppose that*

(i) *the action $(T_{\psi_k(h)})_{h\in H}$ is ergodic for any $k \in H \setminus \{0\}$ and*
(ii) $\max_{1\leqslant i\leqslant d} \|\psi_k(e_i)\|_\infty \geqslant \delta\|k\|_\infty$ *for some $\delta > 0$ and every $k \in H$.*

Then T is mixing.

Proof. By Proposition 4.4, T has restricted growth and (3.1) holds. Take a sequence $(h_n)_{n=1}^{\infty}$ in H such that $h_n \neq 0$ for all $n \in \mathbb{N}$ and

$$\liminf_{n\to\infty} \#H_n(h_n)/\#H_n > 0. \tag{4.6}$$

In view of Theorem 3.5, it suffices to show that $\mathcal{M}_{\psi_{h_n}, H_n(h_n)} \to P_0$ strongly. Suppose first that $h_n = k$ for some $k \in H$ and all n. Then just use (i) and apply the mean ergodic theorem to the action $(T_{\psi_k(h)})_{h\in H}$ to conclude that $\mathcal{M}_{\psi_k, H_n(k)} \to P_0$. Hence it remains to consider only the sequences $(h_n)_{n=1}^{\infty}$ with $h_n \to \infty$ in H. Let $i(k)$ stand for an index at which the maximum in (ii) is attained. Then

$$\big\|\psi_{h_n}(e_{i(h_n)})\big\|_\infty \geqslant \delta\|h_n\|_\infty. \tag{4.7}$$

Since $H_n(h_n)$ is a parallelepiped in $\mathbb{Z}_+^d$, there exist an element $h_n' \in H$, a positive integer $r_n' \leqslant r_n$ and two parallelepipeds $J_n \subset \mathbb{Z}_+^{i(h_n)-1}$ and $J_n' \subset \mathbb{Z}_+^{d-i(h_n)}$ such that

$$H_n(h_n) + h_n' = J_n \times \{0, 1, \ldots, r_n' - 1\} \times J_n'.$$

It follows from (4.6) that there exist two reals $\delta' > 0$ and $\delta'' \leqslant 1$ such that

$$r'_n/r_n \geqslant \delta' \quad \text{and} \quad \|h_n\|_\infty/r_n \leqslant \delta'' \quad \text{eventually}. \tag{4.8}$$

We set $V_n := H_n(h_n) + h'_n$ and $g_n := \psi_{h_n}(e_{i(h_n)})$. By (4.7), $\|g_n\|_\infty \to \infty$ (recall that $h_n \to \infty$). Find $p_n \in \mathbb{N}$ with $g_n \in F^\bullet_{p_n+1} \setminus F^\bullet_{p_n}$. Then, of course, $p_n \to \infty$. Let m_n be the smallest positive integer such that $m_n g_n \notin F^\bullet_{p_n+1}$. Then

$$m_n g_n = (m_n - 1)g_n + g_n \in F^\bullet_{p_n+1} + F^\bullet_{p_n+1}.$$

Since $\{g \in G \mid \|g\|_\infty \leqslant a_m\} \supset F^\bullet_m \supset \{g \in G \mid \|g\|_\infty < a_m\}$ for all $m \in \mathbb{N}$, we have

$$a_{p_n} \leqslant \|g_n\|_\infty \leqslant a_{p_n+1} \quad \text{and} \quad a_{p_n+1} \leqslant m_n\|g_n\|_\infty \leqslant 2a_{p_n+1}. \tag{4.9}$$

Moreover, it is straightforward (see (4.3)) that

$$\underbrace{Q + \cdots + Q}_{l \text{ times}} = \big\{h \in H \mid h \geqslant 0 \text{ and } \|h\|_\infty \leqslant l\big\}$$

and

$$Q_+ \cap \big\{h \in H \mid \|h\|_\infty = 1 \text{ and } -h \geqslant 0\big\} = \emptyset.$$

Recall that Q_+ denotes the minimal (finite) subset of H such that $(F^\bullet_n + F^\bullet_n) \setminus F^\bullet_n \subset \phi_n(Q_+) + F_n$ for all $n \in \mathbb{N}$. Hence

$$-l\,Q_+ \cap \big(\underbrace{Q + \cdots + Q}_{l \text{ times}}\big) = \emptyset.$$

By (i) and Corollary 2.6(i), the sequence $(\phi_n(k))_{n=1}^\infty$ is mixing for every $0 \neq k \in H$. Thus we may apply Lemma 2.9 and conclude that the sequence $(lm_n g_n)_{n=1}^\infty$ is mixing for every $l \in \mathbb{N}$. Notice that

$$\mathcal{M}_{\psi_{h_n}, H_n(h_n)} = \mathcal{M}_{\psi_{h_n}, V_n} U_n = U_n \mathcal{M}_{\psi_{h_n}, V_n},$$

where U_n is the unitary operator given by $U_n f := f \circ T_{\psi_{h_n}(-h'_n)}$. Therefore, $\mathcal{M}_{\psi_{h_n}, H_n(h_n)} \to P_0$ strongly if and only if $\mathcal{M}_{\psi_{h_n} V_n} \to P_0$ strongly as $n \to \infty$. We set $I_n := \{0, e_{i(h_n)}, \ldots, (r'_n - 1)e_{i(h_n)}\}$. To complete the proof it remains only to establish that $m_n/\#I_n \to 0$ and apply Lemma 3.10. By (4.5),

$$\|g_n\|_\infty = \big\|\psi_{h_n}(e_{i(h_n)})\big\|_\infty \leqslant \gamma\|h_n\|_\infty$$

eventually in n. Using that, (4.8) and (4.9) we obtain

$$\frac{m_n}{r_n} \leqslant \frac{2\gamma\delta'' a_{p_n+1}}{\|g_n\|^2_\infty} \leqslant 2\gamma\delta'' \frac{a_{p_n+1}}{a^2_{p_n}}$$

for all large enough n. It follows (use also (4.8) plus (4.2))

$$\lim_{n\to\infty} \frac{m_n}{r'_n} \leqslant \frac{1}{\delta'} \lim_{n\to\infty} \frac{m_n}{r_n} \leqslant \frac{2\gamma\delta''}{\delta'} \lim_{n\to\infty} \frac{a_{p_n+1}}{a^2_{p_n}} = \frac{2\gamma\delta''}{\delta'} \lim_{n\to\infty} \frac{r_{p_n}}{a_{p_n}} = 0.$$

(Notice that it follows from (1.6) that $\lim_{n\to\infty} a_{n+1}/(a_n r_n) = 1$.) □

Fix a family of reals $\xi_1, \ldots, \xi_m$. For a non-empty subset $J \subset \{1, \ldots, m\}$, we let $\xi_J := \prod_{i\in J} \xi_i$. We also let $\xi_\emptyset := 1$.

Definition 4.6. If the family of reals ξ_J, J runs all the subsets of $\{1, \ldots, m\}$, is independent over $\mathbb{Q}$ then we say that $\xi_1, \ldots, \xi_m$ is *good*.

It is clear that any subfamily of a good family is good. Moreover, given non-zero rationals $q_1, \ldots, q_m$, the family $q_1\xi_1, \ldots, q_m\xi_m$ is good if and only if so is $\xi_1, \ldots, \xi_m$.

We also let $\mathbb{R}^J := \{g = (g_1, \ldots, g_d) \in \mathbb{R}^d \mid g_i = 0 \text{ for all } i \notin J\}$. In a similar way we define $\mathbb{Z}^J$.

Lemma 4.7. *Let $\xi_1, \dots, \xi_m$ be a good family. Then*

$$\det \begin{pmatrix} q_{1,1}+\xi_1 & q_{1,2} & \dots & q_{1,m} \\ q_{2,1} & q_{2,2}+\xi_2 & \dots & q_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ q_{1,m} & q_{2,m} & \dots & q_{m,m}+\xi_m \end{pmatrix} \neq 0$$

for any $q_{i,j} \in \mathbb{Q}$, $i, j = 1, \dots, m$.

Proof. It is enough to notice that the determinant equals to $\sum_{J\subset\{1,\dots,m\}} r_J \xi_J$ with some coefficients $r_J \in \mathbb{Q}$. If the determinant vanishes then $r_J = 0$ for all J by the definition of a good family. However, it is easy to see that $\xi_{\{1,\dots,m\}} = 1$, a contradiction. □

We also state without proof a couple of well known facts.

Lemma 4.8.

(i) *A weakly mixing action of a l.c.s.c. Abelian group is totally ergodic.*
(ii) *Let V and V' be two mutually commuting actions of l.c.s.c. Abelian groups F and F' respectively. If V is weakly mixing and V' is ergodic then V' is weakly mixing.*

Now we are ready to prove the main results of the paper. We consider first the case where $d_2 = 0$ and $d_1 > 1$.

Theorem 4.9. *Let $G = \mathbb{R}^d$, $d > 1$, and* (4.2) *and* (4.3) *hold. Let s be the polynomial from Example* 4.2 *and let the parameters α_i, β_i satisfy the following additional conditions*: *$\alpha_i \in \mathbb{Q}$ for all $i = 1, \dots, d$ and the family $\alpha_1 + 2\beta_2, \dots, \alpha_d + 2\beta_d$ is good. Put $s_n := s$ for all $n \in \mathbb{N}$. Then the rank-one action T of G associated with $(H_n, \phi_n, s_n, F_n)_n$ is mixing.*

Proof. It is easy to calculate that

$$\psi_{e_i} = \begin{pmatrix} \alpha_1 & & & & & & & \\ & \ddots & & & & & & \\ & & \alpha_{i-1} & & & & & \\ \alpha_i & \dots & \alpha_i & 2(\alpha_i+\beta_i) & \alpha_i & \dots & \alpha_i & \\ & & & & \alpha_{i+1} & & & \\ & & & & & \ddots & & \\ & & & & & & \alpha_d & \end{pmatrix}, \tag{4.10}$$

where the entries outside the main diagonal and the i-th line are zero, $i = 1, \dots, d$.

We first claim that the group generated by $\bigcup_{i,j=1}^d \psi_{e_i}(e_j)$ is dense in G. For a real $x \in \mathbb{R}$, denote the fractional part of x by $\langle x \rangle$. Identify $\mathbb{T}$ with the interval $[0, 1)$ endowed with addition mod 1. Since $\psi_{e_i}(e_i) = 2(\alpha_i + \beta_i)e_i$ and $\alpha_i + \beta_i \neq 0$, the map

$$\pi : G \ni (x_1, \dots, x_d) \mapsto \left(\left\langle \frac{x_1}{2(\alpha_1+\beta_1)} \right\rangle, \dots, \left\langle \frac{x_d}{2(\alpha_d+\beta_d)} \right\rangle \right) \in \mathbb{T}^d$$

is a quotient of G by the lattice generated by $\bigcup_{i=1}^d \psi_{e_i}(e_i)$. We see that for $i \neq 1$,

$$\pi\big(\psi_{e_1}(e_i)\big) = \Bigg(\underbrace{\left\langle \frac{\alpha_1}{2(\alpha_1+\beta_1)} \right\rangle, 0, \dots, 0, \left\langle \frac{\alpha_i}{2(\alpha_i+\beta_i)} \right\rangle}_{i}, 0, \dots, 0 \Bigg).$$

Since $\alpha_1 + \beta_1 \notin \mathbb{Q}$ and $(\alpha_i + \beta_i)/(\alpha_1 + \beta_1) \notin \mathbb{Q}$ for $i = 2, \dots, d$, the group generated by $\pi(\psi_{e_1}(e_i))$ is dense in the subgroup

$$\underbrace{\mathbb{T} \times \{0\} \times \dots \times \{0\} \times \mathbb{T}}_{i} \times \{0\} \times \dots \times \{0\} \subset \mathbb{T}.$$

Thus our first claim follows.

Now let us verify conditions (i) and (ii) from Proposition 4.5.

It follows from (4.10) that $\det \psi_{e_i} \neq 0$. Hence $\psi_{e_i}(H)$ is a lattice in G. Then by Corollary 2.6(iii), the sequence $(\phi_n(e_i))_{n=1}^{\infty}$ is uniformly mixing. Since $\phi_n(e_i) = a_n e_i$, it follows that the transformation T_{e_i} is weakly mixing for every $i = 1, \dots, m$. By Lemma 4.8(ii), the action $(T_g)_{g\in\mathbb{R}^J}$ is also weakly mixing for any non-empty subset $J \subset \{1, \dots, m\}$. Now take any $0 \neq t = (t_1, \dots, t_d) \in H$. If $\det \psi_t \neq 0$ then $\psi_t(H)$ is a lattice in G. By Corollary 2.6(ii), $(T_{\psi_t(h)})_{h\in H}$ is ergodic, as desired. Consider now the second case when $\det \psi_t = 0$. Since $\psi_t = t_1\psi_{e_1} + \cdots + t_d\psi_{e_d}$, it follows from (4.10) that

$$\psi_t = \begin{pmatrix} (\alpha_1 + 2\beta_1)t_1 + \alpha_1 u & \alpha_1 t_1 & \dots & \alpha_1 t_1 \\ \alpha_2 t_2 & (\alpha_2 + 2\beta_2)t_2 + \alpha_2 u & \dots & \alpha_2 t_2 \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_d t_d & \alpha_d t_d & \dots & (\alpha_d + 2\beta_d)t_d + \alpha_d u \end{pmatrix},$$

where $u := \sum_{i=1}^{d} t_i$. Let $J := \{i \mid t_i \neq 0\}$. It follows immediately from Lemma 4.7 that there exists $i \notin J$. Then for any $j \neq i$, the j-th element of the i-th line of ψ_t is 0. Crossing out the i-th line and the i-column for all $i \notin J$ we obtain a $(\#J \times \#J)$-matrix M and $\det \psi_t = u^{d-\#J} \prod_{i\notin J} \alpha_i \det M$. It is easy to see that Lemma 4.7 is applicable to M. Hence $\det M \neq 0$. Since $\det \psi_t = 0$, it follows that $u = 0$. Hence for any $j \notin J$, the entire i-th line in ψ_t vanishes. Therefore $\psi_t(H) \subset \mathbb{R}^J$. Moreover, $\psi_t(H)$ is a lattice in $\mathbb{R}^J$ since $\det M \neq 0$. By Lemma 4.8(i), the action $(T_{\psi_t(h)})_{h\in H}$ is ergodic. Thus, Proposition 4.5(i) holds. The other condition of that theorem is much easier to verify. Let $\|t\|_\infty = |t_j|$ for some j. Take any $l \neq j$. Then we have

$$\big\|\psi_t(e_l)\big\|_\infty \geqslant \alpha_j |t_j| \geqslant \min_{1\leqslant i\leqslant d} \alpha_i \cdot \|t\|_\infty. \qquad \square$$

It is easy to understand that the above construction does not work in case $G = \mathbb{R}$ since the first claim of the proof fails. However, only a slight 'complication' of the construction is needed to cover the exceptional case. We outline it briefly. Let S be a finite family of monotonic polynomials from H to $G = \mathbb{R}^{d_1} \times \mathbb{Z}^{d_2}$ of degree $l > 1$. Let $(s_n)_{n=1}^{\infty}$ be a sequence of elements of S such that every element of S occurs in $(s_n)_{n=1}^{\infty}$ infinitely many times. Suppose that (4.2) and (4.4) are satisfied. Define H_n, ϕ_n, F_n as above. Then the (C, F)-action T associated with $(H_n, \phi_n, s_n, F_n)_n$ is well defined and it satisfies the restricted growth condition (the same proof as in Proposition 4.4 works as well to establish this fact). Let us assume now that $l = 2$. For any $s \in S$, let

$$\partial_t s(h) = \psi_t^s(h) + a_t^s \quad \text{for all } t, h \in H,$$

where $\psi_t^s : H \to H$ is a homomorphism and a_t^s an element of H. Suppose, in addition, that

(i) the action $(T_{\psi_k^s(h)})_{h\in H}$ is ergodic for any $k \in H \setminus \{0\}$ and $s \in S$ and
(ii) $\min_{s\in S} \max_{1\leqslant i\leqslant d} \|\psi_k^s(e_i)\|_\infty \geqslant \delta\|k\|_\infty$.

Then repeating the proof of Proposition 4.5 almost verbatim one can show that T is mixing. Now let $G = \mathbb{R}^d$ for any $d > 0$. Suppose that every polynomial from S satisfies the conditions of Theorem 4.9. Then slightly modifying the proof of this theorem one can establish that T is mixing whenever

$$\text{the group generated by } \bigcup_{s\in S} \bigcup_{i,j=1}^{d} \psi_{e_i}^s(e_j) \text{ is dense in } G. \tag{4.11}$$

The main point of this modification is to replace the references to Corollary 2.6 with the references to Remark 2.7. If $d > 1$ then (4.11) is satisfied (see the proof of Theorem 4.9). The following theorem provides an example where (4.11) holds in case $d = 1$.

Theorem 4.10. *Let $G = \mathbb{R}$ and* (4.2) *and* (4.3) *hold with $l = 2$. Let*

$$s(x) := \alpha x^2 + \beta x, \quad \tilde{s}(x) := \tilde{\alpha} x^2 + \tilde{\beta} x \quad \text{at all } x \in \mathbb{Z},$$

where α and $\tilde{\alpha}$ are rationally independent positive reals and $\alpha + \beta \geqslant 0$, $\tilde{\alpha} + \tilde{\beta} \geqslant 0$. Set $s_n := s$ for even n and $s_n := \tilde{s}$ for odd n. Then the (C, F)-action of $\mathbb{R}$ associated with $(H_n, \phi_n, s_n, F_n)_n$ is mixing.

Proof. In view of the reasoning preceding the statement of the theorem it suffices to notice that s and $\tilde{s}$ are both non-negative and non-decreasing on $\mathbb{Z}_+$ and (4.11) is satisfied since $\psi^s_{e_1}(e_1) = 2\alpha$ and $\psi^{\tilde{s}}_{e_1}(e_1) = 2\tilde{\alpha}$. □

Now we pass to the case $G = \mathbb{Z}^d$.

Theorem 4.11. *Let $G = \mathbb{Z}^d$ and* (4.2) *and* (4.4) *hold with $l = 2$. Let s be the polynomial from Example* 4.2 *with $\alpha_1 = \cdots = \alpha_d = 1$ and $\beta_1 = \cdots = \beta_d = -0.5$. Put $s_n := s$ for all $n \in \mathbb{N}$. Then the rank-one action T of G associated with $(H_n, \phi_n, s_n, F_n)_n$ is mixing.*

Proof. As in the proof of Theorem 4.9 it suffices to show that the group generated by $\bigcup_{i,j=1}^d \psi_{e_i}(e_j)$ is (= dense in) G and the conditions (i) and (ii) of Proposition 4.5 are satisfied. The first claim is trivial since $\psi_{e_i}(e_i) = e_i$ for all $i = 1, \ldots, d$ (see (4.10)). Now take any $t = (t_1, \ldots, t_d) \in G$ and set $J(t) := \{i \mid t_i \neq 0\}$. We will check Proposition 4.5(i) by induction in $\#J(t)$.

If $\#J(t) = 1$ then $t = t_i e_i$ for some $1 \leqslant i \leqslant d$ and a non-zero integer t_i. Hence $\det \psi_t = t_i^d \det \psi_{e_i} \neq 0$. It remains to apply Corollary 2.6(ii).

Suppose now that there exists $p < d$ such that the action $(T_{\psi_t(h)})_{h\in H}$ is ergodic for any $t \in H$ with $1 \leqslant \#J(t) \leqslant p$. Take any $t \in H$ with $\#J(t) = p + 1$. We are going to show that the action $(T_{\psi_t(h)})_{h\in H}$ is ergodic. For convenience, let us assume that $J(t) = \{1, \ldots, p + 1\}$. (In the general case one can argue in a similar way.) Then

$$\psi_t = \begin{pmatrix} \sum_{i=1}^{p+1} t_i & t_1 & \ldots & t_1 & t_1 & \ldots & t_1 \\ t_2 & \sum_{i=1}^{p+1} t_i & \ldots & t_2 & t_2 & \ldots & t_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ t_{p+1} & t_{p+1} & \ldots & \sum_{i=1}^{p+1} t_i & t_{p+1} & \ldots & t_{p+1} \\ 0 & 0 & \ldots & 0 & \sum_{i=1}^{p+1} t_i & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 0 & 0 & \ldots & \sum_{i=1}^{p+1} t_i \end{pmatrix}.$$

We consider now three possible cases. Denote by A the $(p + 1) \times (p + 1)$-submatrix of ψ_t standing at the upper left corner.

Case 1. If $\sum_{i=1}^{p+1} t_i \neq 0$ and $\det A \neq 0$ then $\det \psi_t = (\sum_{i=1}^{p+1} t_i)^{d-p-1} \det A \neq 0$. Hence $\psi_t(H)$ is of finite index in G and we are done (just apply Corollary 2.6(ii)).

Case 2. If $\sum_{i=1}^{p+1} t_i \neq 0$ but $\det A = 0$ then it is easy to verify that $\operatorname{rk} A \geqslant 2$. Hence $\operatorname{rk} \psi_t \geqslant d - (p + 1) + 2 > d - p$. Therefore

$$\psi_t(H) \cap \mathbb{Z}^{J(t)} \neq \{0\}.$$

On the other hand, it follows from the inductive assumption and Corollary 2.6(i) that the sequence $(\phi_n(h))_{n=1}^{\infty}$ is mixing for T for any non-zero $h \in \mathbb{Z}e_1 + \cdots + \mathbb{Z}e_p$. Hence the transformation T_h is weakly mixing for all such h (recall that $\phi_n(h) = a_n h$). Thus the transformation group $(T_{\psi_t(h)})_{h\in H}$ contains a weakly mixing transformation and therefore it is ergodic.

Case 3. If $\sum_{i=1}^{p+1} t_i = 0$ then it is easy to see that $\psi_t(H) \subset \mathbb{Z}^{J(t)}$. Moreover,

$$\det A = t_1 \cdots t_{p+1} \det \begin{pmatrix} 0 & 1 & 1 & \ldots & 1 \\ 1 & 0 & 1 & \ldots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \ldots & 0 \end{pmatrix} \neq 0.$$

Hence $\psi_t(H)$ is of finite index in $\mathbb{Z}^{J(h)}$. Since the transformation T_{e_1} is weakly mixing, the action $(T_g)_{g\in\mathbb{Z}^{J(h)}}$ is weakly mixing by Lemma 4.8(ii). It remains to make use of Lemma 4.8(i).

The condition (ii) of Proposition 4.5 is checked in a straightforward way (as in the proof of Theorem 4.9). □

Remark 4.12. Note that if we put $\gamma_i = \delta_i = 0$ for all i in the statement of Theorem 4.10 (see the definition of s in Example 4.2) then we obtain exactly the main result of [3]. However, it was proved there completely only for $G = \mathbb{Z}^2$. Roughly speaking, it was assumed implicitly in [3] that $\det \psi_t \neq 0$ for all $t \in \mathbb{Z}^d \setminus \{0\}$. While being the case for $d = 2$, it is no longer true for $d > 2$ (see Cases 2 and 3 in the proof of Theorem 4.10).

Combining the arguments from Theorems 4.9–4.11, one can prove the following.

Theorem 4.13. *Let $G = \mathbb{R}^{d_1} \times \mathbb{Z}^{d_2}$ and* (4.2) *and* (4.4) *hold. Let $\tilde{s}_n : \mathbb{Z}^{d_1} \to \mathbb{R}^{d_1}$ and $\hat{s}_n : \mathbb{Z}^{d_2} \to \mathbb{Z}^{d_2}$ stand for the sequences of polynomials defined in the statements of Theorem* 4.9 *if $d_1 > 1$ (or Theorem* 4.10 *if $d_1 = 1$) and Theorem* 4.11 *respectively. We set*

$$s_n(t_1, \dots, t_{d_1+d_2}) := \big(\tilde{s}_n(t_1, \dots, t_{d_1}), \hat{s}_n(t_{d_1+1}, \dots, t_{d_1+d_2})\big)$$

for all $(t_1, \dots, t_{d_1+d_2}) \in H = \mathbb{Z}^{d_1} \times \mathbb{Z}^{d_2}$. Then the rank-one (C, F)-action of G associated with $(H_n, \phi_n, s_n, F_n)_n$ is mixing.

Now we consider a more general case where the 'spacer mapping' $s : H \to G$ is a polynomial of arbitrary degree $l > 2$. Then for any $t_1, \dots, t_{l-1} \in H$, there exist a homomorphism $\psi_{t_1,\dots,t_{l-1}} : H \to G$ and an element $a_{t_1,\dots,t_{l-1}} \in G$ such that

$$\partial_{t_1} \cdots \partial_{t_{l-1}} s(h) = \psi_{t_1,\dots,t_{l-1}}(h) + a_{t_1,\dots,t_{l-1}} \quad \text{for all } h \in H.$$

It is easy to verify that the map

$$\underbrace{H \times \cdots \times H}_{l \text{ times}} \ni (t_1, \dots, t_l) \mapsto \psi_{t_1,\dots,t_{l-1}}(t_l) \in G$$

is a 'polyhomomorphic' form and there exists $\gamma > 0$ such that

$$\|\psi_{t_1,\dots,t_{l-1}}\|_\infty \leqslant \gamma \|t_1\|_\infty \cdots \|t_l\|_\infty \quad \text{for all } t_1, \dots, t_l \in H.$$

The following statement is a higher degree analogue of Proposition 4.5.

Proposition 4.14. *Let $s : H \to G$ be a monotonic polynomial of degree $l > 2$. Assume that* (4.2) *and* (4.4) *hold. Set $s_n := s$ for all n and denote by T the (C, F)-action of G associated with $(H_n, \phi_n, s_n, F_n)_n$. Suppose that*

(i) *the action $(T_{\psi_{t_1,\dots,t_{l-1}}(h)})_{h \in H}$ is ergodic for any family $t_1, \dots, t_{l-1} \in H \setminus \{0\}$ and*
(ii) $\max_{1 \leqslant i \leqslant d} \|\psi_{t_1,\dots,t_{l-1}}(e_i)\|_\infty \geqslant \delta \|t_1\|_\infty \cdots \|t_{l-1}\|_\infty$ *for some $\delta > 0$ and all $t_1, \dots, t_{l-1} \in H \setminus \{0\}$.*

Then T is mixing.

Proof. (Cf. with the proof of Proposition 4.5.) By Proposition 4.4, T has restricted growth. Take a sequence $(h_n)_{n=1}^\infty$ in $H \setminus \{0\}$ such that (4.6) holds. In view of Theorem 3.5, it suffices to show that $\mathcal{M}_{s'_n, H_n(h_n)} \to P_0$ strongly, where s'_n is the h_n-derivative of s. Applying Corollary 3.12 $l - 2$ times, we obtain that the latter holds whenever

$$\mathcal{M}_{\psi_{t_1,\dots,t_{l-2},h_n}, H_n(h_n)} \to P_0 \quad \text{strongly for any family } t_1, \dots, t_{l-2} \in H \setminus \{0\}. \tag{4.12}$$

Suppose first that $h_n = k$ for some $k \in H$ and all n. Then just use (i) and apply the mean ergodic theorem to the action $(T_{\psi_{t_1,\dots,t_{l-2},k}(h)})_{h \in H}$ to deduce (4.11).

It remains to consider the second case where $h_n \to \infty$ in H. Notice first that since $\mathcal{M}_{\partial_k s, H_n(k)} \to P_0$ strongly, it follows from Corollary 2.3(ii) that the sequence $(\phi_n(k))_{n=1}^\infty$ is uniformly mixing for every $k \in H \setminus \{0\}$. The rest of the argument is almost a literal repetition of that from the proof of Proposition 4.5. □

As an application we construct a family of mixing (C, F)-actions with polynomial 'spacer map' of higher degree in the simplest case when $G = \mathbb{Z}$.

Example 4.15. Let $G=\mathbb{Z}$. Suppose that the sequence $(r_n)_{n=1}^{\infty}$ is of sub-exponential growth and $\lim_{n\to\infty} r_n=\infty$. Then, as was noticed above, (4.2) and (4.4) hold. Let $p:\mathbb{Z}\to\mathbb{Z}$ be a polynomial of degree $l>0$ which non-decreases on $\mathbb{Z}_+$. Assume also that $p(0)=0$. Suppose that the image of p is not contained in any proper subgroup of $\mathbb{Z}$. For instance, if $p(\mathbb{Z})\ni 1$ then p enjoys this property. Define a polynomial $s:\mathbb{Z}\to\mathbb{Z}$ by setting

$$s(t):=p(0)+p(1)+\cdots+p(t-1)\quad\text{for } t>0.$$

Clearly, s is of degree $l+1$ and s non-decreases on $\mathbb{Z}_+$. Let $s_n:=s$ for all $n\in\mathbb{N}$. Denote by T the corresponding $\mathbb{Z}$-action. We will show that it is mixing. To this end it suffices to verify that the conditions (i) and (ii) of Proposition 4.14 hold for T. It is obvious that (ii) holds. The condition (i) will follow from the fact that T is totally ergodic. To establish the latter we are going to apply Proposition 2.5. Let $h_n=1$ for all $n\in\mathbb{N}$. Then $\partial_{h_n}s(t)=s(t+1)-s(t)=p(t)$ at all $t\geqslant 0$. If T is not totally ergodic then by Proposition 2.5 there exist $k>0$ and $j\in\{0,\dots,k-1\}$ such that

$$\frac{\#\{i\in\{1,\dots,r_n-1\}\mid p(i)\equiv j \bmod k\}}{r_n-1}\to 1\quad\text{as } n\to\infty.$$

Hence if n is large enough we can find $0<i<r_n-2-l$ such that

$$p(i)\equiv p(i+1)\equiv\cdots\equiv p(i+l)\bmod k.$$

Since $p(0)=0$ and l is the degree of p, it follows from [16, Corollary 1.17] that $p(\mathbb{Z})\subset k\mathbb{Z}$, a contradiction.

In particular, putting $r_n=n$ and $p(t)=t^l$, $t\in\mathbb{Z}$, we obtain the family of polynomial staircases first proved to be mixing in [1] (when $l=1$) and [5] (when $l>1$).

5. Entropic properties of rank-one actions

Let $G=\mathbb{R}^{d_1}\times\mathbb{Z}^{d_2}$ with arbitrary $d_1,d_2\geqslant 0$ and T a (C,F)-action of G on a probability space $(X,\mathfrak{B},\mu)$. We will assume that

$$F_n=\bigl\{g\in G\mid g\geqslant 0 \text{ and } \|g\|_\infty<a_n\bigr\}$$

for a sequence $a_n\in\mathbb{R}_+$. This implies that T is rank-one by cubes. It is well known that the entropy of any rank-one $\mathbb{Z}$-action is zero. This fact extends easily to the rank-one (by cubes) actions of any group G. However in case of higher dimensional groups, say $G=\mathbb{Z}^2$, there exist rank-one (by rectangles) actions $S=(S_g)_{g\in G}$ of G such that the transformation $S_{(1,0)}$ is Bernoulli (see [20]). Then one has $h(S)=0$ but $h(S_{(1,0)})>0$. We show now that this is impossible for rank-one (by cubes) actions.

Theorem 5.1. *Let T be a rank-one (by cubes) action of G. Then $h(T_g)=0$ for each $g\in G$.*

Proof. We consider only the case $G=\mathbb{R}^d$ (in the general case one can argue in a similar way). Let $(M_n)_{n=1}^{\infty}$ be a sequence of positive reals such that

$$\lim_{n\to\infty}\frac{M_n}{a_n}=0\quad\text{and}\quad\lim_{n\to\infty}\frac{\log a_n}{M_n}=0. \tag{5.1}$$

Fix $g\in G\setminus\{0\}$ and take a family $v_1,\dots,v_d$ of mutually orthogonal (with respect to Euclidean inner product in $\mathbb{R}^d$) vectors such that $v_1=g$. Consider the finest partition $\mathcal{F}$ of G into parallelepipeds whose vertices belong to the lattice generated by $v_1,\dots,v_d$. Let $\mathcal{F}_n$ be a maximal subset of $\mathcal{F}$ such that the atoms $ig+A$, A runs $\mathcal{F}_n$ and $0\leqslant i<M_n$, are mutually disjoint and all of the are contained in F_n. It follows from the first limit in (5.1) that

$$\frac{\#\mathcal{F}_n\cdot M_n\cdot\lambda_G(A)}{\lambda_G(F_n)}\to 1. \tag{5.2}$$

Let $\mathcal{P}_n$ be a finite partition of X consisting of the n-cylinders $[A]_n$, where A runs $\mathcal{F}_n$ and the complement B_n to the union of these n-cylinders. Then

$$H(\mathcal{P}_n)=-\#\mathcal{F}_n\mu\bigl([A]_n\bigr)\log\mu\bigl([A]_n\bigr)-\mu(B_n)\log\mu(B_n), \tag{5.3}$$

where A is an atom of $\mathcal{F}_n$. Since $\mu(X \setminus X_n) \to 0$ and (5.2) holds, it is easy to show that the second term in the right-hand side of (5.3) goes to 0. Substituting (1.11) to (5.3) and making use of (5.2) we now obtain

$$H(\mathcal{P}_n) \leqslant \#\mathcal{F}_n \frac{\lambda_G(A)}{\lambda_G(F_n)} \log \frac{\lambda_G(F_n)}{\lambda_G(A)} + \bar{o}(1) \leqslant \frac{2d}{M_n} \log \frac{a_n}{\lambda_G(A)} + \bar{o}(1)$$

(notice that $\lambda_G(F_n) = a_n^d$). It follows from the second limit in (5.1) that $H(\mathcal{P}_n) \to 0$ as $n \to \infty$. Denote by $\mathcal{Q}_n$ the finite partition $\bigvee_{i=0}^{M_n-1} T_g^i \mathcal{P}_n$ of X. It is easy to deduce from the definition of $\mathcal{P}_n$ that the sequence $(\mathcal{Q}_n)_{n=1}^{\infty}$ generates $\mathfrak{B}$, i.e., for any $B \in \mathfrak{B}$ and $n \in \mathbb{N}$, there exists a $\mathcal{Q}_n$-measurable subset Q_n such that $\mu(B \Delta Q_n) \to 0$ as $n \to \infty$. Hence

$$h(T_g) = \lim_{n\to\infty} h(T_g, \mathcal{Q}_n) = \lim_{n\to\infty} h(T_g, \mathcal{P}_n) \leqslant \lim_{n\to\infty} H(\mathcal{P}_n) = 0. \qquad \square$$

We see, in particular, that for all mixing actions T constructed in Theorems 4.9–4.11 and 4.13, $h(T_g) = 0$ for all $g \in G$.

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