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Sinaĭ's condition for real valued Lévy processes

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Abstract

We prove that the upward ladder height subordinator H associated to a real valued Lévy process ξ has Laplace exponent φ that varies regularly at ∞ (respectively, at 0) if and only if the underlying Lévy process ξ satisfies Sinaĭ's condition at 0 (respectively, at ∞). Sinaĭ's condition for real valued Lévy processes is the continuous time analogue of Sinaĭ's condition for random walks. We provide several criteria in terms of the characteristics of ξ to determine whether or not it satisfies Sinaĭ's condition. Some of these criteria are deduced from tail estimates of the Lévy measure of H, here obtained, and which are analogous to the estimates of the tail distribution of the ladder height random variable of a random walk which are due to Veraverbeke and Grübel.

Résumé

Nous montrons que l'exposant de Laplace du subordinateur d'échelle H, associé à un processus de Lévy ξ, à valeurs réelles, est à variation régulière à l'infini (respectivement, en 0) si et seulement si le processus de Lévy sous-jacent vérifie la condition de Sinaĭ en 0 (respectivement, en $+\infty$). Cette dernière est l'analogue pour les processus de Lévy de la condition de Sinaĭ pour les marches aléatoires. Nous obtenons des estimations de la mesure de Lévy de H qui nous permettent d'établir des critères, en termes des caractéristiques de ξ, pour déterminer quand celui-ci vérifie la condition de Sinaĭ. Certaines de ces estimations sont l'analogue de celles obtenues par Veraverbeke et Grübel pour la queue de la fonction de répartition de la variable aléatoire d'échelle d'une marche aléatoire.

MSC: 60G30; 60G51

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1. Introduction and main results

Let $\xi = \{\xi_t,\ t \geqslant 0\}$ be a real valued Lévy process, $S = (S_t,\ t \geqslant 0)$ its current supremum and $L = (L_t,\ t \geqslant 0)$ the local time at 0 of the strong Markov process ξ reflected at its current supremum, that is to say $(S_t - \xi_t,\ t \geqslant 0)$. The first purpose of this work is to obtain some asymptotic properties of the ascending ladder height subordinator H associated to ξ (that is, the current supremum of ξ evaluated at the inverse of the local time at 0, i.e. L^{-1}, $H \equiv (S_{L_t^{-1}},\ t \geqslant 0)$).

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According to Fristedt [11] the ascending ladder process (L^{-1}, H) is a bivariate subordinator, that is, a Lévy process in $\mathbb{R}^2$ with increasing paths (coordinatewise) whose bivariate Laplace exponent κ,

$$\mathrm{e}^{-\kappa(\lambda_1,\lambda_2)} \equiv \mathbf{E}\big(\mathrm{e}^{-\lambda_1 L_1^{-1} - \lambda_2 H_1}\big), \quad \lambda_1, \lambda_2 \geqslant 0,$$

with the assumption $\mathrm{e}^{-\infty} = 0$, is given by

$$\kappa(\lambda_1, \lambda_2) = k \exp\left\{ \int_0^\infty \frac{\mathrm{d}t}{t} \int_{[0,\infty[} \big(\mathrm{e}^{-t} - \mathrm{e}^{-\lambda_1 t - \lambda_2 x}\big) \mathbf{P}(\xi_t \in \mathrm{d}x) \right\}, \quad \lambda_1, \lambda_2 \geqslant 0,$$

with k a constant that depends on the normalization of the local time. (See Doney [7], for a survey, and Bertoin [2] VI, for a detailed exposition of the fluctuation theory of Lévy processes and Vigon [24] for a description of the Lévy measure of H.)

The fact that the ladder process (L^{-1}, H) is a bivariate subordinator is central in the fluctuation theory of Lévy processes because it enables to obtain several properties of the underlying Lévy process using results for subordinators, which are objects simpler to manipulate. Among the various properties that can be obtained using this fact, there is a well known arc-sine law in the time scale for Lévy processes, see Theorem VI.3.14 in Bertoin's book [2] for a precise statement. That result tell us that Spitzer's condition is a condition about the underlying Lévy process ξ which ensures that the Laplace exponent $\kappa(\cdot, 0)$ of the ladder time subordinator L^{-1} is regularly varying and which in turn permits to obtain an arc-sine law in the time scale for Lévy processes. Now, if we want to establish an analogous result in the space scale we have to answer the question: *What is the analogue of Spitzer's condition for the upward ladder height process H*? or put another way: *What do we need to assume about ξ to ensure that the Laplace exponent*

$$\varphi(\lambda) \equiv \kappa(0, \lambda) = k \exp\left\{ \int_0^\infty \frac{\mathrm{d}t}{t} \int_{[0,\infty[} \big(\mathrm{e}^{-t} - \mathrm{e}^{-\lambda x}\big) \mathbf{P}(\xi_t \in \mathrm{d}x) \right\}, \quad \lambda \geqslant 0,$$

of H varies regularly?

A now classical limit theorem for random walks due to Greenwood, Omey and Teugels [15], Dynkin [9] and Rogozin [20] tell us that for random walks the answer to these questions is Sinaĭ's condition; see also [4] Theorem 8.9.17. So given that the fluctuation theory for Levy processes mirrors that of random walks, it is natural to hope that the answer to the questions posed above is the continuous time version of Sinaĭ's condition. We will say that a Lévy process ξ satisfies Sinaĭ's condition at ∞ (respectively, at 0) if

(Sinaĭ) There exists a $0 \leqslant \beta \leqslant 1$ such that

$$\int_0^\infty \frac{\mathrm{d}t}{t} \mathbf{P}(z < \xi_t \leqslant \lambda z) \to \beta \log(\lambda) \quad \text{as } z \to +\infty \text{ (respectively, } z \to 0+), \ \forall \lambda > 1.$$

The term β will be called Sinaĭ's index of ξ.

Example 1. A Lévy process, ξ, which satisfies Sinaĭ's condition is the strictly stable process with index $0 < \alpha \leqslant 2$. Indeed, for every $z > 0$ and $\lambda > 1$ we have by the scaling property of ξ that

$$\begin{aligned} \int_0^\infty \frac{\mathrm{d}t}{t} \mathbf{P}(z < \xi_t \leqslant \lambda z) &= \int_0^\infty \frac{\mathrm{d}t}{t} \mathbf{P}\big(z < t^{1/\alpha} \xi_1 \leqslant \lambda z\big) = \mathbf{E}\left(1_{\{\xi_1 > 0\}} \int_0^\infty \frac{\mathrm{d}t}{t} 1_{\{(z/\xi_1)^\alpha < t \leqslant (z\lambda/\xi_1)^\alpha\}} \right) \\ &= \mathbf{E}\big(1_{\{\xi_1 > 0\}} \log\big(\lambda^\alpha\big)\big) = \alpha \mathbf{P}(\xi_1 > 0) \log(\lambda). \end{aligned}$$

Thus any stable process ξ does satisfies Sinaĭ's condition at infinity and at 0 with index $\alpha\rho$, where ρ is the positivity parameter of ξ, $\rho = \mathbf{P}(\xi_1 \geqslant 0)$.

We recall that a measurable function $f:[0,\infty[\to[0,\infty[$ varies regularly at infinity (respectively, at 0) with index $\alpha\in\mathbb{R}$, $f\in RV_\alpha^\infty$ (respectively, $\in RV_\alpha^0$), if for any $\lambda>0$,

$$\lim\frac{f(\lambda x)}{f(x)}=\lambda^\alpha\quad\text{at }\infty\text{ (respectively, at 0).}$$

We have all the elements to state our main result, which provides an answer to the questions above.

Theorem 1. *For $\beta\in[0,1]$, the following are equivalent*

(i) *The Lévy process ξ satisfies Sinaĭ's condition at ∞ (respectively, at 0) with index β.*
(ii) *The Laplace exponent of the ladder height subordinator H varies regularly at 0 (respectively, at ∞) with index β.*

Proof. By the fluctuation identity of Bertoin and Doney [3] we have that for any $z>0$, $\lambda>1$,

$$\int_0^\infty\frac{\mathrm{d}t}{t}\mathbf{P}(z<\xi_t\leqslant\lambda z)=\int_0^\infty\frac{\mathrm{d}t}{t}\mathbf{P}(z<H_t\leqslant\lambda z).$$

As a consequence, Sinaĭ's condition is satisfied by the Lévy process ξ if and only if it is satisfied by the ascending ladder height subordinator H. The result then follows from Theorem 4, which establishes that the Laplace exponent ϕ of any given subordinator, say σ, varies regularly if and only if σ satisfies Sinaĭ's condition. □

Assuming that the Lévy process ξ satisfies Sinaĭ's condition and applying known results for subordinators, when its Laplace exponent is regularly varying, we can deduce the behavior at 0 or ∞ of ξ from that of H. (See Bertoin [2] Chapter III for an account on the short and long time behavior of subordinators.) The following *spatial arc-sine law* for Lévy processes is an example of the results that can be obtained.

Corollary 1. *For $r>0$, denote the first exit time of ξ out of $]-\infty,r]$ by $T_r=\inf\{t>0\colon\ \xi_t>r\}$, the undershoot and overshoot of the supremum of ξ by $U(r)=r-S_{T_r-}$ and $O(r)=S_{T_r}-r=\xi_{T_r}-r$. For any $\beta\in[0,1]$, the conditions* (i) *and* (ii) *in Theorem* 1 *are equivalent to the following conditions*:

(iii) *The random variables $r^{-1}(U(r),O(r))$ converge in distribution as $r\to\infty$ (respectively, as $r\to0$).*
(iv) *The random variables $r^{-1}O(r)$ converge in distribution as $r\to\infty$ (respectively, as $r\to0$).*
(v) *The random variables $r^{-1}S_{T_r-}$ converge in distribution as $r\to\infty$ (respectively, as $r\to0$).*
(vi) $\lim r^{-1}\mathbf{E}(S_{T_r-})=\beta\in[0,1]$ *as $r\to\infty$ (respectively, as $r\to0$).*

In this case, the limit distribution in (iii) *is determined as follows: if $\beta=0$ (respectively, $\beta=1$), it is the Dirac mass at $(1,\infty)$ (respectively, at $(0,0)$). For $\beta\in]0,1[$, it is the distribution with density*

$$p_\beta(u,w)=\frac{\beta\sin\beta\pi}{\pi}(1-u)^{\beta-1}(u+w)^{-1-\beta},\quad 0<u<1,\ w>0.$$

In particular, the limit law in (v) *is the generalized arc-sine law of parameter β.*

Proof. We recall that for every $r>0$, the random variables $(U(r),O(r))$ are almost surely equal to the undershoot and overshoot, $(U_H(r),O_H(r))$, of the ladder height subordinator H. Thus the result is a straightforward consequence of the Dynkin–Lamperti arc-sine law for subordinators, Theorem III.3.6 in [2], using the elementary relations: for every $r>0$

$$\mathbf{P}\big(U_H(r)>y\big)=\mathbf{P}\big(O_H(r-y)>y\big),\quad r>y>0,$$
$$\mathbf{P}\big(O_H(r)>x,U_H(r)>y\big)=\mathbf{P}\big(O_H(r-y)>x+y\big),\quad r>y>0,\ x>0.\qquad\square$$

To summarize, in Theorem 1 we provided a necessary and sufficient condition in terms of the marginal laws of ξ which completely answers the questions posed at the beginning of this work. However, the possible drawback of this result is that in most of the cases we only know the characteristics of the Lévy process ξ, that is, its linear and Gaussian terms and Lévy measure, and so it would be suitable to have a condition in terms of the characteristics of the process. That is the purpose of the second part of this work.

One case at which Sinaĭ's condition can be verified using the characteristics of the process is the case at which the underlying Lévy process belongs to the domain of attraction at infinity (respectively, at 0) of a strictly stable law of index $0<\alpha\leqslant 2$, and which does not require a centering function. That is, whenever there exists a deterministic function $b:]0,\infty[\to]0,\infty[$ such that

$$\frac{\xi_t}{b(t)} \xrightarrow{\mathcal{D}} X(1) \quad \text{as } t\to\infty \text{ (respectively, as } t\to 0), \tag{1}$$

with $X(1)$ a strictly stable random variable of parameter $0<\alpha\leqslant 2$. It is well known that if such a function b exists, it is regularly varying at infinity (respectively, at 0) with index $\beta=1/\alpha$. Plainly, the convergence in (1) can be determined in terms of the characteristic exponent Ψ of ξ, i.e. $\mathbf{E}(\mathrm{e}^{i\lambda\xi_t})=\exp\{t\Psi(\lambda)\},\ \lambda\in\mathbb{R}$, since the latter convergence in distribution is equivalent to the validity of the limit

$$\lim t\Psi\left(\frac{\lambda}{b(t)}\right)=\Psi_\alpha(\lambda) \quad \text{as } t\to\infty \text{ (respectively, as } t\to 0) \text{ for } \lambda\in\mathbb{R}, \tag{2}$$

where Ψ_α is the characteristic exponent of a strictly stable law and is given by

$$\Psi_\alpha(\lambda)=\begin{cases} -c|\lambda|^\alpha(1-\mathrm{i}\delta\,\mathrm{sgn}(\lambda)\tan(\pi\alpha/2)), & 0<\alpha<1 \text{ or } 1<\alpha<2;\\ -c|\lambda|^\alpha(1-\mathrm{i}\delta\,\mathrm{sgn}(\lambda)\tan(\pi\alpha/2)\ln(|\lambda|)), & \alpha=1;\\ -q^2\lambda^2/2, & \alpha=2; \end{cases}$$

for $\lambda\in\mathbb{R}$, where $c>0$ and the term $\delta\in[-1,1]$ is the so called skewness parameter. We have the following theorem whose proof will be given in Section 3.

Theorem 2. *Let $0<\alpha\leqslant 2$ and $\delta\in[-1,1]$. Assume that there exists a function $b:]0,\infty[\to]0,\infty[$ such that the limit in Eq.* (2) *holds as t goes to infinity* (*respectively, as $t\to 0$*). *Then the Lévy process ξ satisfies Sinai's condition at ∞* (*respectively, at* 0) *with index $\alpha\rho$, where ρ is given by $\rho=1/2+(\pi\alpha)^{-1}\arctan(\delta\tan(\alpha\pi/2))$.*

The converse of this theorem is not true in general, see Remark 2 below.

With the aim of providing some other criteria in terms of the characteristics of the underlying Lévy process ξ to determine whether or not it satisfies Sinaĭ's condition we recall that the regular variation of the Laplace exponent, φ, of H is closely related to the regular variation of the right tail of its Lévy measure, cf. [2] p. 82. Owing to this, we will next restrict ourselves to studying the behavior of the right tail of the Lévy measure of H, say $\overline{po}$. To that end, we should be able to control the behavior of the dual ladder height subordinator $\widehat{H}$, that is, the ladder height subordinator of the dual Lévy process $\hat{\xi}=-\xi$. This is due to the fact, showed by Vigon [24], that the Lévy measure of the ladder height subordinator H is determined by the Lévy measure of ξ and the potential measure of $\widehat{H}$. (See the Lemma 1 below for a precise statement.)

Thus, under some assumptions on the dual ladder height process, that can be verified directly from the characteristics of ξ, below we will provide some tail estimates of the right tail of the Lévy measure of H which in turn will allow us to furnish some NASC for the regular variation, at infinity or 0, of the Laplace exponent of the ladder height subordinator H. But first we need to introduce some supplementary notation.

We will assume hereafter that the underlying Lévy process is not spectrally negative, that is $\Pi]0,\infty[>0$, since in that case the ladder height process H is simply a drift, $H_t=\mathrm{d}t,\ t\geqslant 0$. We will denote by (k_0,d,po) (respectively, $(\hat{k}_0,\hat{d},ne)$) the characteristics of the subordinator H (respectively, $\widehat{H}$) that is, its killing and drift terms and Lévy measure, respectively. Let $\widehat{V}$ be the potential measure of $\widehat{H}$, that is $\widehat{V}(\mathrm{d}x)=\mathbf{E}(\int_0^\infty 1_{\{\widehat{H}_t\in \mathrm{d}x\}}\,\mathrm{d}t)$. Furthermore, we will denote by (a,q,Π) the characteristics of the Lévy process ξ. Finally, by the symbols $\overline{po}, \overline{ne}, \overline{\Pi}^+$, we denote the right tail of the Lévy measures of H, $\widehat{H}$ and ξ respectively, that is

$$\overline{po}(x)=po]x,\infty[,\quad \overline{ne}(x)=ne]x,\infty[,\qquad \overline{\Pi}^+(x)=\Pi]x,\infty[,\ x>0,$$

and by Π^+ the restriction of Π to $]0,\infty[$, $\Pi^+=\Pi|_{]0,\infty[}$.

As we mentioned before, Vigon [24] established some identities "equations amicales" relying the Lévy measures po, ne and Π; these are quoted below for ease of reference.

Lemma 1. (Vigon [24], Equations amicales.) *We have the following relations*

(EAI) $\overline{po}(x)=\int_0^\infty \widehat{V}(\mathrm{d}y)\overline{\Pi}(x+y),\ x>0.$
(EA) $\overline{\Pi}^+(x)=\int_{]x,\infty[} po(\mathrm{d}y)\overline{ne}(y-x)+\hat{d}\bar{p}(x)+\hat{k}_0\overline{po}(x)$, *for any* $x>0$; *where* $\bar{p}(x)$ *is the density of the measure* po, *which exists if* $\hat{d}>0$.

We will say that a measure M on $[0,\infty[$ belongs to the class $\mathcal{L}^0$ of *long tailed* measures if its tail $\overline{M}(x)=M]x,\infty[$, is such that $0<\overline{M}(x)<\infty$ for each $x>0$ and

$$\lim_{x\to\infty}\frac{\overline{M}(x+t)}{\overline{M}(x)}=1,\quad \text{for each } t\in\mathbb{R}.$$

It is well known that this family includes the subexponential measures and the cases at which $\overline{M}$ is regularly varying.

We have now all the elements to state our results that relate the behavior of $\overline{\Pi}^+$ with that of $\overline{po}$ at infinity, they are the continuous time analogue of the result of Veraverbeke [22] and Grübel [16] for random walks.

Theorem 3.

(a) *Assume that the dual ladder height subordinator has a finite mean* $\mu=\mathbf{E}(\widehat{H}_1)<\infty$, *which implies that* ξ *does not drifts to* ∞. *The following are equivalent*
  (a-1) *The measure* Π_I^+ *on* $[0,\infty[$ *with tail* $\overline{\Pi}_I^+(x)=\int_x^\infty \overline{\Pi}^+(z)\,\mathrm{d}z,\ x\geqslant 0$, *belongs to* $\mathcal{L}^0$.
  (a-2) $po\in\mathcal{L}^0$.
  (a-3) $\overline{po}(x)\sim\frac{1}{\mu}\overline{\Pi}_I^+(x)$, *as* $x\to\infty$.

(b) *Assume that the dual ladder height subordinator* $\widehat{H}$ *has killing term* $\hat{k}_0>0$ *or equivalently that* ξ *drifts to* ∞. *The following are equivalent*
  (b-1) $\Pi^+=\Pi|_{]0,\infty[}\in\mathcal{L}^0$.
  (b-2) $po\in\mathcal{L}^0$,

$$\frac{\hat{d}\bar{p}(x)}{\overline{po}(x)}\to 0\quad \textit{and}\quad \int_0^1\left(\frac{\overline{po}(x)-\overline{po}(x+y)}{\overline{po}(x)}\right)ne(\mathrm{d}y)\to 0\quad \textit{as } x\to\infty. \tag{3}$$

  (b-3) $\overline{po}(x)\sim\frac{1}{\hat{k}_0}\overline{\Pi}^+(x)$ *as* $x\to\infty$.

In Remarks 3 and 4 below we make some comments on NASC for the assumptions in Theorem 3 and on some related results, respectively.

As a corollary of the previous theorem we have the following criterions to determine whether or not the tail of the Lévy measure of H is regularly varying.

Corollary 2.

(a) *Under the assumptions of* (a) *in Theorem* 3 *and for any* $\alpha\in\,]0,1]$ *we have that* $\overline{\Pi}^+\in RV^\infty_{-1-\alpha}$ *if and only if* $\overline{po}\in RV^\infty_{-\alpha}$. *Both imply that*

$$\overline{po}(x)\sim\frac{1}{\alpha\mu}x\overline{\Pi}^+(x),\quad x\to\infty.$$

(b) *Under the assumptions of* (b) *in Theorem* 3 *and for any* $\alpha\in\,]0,1[$ *we have that* $\overline{\Pi}^+\in RV^\infty_{-\alpha}$ *if and only if* $\overline{po}\in RV^\infty_{-\alpha}$ *and*

$$\int_0^1\left(\frac{\overline{po}(x)-\overline{po}(x+y)}{\overline{po}(x)}\right)ne(\mathrm{d}y)\to 0\quad \textit{as } x\to\infty.$$

Proof. The proof of (a) in Corollary 2 follows from the fact that under these hypotheses

$$\frac{x\overline{\Pi}^{+}(x)}{\int_x^\infty \overline{\Pi}^{+}(z)\,\mathrm{d}z} \to \alpha \quad \text{as } x \to \infty,$$

which is a consequence of Theorem 1.5.11 in [4]. The proof of (b) is straightforward. □

The behavior at 0 of $\overline{po}$ was studied by Vigon in [23] Theorems 6.3.1 and 6.3.2. He obtained several estimations that we will use here to provide an analogue of Corollary 2 for the behavior at 0 of $\overline{po}$.

Proposition 1.

(a) *Assume that $\widehat{H}$ has a drift $\hat{d} > 0$. For any $\alpha \in]0,1]$ we have that $\overline{po} \in RV^0_{-\alpha}$ if and only if $\overline{\Pi}^{+} \in RV^0_{-\alpha-1}$.*

(b) *Assume that $\widehat{H}$ has a drift $\hat{d} = 0$ and that the total mass of the measure ne is finite, equivalently,* $\lim_{x\to 0+}\overline{ne}(x) < \infty$. *Then for any $\alpha \in]0,1]$ we have that $\overline{po} \in RV^0_{-\alpha}$ if and only if $\overline{\Pi}^{+} \in RV^0_{-\alpha}$. Moreover, the same assertion holds if furthermore $\alpha = 0$ and* $\lim_{x\to 0+}\overline{po}(x) = \infty$.

Cf. [24] and [23] Chapter 10 for NASC for the assumptions in Proposition 1.

Before finishing this section we would like to make some remarks.

Remark 1. As a consequence to Theorem 1, in the case where Sinaĭ's condition hold for ξ with index $\beta = 1$ at ∞ (respectively, at 0) we have that $r^{-1}\xi_{T_r}$ converges in law to 1 as $r \to \infty$ (respectively, to 0). The almost sure convergence of this random variable was studied by Doney and Maller [8]. Precisely, Theorem 8 of Doney and Maller [8] provide necessary and sufficient conditions, on the characteristics of ξ, according to which $r^{-1}\xi_{T_r}$ converges a.s. to 1 as $r \to \infty$. Moreover, Corollary 1 in [8] establishes that the latter sequence of r.v. converges a.s. to 1 as $r \to 0$ if and only if ξ creeps upward.

Remark 2. Observe that if ξ satisfies the hypotheses of Theorem 2 at infinity so it does $\hat{\xi} = -\xi$ and as a consequence $\hat{\xi}$ satisfies Sinaĭ's condition. Thus, given that the downward ladder height subordinator, $\widehat{H}$, associated to ξ is the upward ladder height subordinator associated to $\hat{\xi}$, then under the hypotheses of Theorem 2 the Laplace exponent of H and $\widehat{H}$, respectively, is regularly varying at 0 with index $\alpha\rho$ and $\alpha(1-\rho)$, respectively. This fact allow us to realize that the reciprocal of Theorem 2 is not true in general. To construct a counterexample, let $h = \{h_t,\ t \geqslant 0\}$ be a stable subordinator with parameter $\alpha \in]0,1[$, and $\hat{h} = \{\hat{h}_t,\ t \geqslant 0\}$, be a subordinator with infinite lifetime, without drift, such that its Lévy measure is absolutely continuous with a decreasing density and assume that its Laplace exponent is not regularly varying at 0. Observe that according to Theorem 4, h satisfies Sinaĭ's condition at infinity but $\hat{h}$ does not. According to the results in Section 7.3 in [23] there exists a real valued Lévy process, say $\tilde{\xi}$, such that its upward and downward ladder height subordinators are equal in law to h and $\hat{h}$, respectively. Thus the process $\tilde{\xi}$ satisfies Sinaĭ's condition at infinity, since h does, but there does not exists any function such that the limit in Eq. (2) holds as t goes to infinity, because if this were indeed the case it would imply that $\hat{h}$ satisfies Sinaĭ's condition at infinity, which is a contradiction.

Remark 3. The assumptions in Theorem 3 can be verified using only the characteristics of the underlying Lévy process ξ. According to a result due to Chow [5] necessary and sufficient conditions on ξ to be such that $\mathbf{E}(\widehat{H}_1) < \infty$, are either $0 < \mathbf{E}(-\xi_1) \leqslant \mathbf{E}|\xi_1| < \infty$ or $0 = \mathbf{E}(-\xi_1) < \mathbf{E}|\xi_1| < \infty$ and

$$\int_{[1,\infty[} \left(\frac{x\overline{\Pi}^{-}(x)}{1+\int_0^x \mathrm{d}y \int_y^\infty \overline{\Pi}^{+}(z)\,\mathrm{d}z}\right)\mathrm{d}x < \infty \quad \text{with } \overline{\Pi}^{-}(x) = \Pi]-\infty,-x[,\ x > 0.$$

Observe that under such assumptions the Lévy process ξ does not drift to ∞, i.e. $\liminf_{t\to\infty}\xi_t = -\infty$, $\mathbf{P}$-a.s. The case where the Lévy process ξ drifts to ∞, $\lim_{t\to\infty}\xi_t = \infty$, $\mathbf{P}$-a.s. or equivalently $\hat{k}_0 > 0$ is considered in (b). Kesten and Erickson's criteria state that ξ drifts to ∞ if and only if

$$\int_{]-\infty,-1[} \left(\frac{|y|}{\overline{\Pi}^{+}(1)+\int_{1}^{|y|}\overline{\Pi}^{+}(z)\,\mathrm{d}z}\right)\Pi(\mathrm{d}y) < \infty = \int_{1}^{\infty}\overline{\Pi}^{+}(x)\,\mathrm{d}x \quad \text{or} \quad 0 < \mathbf{E}(\xi_1) \leqslant \mathbf{E}|\xi_1| < \infty,$$

cf. [17] and [10]. (Actually, Chow, Kesten and Erickson proved the results above for random walks, its translation for real valued Lévy processes can be found in [8] and [23].)

Remark 4. The results in Theorem 3 are close in spirit to those obtained by Klüppelberg, Kyprianou and Maller [19]. In that work the authors assume that the Lévy process ξ drifts to $-\infty$, i.e. $\lim_{t\to\infty}\xi_t = -\infty$, $\mathbf{P}$-a.s. and obtain several asymptotic estimates of the function $\overline{po}$ in terms of the Lévy measure Π. In our setting we permit any behavior of ξ at the price of making some assumptions on the dual ladder height subordinator. Furthermore, the results in Theorem 3 concern only the case at which the underlying Lévy process does not have exponential moments and so it extends to Lévy processes Theorem 1-(B,C) of Veraverbeke [22]. The case at which the Lévy process has exponential moments has been considered by Klüppelberg et al. [19] Proposition 5.3 under the assumption that the underlying Lévy process has positive jumps and drifts to $-\infty$, but actually the latter hypothesis is not used in their proof, and so their result is still true in this more general setting, which extends Theorem 1-A in [22].

Remark 5. The estimate of $\overline{po}$ obtained in Theorem 3(a) holds whenever the function $\overline{\Pi}^{+}_{I}$ belongs to the class $\mathcal{L}^0$, but it is known that it occurs even if $\Pi^{+} \notin \mathcal{L}^0$, see e.g. Klüppelberg [18]. A question arises: Is it possible to sharpen the estimate of $\overline{po}$ provided in Theorem 3(a) when moreover $\Pi^{+} \in \mathcal{L}^0$? The following result answers this question in affirmatively and will be proved below.

Proposition 2. *Assume that $\mu = \mathbf{E}(\widehat{H}_1) < \infty$. The following are equivalent*

(i) $\Pi^{+} \in \mathcal{L}^0$.
(ii) *For any $g : \mathbb{R}^{+} \to \mathbb{R}^{+}$ directly Riemann integrable,*

$$\lim_{x\to\infty}\frac{1}{\overline{\Pi}^{+}(x)}\int_{x}^{\infty} po(\mathrm{d}y)g(y-x) = \frac{1}{\mu}\int_{0}^{\infty} g(z)\,\mathrm{d}z.$$

To our knowledge the discrete time analogue of this result, that we state below, is unknown, although it can be easily deduced from the arguments in Asmussen et al. [1] Lemma 3.

Proposition 3. *Let $X_1, X_2, \ldots$ be a sequence of independent and identically distributed random variables, Z its associated random walk $Z_0 = 0$, $Z_n = \sum_{k=1}^{n} X_k, n > 0$ and define the pair of random variables (N, Z_N) where N is the first ladder epoch of the random walk Z, $N = \min\{k\colon Z_k > 0\}$, and Z_N, is the position of Z at the instant N. Assume that $m = \mathbf{E}(Z_{\widehat{N}}) < \infty$, where $\widehat{N} = \inf\{n > 0\colon Z_n \leqslant 0\}$ and that the law of X_1 is non-lattice. The following are equivalent*

(i) *The law of X_1 belongs to the class $\mathcal{L}^0$.*
(ii) *For any $g : \mathbb{R}^{+} \to \mathbb{R}^{+}$ directly Riemann integrable,*

$$\lim_{x\to\infty}\frac{1}{\overline{F}(x)}\int_{x}^{\infty} g(y-x)\mathbf{P}(Z_N \in \mathrm{d}y) = \frac{1}{m}\int_{0}^{\infty} g(z)\,\mathrm{d}z,$$

where $\overline{F}(x) = \mathbf{P}(X_1 > x)$, $x > 0$.

The proof of this result is quite similar to that of Proposition 2 and so we will omit it.

The forthcoming sections are organized as follows. In Section 2, we focus all our efforts into prove an equivalent form of the Dynkin and Lamperti theorem for subordinators (see e.g. [2] Theorem III.6) which is one of the key tools in the proof of Theorem 1 and it is interesting in itself. In Section 3, we prove Theorem 2. Finally, Section 4 is devoted to the proof of Theorem 3 and Propositions 1 and 2.

2. A result for subordinators

In the proof of Theorem 1 we have seen that a fluctuation identity due to Bertoin and Doney allows us to simplify our problem for general Levy processes into one for subordinators. Namely, that for subordinators Sinaĭ's condition is equivalent to the regular variation of the associated Laplace exponent. The purpose of this section is to prove the latter assertion, and so throughout this section we will only deal with subordinators.

Let $\sigma = (\sigma_t,\ t \geqslant 0)$ be a subordinator, possibly killed, with life time ζ, and denote by ϕ its Laplace exponent,

$$\phi(\lambda) \equiv -\log \mathbf{E}\big(\mathrm{e}^{-\lambda\sigma_1}, 1 < \zeta\big), \quad \lambda \geqslant 0.$$

It is well known that the Laplace exponent ϕ can be represented as

$$\phi(\lambda) = \kappa + \lambda d + \int_{]0,\infty[} \big(1 - \mathrm{e}^{-\lambda y}\big)\nu(\mathrm{d}y), \quad \lambda \geqslant 0,$$

where $\kappa, d \geqslant 0$ are the killing rate and drift coefficient of σ, respectively, and ν is the Lévy measure of σ, that is, a measure on $]0, \infty[$ such that $\int_{]0,\infty[} \min\{1, y\}\nu(\mathrm{d}y) < \infty$.

The main result of this section is the following equivalent form of the Dynkin and Lamperti theorem for subordinators (see e.g. [2] Theorem III.6, see also page 82 therein for an account on necessary and sufficient conditions according to which a subordinator has a Laplace exponent that is regularly varying either at infinity or at 0).

Theorem 4. *For $\beta \in [0, 1]$, the following are equivalent*:

(i) *The subordinator σ satisfies Sinaĭ's condition at $0+$ (respectively, at $+\infty$) with index β.*
(ii) *The Laplace exponent ϕ is regularly varying at $+\infty$ (respectively, at $0+$) with index β.*

The proof of this result relies on the following elementary remark.

Remark 6. Write

$$\phi(\theta) = \big(1 + \phi(\theta)\big)\frac{\phi(\theta)}{1 + \phi(\theta)}, \quad \theta \geqslant 0. \tag{4}$$

The first (respectively, second) factor in the right-hand term of the previous equality can be used to determine the behavior at infinity (respectively, at 0) of ϕ. More precisely, $\phi \in RV_\beta^\infty$ (respectively, $\in RV_\beta^0$) if and only

$$1 + \phi(\cdot) \in RV_\beta^\infty \quad \left(\text{respectively, } \frac{\phi(\cdot)}{1 + \phi(\cdot)} \in RV_\beta^0\right).$$

This is based on the fact that for $\lambda > 1$,

$$\int_0^\infty \frac{\mathrm{d}t}{t}\mathbf{P}(z < \sigma_t \leqslant \lambda z) \sim \int_0^\infty \frac{\mathrm{d}t}{t}\mathrm{e}^{-t}\mathbf{P}(z < \sigma_t \leqslant \lambda z) \sim \int_0^1 \frac{\mathrm{d}t}{t}\mathbf{P}(z < \sigma_t \leqslant \lambda z), \quad z \to 0+,$$

and

$$\int_0^\infty \frac{\mathrm{d}t}{t}\mathbf{P}(z < \sigma_t \leqslant \lambda z) \sim \int_0^\infty \frac{\mathrm{d}t}{t}\big(1 - \mathrm{e}^{-t}\big)\mathbf{P}(z < \sigma_t \leqslant \lambda z) \sim \int_1^\infty \frac{\mathrm{d}t}{t}\mathbf{P}(z < \sigma_t \leqslant \lambda z), \quad z \to \infty,$$

since only the small time (respectively, long time) behavior of σ is relevant to estimate the value at 0 (respectively, at ∞) of the leftmost integral in the former (respectively, latter) equation. In general, for studying properties related to the small time behavior of σ it may be useful to consider

$$\int_0^\infty \frac{\mathrm{d}t}{t}\mathrm{e}^{-t}\mathbf{P}(\sigma_t \in \mathrm{dz}) \quad \text{or} \quad \int_0^1 \frac{\mathrm{d}t}{t}\mathbf{P}(\sigma_t \in \mathrm{d}z),$$

instead of $\int_0^\infty \frac{\mathrm{d}t}{t}\mathbf{P}(\sigma_t \in \mathrm{d}z)$. The analogous holds also for large time behavior.

The proof of Theorem 4 will be given using the previous remark and via three lemmas whose proof will be given at the end of this section. The first of them will enable us to relate the factors in Eq. (4) with a transformation of the type Mellin's convolution.

Lemma 2. *We have that*

(i) $1+\phi(\theta)=\exp\{\widehat{G}_1(\theta)\}$, *for* $\theta>0$; *where the function* $\widehat{G}_1$ *is the Mellin convolution of the non-decreasing function*

$$G_1(y)=\int_0^\infty \frac{\mathrm{d}t}{t}\,\mathrm{e}^{-t}\mathbf{P}\left(\sigma_t>\frac{1}{y}\right),\quad y>0,$$

and the kernel $k(x)=x\,\mathrm{e}^{-x},\ x>0$; *that is,*

$$\widehat{G}_1(\theta)=k^M * G_1(\theta):=\int_0^\infty \frac{\mathrm{d}x}{x}k\left(\frac{\theta}{x}\right)G_1(x),\quad \theta>0.$$

(ii) $\phi(\theta)/(1+\phi(\theta))=\exp\{-\widehat{G}_2(\theta)\}$, *for* $\theta>0$; *where* $\widehat{G}_2$ *is the Laplace transform of the measure*

$$G_2(\mathrm{d}x)=\int_0^\infty \frac{\mathrm{d}t}{t}\left(1-\mathrm{e}^{-t}\right)\mathbf{P}(\sigma_t\in \mathrm{d}x),\quad x>0;$$

which is in fact the harmonic renewal measure associated to the law $F(\mathrm{d}x)=\mathbf{P}(\sigma_\Theta\in\mathrm{d}x)$ *with* Θ *an independent random variable with exponential law of parameter* 1.

A consequence of Lemma 2 is that $1+\phi\in RV^\infty_\beta$ if and only if

$$\lim_{\theta\to\infty}\widehat{G}_1(\lambda\theta)-\widehat{G}_1(\theta)=\beta\log\lambda,\quad \forall\lambda>0. \tag{5}$$

Moreover, $\phi(\cdot)/(1+\phi(\cdot))\in RV^0_\beta$ if and only if

$$\lim_{\theta\to 0}\widehat{G}_2(\lambda\theta)-\widehat{G}_2(\theta)=-\beta\log\lambda,\quad \forall\lambda>0. \tag{6}$$

The second of these lemmas enable us to relate Sinai's condition with the behavior at infinity of the differences of the function G_1, and those of the function $G_2(x)\equiv G_2[0,x]$, $x>0$.

Lemma 3. *Let* $\beta\in[0,1]$.

(i) *Sinaĭ's condition holds at 0 with index* β *if and only if*

$$\lim_{z\to\infty}G_1(\lambda z)-G_1(z)=\beta\log(\lambda),\quad \forall\lambda>1.$$

(ii) *Let* $G_2(z):=G_2[0,z]$, $z>0$. *Sinaĭ's condition holds at infinity with index* β *if and only if*

$$\lim_{z\to\infty}G_2(\lambda z)-G_2(z)=\beta\log(\lambda),\quad \forall\lambda>1.$$

The last ingredient to achieve the proof of Theorem 4 is an Abelian–Tauberian type result relating the behavior of the differences of G_1 (respectively, G_2) with those of the functions $\widehat{G}_1$ (respectively, $\widehat{G}_2$).

Lemma 4.

(i) *The following are equivalent*

$$\lim_{y\to\infty}G_1(\lambda y)-G_1(y)=\beta\log(\lambda),\quad \forall\lambda>0. \tag{7}$$

$$\lim_{\theta\to\infty}\widehat{G}_1(\lambda\theta)-\widehat{G}_1(\theta)=\beta\log(\lambda),\quad \forall\lambda>0. \tag{8}$$

Both imply that

$$G_1(\theta) - \widehat{G}_1(\theta) \underset{\theta\to\infty}{\longrightarrow} \beta\gamma.$$

(ii) *The following are equivalent*

$$\lim_{y\to\infty} G_2(\lambda y) - G_2(y) = \beta \log(\lambda), \quad \forall \lambda > 0. \tag{9}$$

$$\lim_{\theta\to 0} \widehat{G}_2(\lambda\theta) - \widehat{G}_2(\theta) = -\beta \log(\lambda), \quad \forall \lambda > 0. \tag{10}$$

Both imply that

$$G_2(\theta) - \widehat{G}_2\left(\frac{1}{\theta}\right) \underset{\theta\to\infty}{\longrightarrow} \beta\gamma.$$

Where γ is Euler's constant $\gamma = \int_0^\infty \mathrm{e}^{-v} \log(v)\,\mathrm{d}v$.

Tacking for granted Lemmas 2, 3 and 4 the proof of Theorem 4 is straightforward.

A consequence of Lemma 4 is that quantities related to Sinaĭ's condition can be used to determine some parameters of σ. That is the content of the following corollary.

Corollary 3.

(i) *σ has a finite lifetime a.s. if and only if*

$$r \equiv \lim_{\theta\to\infty} G_2(\theta) < \infty.$$

In this case, $\phi(0) = (\mathrm{e}^r - 1)^{-1}$. In particular, Sinaĭ's condition is satisfied at infinity with index $\beta = 0$.

(ii) *σ is a compound Poisson process if and only if*

$$\tilde{r} \equiv \lim_{\theta\to\infty} G_1(\theta) < \infty.$$

In this case, $\nu]0,\infty[= \mathrm{e}^{\tilde{r}} - 1$. In particular, Sinaĭ's condition is satisfied at 0 with index $\beta = 0$.

(iii) *Assume that Sinaĭ's condition holds at infinity with index $\beta = 1$. Then σ has a finite mean if and only if*

$$R \equiv \lim_{\theta\to\infty} \log(\theta) - G_2(\theta) < \infty.$$

In this case, $\mathbf{E}(\sigma_1) = \mathrm{e}^{\gamma+R}$.

(iv) *Assume that Sinaĭ's condition holds at 0 with index $\beta = 1$. Then σ has a strictly positive drift d if and only if*

$$\widetilde{R} \equiv \lim_{\theta\to\infty} G_1(\theta) - \log(\theta) < \infty$$

In this case, $d = \mathrm{e}^{\gamma+\widetilde{R}}$.

Before we pass to the proof of Lemmas 2, 3 and 4 we make a final remark.

Remark 7. For $\beta \in]0,1[$, it is well known that $\phi \in RV_\beta^\infty$ if and only if the sequence of subordinators σ^z defined by $(\sigma_t^z = z\sigma_{t/\phi(z)},\ t \geqslant 0)$ converge as $z \to \infty$, in the sense of finite dimensional distributions and in Skorohod's topology, to a stable subordinator $\tilde{\sigma}$ of parameter β. This is equivalent to say that for any $t > 0$

$$\mathbf{P}(1 < z\sigma_{t/\phi(z)} \leqslant \lambda) \underset{z\to\infty}{\longrightarrow} \mathbf{P}(1 < \tilde{\sigma}_t \leqslant \lambda), \quad \lambda > 1, \tag{11}$$

and

$$\mathbf{P}(\lambda < z\sigma_{t/\phi(z)} \leqslant 1) \underset{z\to\infty}{\longrightarrow} \mathbf{P}(\lambda < \tilde{\sigma}_t \leqslant 1), \quad 0 < \lambda < 1. \tag{12}$$

On the other hand, Theorem 4 ensures that the latter condition on ϕ holds if and only if Sinaĭ's condition holds at 0, which can be written as follows: for any $\lambda > 1$

$$\int_0^\infty \frac{\mathrm{d}s}{s}\mathbf{P}\left(1 < z^{-1}\sigma_{s/\phi(z^{-1})} \leqslant \lambda\right) = \int_0^\infty \frac{\mathrm{d}t}{t}\mathbf{P}(z < \sigma_t \leqslant \lambda z) \underset{z\to 0}{\longrightarrow} \beta\ln(\lambda) = \int_0^\infty \frac{\mathrm{d}s}{s}\mathbf{P}(1 < \tilde{\sigma}_s \leqslant \lambda), \tag{13}$$

where the first equality is justified by a change of variables $s = t\phi(z^{-1})$ and the last one follows from the scaling property of the stable subordinator $\tilde{\sigma}$; and for any $0 < \lambda < 1$

$$\int_0^\infty \frac{\mathrm{d}s}{s}\mathbf{P}\left(\lambda < z^{-1}\sigma_{s/\phi(z^{-1})} \leqslant 1\right) \underset{z\to 0}{\longrightarrow} \int_0^\infty \frac{\mathrm{d}s}{s}\mathbf{P}(\lambda < \tilde{\sigma}_s \leqslant 1), \tag{14}$$

Putting the pieces together we get that the result in Theorem 4 can be viewed as an equivalence between the convergence of the uni-dimensional laws of σ^z in (11) and (12) and the convergence of the integrated ones in (13) and (14). An analogous fact can be deduced for the convergence of σ^z as z goes to 0 whenever Sinaĭ's condition holds at infinity.

2.1. Proof of Lemmas 2, 3 and 4

Proof of (i) in Lemma 2. We have by Frullani's formula that for every $\theta > 0$

$$1+\phi(\theta) = \exp\left\{\int_0^\infty \frac{\mathrm{d}t}{t}\,\mathrm{e}^{-t}\left(1-\mathrm{e}^{-t\phi(\theta)}\right)\right\} = \exp\left\{\theta\int_0^\infty \mathrm{d}y\,\mathrm{e}^{-\theta y}\int_0^\infty \frac{\mathrm{d}t}{t}\,\mathrm{e}^{-t}\mathbf{P}(\sigma_t > y)\right\} = \exp\left\{\widehat{G}_1(\theta)\right\}. \qquad \square$$

Proof of (ii) in Lemma 2. The equation relating ϕ and the measure G_2 can be obtained using Frullani's formula but to prove moreover that this measure is in fact is an harmonic renewal measure we proceed as follows. Let $(\mathrm{e}_k,\ k \geqslant 1)$ be a sequence of independent identically distributed random variables with exponential law of parameter 1 and independents of σ. Put $\Theta_l = \sum_{k=1}^{l}\mathrm{e}_k,\ l \geqslant 1$. It was proved by Bertoin and Doney [3], and it is easy to prove, that $(\sigma_{\Theta_l},\ l \geqslant 1)$ forms a renewal process. The harmonic renewal measure associated to $(\sigma_{\Theta_l},\ l \geqslant 1)$ is $G_2(\mathrm{d}x)$. Indeed,

$$\sum_{l=1}^\infty \frac{1}{l}\mathbf{P}(\sigma_{\Theta_l} \in \mathrm{d}x) = \sum_{l=1}^\infty \frac{1}{l}\int_0^\infty \mathrm{d}t\frac{t^{l-1}}{(l-1)!}\,\mathrm{e}^{-t}\mathbf{P}(\sigma_t \in \mathrm{d}x) = \int_0^\infty \frac{\mathrm{d}t}{t}\,\mathrm{e}^{-t}\left(\mathrm{e}^{t}-1\right)\mathbf{P}(\sigma_t \in \mathrm{d}x) = G_2(\mathrm{d}x).$$

Moreover, since the l-convolution of $F(\mathrm{d}x) = \mathbf{P}(\sigma_{\Theta_1} \in \mathrm{d}x)$ is such that

$$F^{*l}(\mathrm{d}x) = \mathbf{P}(\sigma_{\Theta_l} \in \mathrm{d}x)$$

we have that the Laplace transform $\widehat{F}(\theta)$ of F is related to that of G_2 by the formula

$$1-\widehat{F}(\theta) = \exp\left\{-\widehat{G}_2(\theta)\right\}, \quad \theta > 0.$$

Which finish the proof since $\widehat{F}(\theta) = (1+\phi(\theta))^{-1}$ for $\theta > 0$. $\square$

Proof of (i) in Lemma 3. We can suppose without loss of generality that $\lambda > 1$. Given that

$$\int_0^\infty \frac{\mathrm{d}t}{t}\mathbf{P}(z < \sigma_t \leqslant \lambda z) = \int_0^\infty \frac{\mathrm{d}t}{t}\mathrm{e}^{-t}\mathbf{P}(z < \sigma_t \leqslant \lambda z) + \int_0^\infty \frac{\mathrm{d}t}{t}(1-\mathrm{e}^{-t})\mathbf{P}(z < \sigma_t \leqslant \lambda z),$$

for every $z > 0$, and

$$G_1(\lambda z) - G_1(z) = \int_0^\infty \frac{\mathrm{d}t}{t}\mathrm{e}^{-t}\mathbf{P}\left(\frac{1}{\lambda z} \leqslant \sigma_t < \frac{1}{z}\right), \quad z > 0,$$

in order to prove (i) in Lemma 3 we only need to check that

$$\lim_{z\to 0}\int_0^\infty \frac{\mathrm{d}t}{t}\left(1-\mathrm{e}^{-t}\right)\mathbf{P}(z<\sigma_t\leqslant \lambda z)=0,\quad \forall\lambda>1.$$

Indeed, given that for any $0<z<\infty$

$$\int_0^1 \frac{\mathrm{d}t}{t}\left(1-\mathrm{e}^{-t}\right)\mathbf{P}(\sigma_t\leqslant \lambda z)\leqslant \int_0^1 \frac{\mathrm{d}t}{t}\left(1-\mathrm{e}^{-t}\right)<\infty,$$

we have by the monotone convergence theorem that

$$\int_0^1 \frac{\mathrm{d}t}{t}\left(1-\mathrm{e}^{-t}\right)\mathbf{P}(z<\sigma_t\leqslant \lambda z)\leqslant \int_0^1 \frac{\mathrm{d}t}{t}\left(1-\mathrm{e}^{-t}\right)\mathbf{P}(\sigma_t\leqslant \lambda z)\to 0\quad \text{as } z\to 0,\ \forall\lambda>1.$$

Furthermore, for any $0<z<\infty$ and $1<\lambda$,

$$\int_1^\infty \frac{\mathrm{d}t}{t}\mathbf{P}(\sigma_t\leqslant \lambda z)\leqslant \int_1^\infty \mathrm{d}t\,\mathbf{P}(\sigma_t\leqslant \lambda z)<\infty,$$

owing to the fact that the renewal measure associated to σ of any interval $[0,z]$, $z>0$, is finite, see e.g. [2] Proposition III.1. Thus, proceeding as in the case $\int_0^1$ we obtain that for any $\lambda>1$,

$$\lim_{z\to 0}\int_1^\infty \frac{\mathrm{d}t}{t}\left(1-\mathrm{e}^{-t}\right)\mathbf{P}(z<\sigma_t\leqslant \lambda z)=0.\qquad \square$$

Proof of (ii) in Lemma 3. As in the proof of (i) it is enough to prove that

$$\lim_{z\to \infty}\int_0^\infty \frac{\mathrm{d}t}{t}\,\mathrm{e}^{-t}\mathbf{P}(z<\sigma_t\leqslant \lambda z)=0,\quad \forall\lambda>1.$$

Indeed, it is straightforward that

$$\lim_{z\to \infty}\int_1^\infty \frac{\mathrm{d}t}{t}\,\mathrm{e}^{-t}\mathbf{P}(z<\sigma_t\leqslant \lambda z)=0,\quad \forall\lambda>1.$$

To prove that

$$\lim_{z\to \infty}\int_0^1 \frac{\mathrm{d}t}{t}\,\mathrm{e}^{-t}\mathbf{P}(z<\sigma_t\leqslant \lambda z)=0,\quad \forall\lambda>1,$$

we will use the inequality (6) in Lemma 1 of [12] which enable us to ensure that for any $u>0$ and $z>0$

$$\mathbf{P}(z<\sigma_t<\infty)\leqslant \frac{t\tilde{\phi}(u)\,\mathrm{e}^{-\kappa t}}{1-\mathrm{e}^{-uz}},\quad \text{with } \tilde{\phi}(u)=ud+\int_{]0,\infty[}\left(1-\mathrm{e}^{-ux}\right)\nu(\mathrm{d}x).$$

Applying this inequality we get that, for any $u,z>0$ and $\lambda>1$,

$$\int_0^1 \frac{\mathrm{d}t}{t}\,\mathrm{e}^{-t}\mathbf{P}(z<\sigma_t\leqslant \lambda z)\leqslant \frac{\tilde{\phi}(u)}{1-\mathrm{e}^{-uz}}\int_0^1 \mathrm{e}^{-(1+\kappa)t}\,\mathrm{d}t.$$

Making, first $z \to \infty$ and then $u \to 0$ in the previous inequality, we obtain the estimate

$$\lim_{z\to\infty} \int_0^1 \frac{\mathrm{d}t}{t} \mathrm{e}^{-t} \mathbf{P}(z < \sigma_t \leqslant \lambda z) \leqslant \tilde{\phi}(0) \int_0^1 \mathrm{e}^{-(1+\kappa)t} \mathrm{d}t,$$

valid for any $\lambda > 1$. Which in fact ends the proof since $\tilde{\phi}(0) = \phi(0) - \kappa = 0$. □

Proof of Lemma 4. The equivalence in (ii) of Lemma 4 follows from Theorem 3.9.1 in [4]. The equivalence in (i) of Lemma 4 is obtained by applying Abelian–Tauberian theorems relying the behavior of the differences of a non-decreasing function and those of their Mellin transform. Indeed, to prove that (7) implies (8) we apply an Abelian theorem that appears in [4] Section 4.11.1. To that end we just need to verify that the Mellin transform $\check{k}$ of the kernel k is finite in a set $A = \{x \in \mathbb{C}\colon a \leqslant \Re(x) \leqslant b\}$ with $a < 0 < b$. This is indeed the case since the Mellin transform of k,

$$\check{k}(x) := \int_0^\infty t^{-x} k(t) \frac{\mathrm{d}t}{t} = \int_0^\infty t^{-x} \mathrm{e}^{-t} \, \mathrm{d}t, \quad x \in \mathbb{C},$$

is finite in the strip $\Re(x) < 1$. That (8) implies (7) is a direct consequence of a Tauberian theorem for differences established in [13] Theorem 2.35. □

Proof of Corollary 3. To prove the assertion in (i) observe that

$$\lim_{\theta\to\infty} G_2(\theta) = \int_0^\infty \frac{\mathrm{d}t}{t} \left(1 - \mathrm{e}^{-t}\right) \mathbf{P}(\sigma_t < \infty) = \int_0^\infty \frac{\mathrm{d}t}{t} \left(1 - \mathrm{e}^{-t}\right) \mathrm{e}^{-t\phi(0)},$$

which is finite if and only if $\phi(0) > 0$, so if and only if σ has finite lifetime a.s. In this case, by Frullani's formula

$$\lim_{\theta\to\infty} G_2(\theta) = \ln\left(\frac{1+\phi(0)}{\phi(0)}\right).$$

In particular, Sinaĭ's condition is satisfied at infinity with index $\beta = 0$.

We next prove (ii). To that end observe that

$$\lim_{\theta\to\infty} G_1(\theta) = \int_0^\infty \frac{\mathrm{d}t}{t} \mathrm{e}^{-t} \mathbf{P}(\sigma_t > 0) = \int_0^\infty \frac{\mathrm{d}t}{t} \mathrm{e}^{-t} \left(1 - \mathbf{P}(\sigma_t = 0)\right).$$

If the latter quantity is finite it implies that for $t > 0$, $\sigma_s = 0$, for all $s \leqslant t$, with probability > 0. So σ is compound Poisson. Reciprocally, if the latter holds then

$$\mathbf{P}(\sigma_t = 0) = \mathrm{e}^{-t\nu]0,\infty[}, \quad t \geqslant 0,$$

and thus

$$\lim_{\theta\to\infty} G_1(\theta) = \ln\left(1 + \nu]0, \infty[\right).$$

In particular, Sinaĭ's condition is satisfied at 0 with index $\beta = 0$.

The proof of the assertion in (iii) and (iv) in Corollary 3 are quite similar so we will only prove the assertion in (iii) and indicate the tools needed to prove (iv). Observe that owing to ξ satisfies Sinaĭ's condition at infinity with index $\beta = 1$ and the assertion in (i) in Corollary 3 then its lifetime is a.s. infinite and so $\phi(0) = 0$. It is well known that any subordinator has finite mean if and only if its Laplace exponent is derivable at 0. Since σ is assumed to have infinite lifetime and the following relations, which are a consequence of the Lemmas 2, 3 and 4,

$$\frac{\phi(\theta)}{\theta} \sim \frac{\frac{\phi(\theta)}{1+\phi(\theta)}}{\theta} \sim \exp\left\{-\widehat{G}_2(\theta) - \log(\theta)\right\} \sim \exp\left\{\gamma - G_2\left(\frac{1}{\theta}\right) + \log\left(\frac{1}{\theta}\right)\right\} \quad \text{as } \theta \to 0,$$

we have that ϕ is derivable at 0 if and only if the limit in Corollary 3(iii) holds. The proof of the assertion in Corollary 3(iv) uses the fact that σ has a strictly positive drift if and only if $\lim_{\theta\to\infty} \phi(\theta)/\theta > 0$. □

3. Proof of Theorem 2

We will prove that under the assumptions of Theorem 2, for $t \to \infty$ in Eq. (2), the assertion in Corollary 1(iv) holds. (The proof of the case $t \to 0$ in Eq. (2) follows in a similar way and so we omit the proof.) To that end, let $(\xi^r)_{r>0}$ be the family of Lévy processes defined by, $(\xi^r(t) = \xi_{rt}/b(r),\ t \geqslant 0)$ for $r > 0$. The hypothesis of Theorem 2 is equivalent to the convergence, in the sense of finite dimensional distributions, of the sequence of Lévy processes ξ^r to a stable Lévy process X with characteristic exponent given by the formula (2). By Corollary 3.6 in Jacod–Shiryaev we have that this convergence holds also in the Skorohod topology and Theorem IV.2.3 in Gihman and Skorohod [14] enable us to ensure that there is also convergence of the first passage time above the level x and the overshoot at the first passage time above the level x by ξ^r to the corresponding objects for X. That is, for any $x > 0$

$$\tau_x^r = \inf\{t > 0\colon\ \xi^r(t) > x\}, \qquad \gamma_x^r = \xi^r(\tau_x^r) - x,$$
$$\tau_x = \inf\{t > 0\colon\ X(t) > x\}, \qquad \gamma_x = X(\tau_x) - x,$$

we have that

$$(\tau_x^r, \gamma_x^r) \xrightarrow[r\to\infty]{\mathcal{D}} (\tau_x, \gamma_x).$$

In particular, for $x = 1$, we have that $\tau_1^r = r^{-1}T_{b(r)}$ and $\gamma_1^r = (\xi_{T_{b(r)}} - b(r))/b(r)$, in the notation of Corollary 1, and thus that

$$\left(r^{-1}T_{b(r)}, \frac{(\xi_{T_{b(r)}} - b(r))}{b(r)}\right) \xrightarrow[r\to\infty]{\mathcal{D}} (\tau_1, \gamma_1).$$

We will next prove that

$$\frac{\xi_{T_r} - r}{r} \xrightarrow[r\to\infty]{\mathcal{D}} \gamma_1,$$

which implies that the assertion (iv) in Corollary 1 holds. To that end, we introduce the generalized inverse of b, $b^{\leftarrow}(t) = \inf\{r > 0\colon\ b(r) > t\}$ for $t > 0$. Given that b is regularly varying at infinity it is known that $b(b^{\leftarrow}(t)) \sim t$ as $t \to \infty$, see e.g. [4] Theorem 1.5.12. Owing the following relations valid for any $\epsilon > 0$ fixed and small enough,

$$b(b^{\leftarrow}(r) - \epsilon) \leqslant r \leqslant b(b^{\leftarrow}(r)), \quad r > 0,$$

we have that for any $x > 0$

$$\begin{aligned}\mathbf{P}\left(\frac{\xi_{T_{b(b^{\leftarrow}(r))}}}{b(b^{\leftarrow}(r))}\frac{b(b^{\leftarrow}(r))}{r} \leqslant x+1\right) &\leqslant \mathbf{P}\left(\frac{\xi_{T_r}}{r} \leqslant x+1\right)\\ &\leqslant \mathbf{P}\left(\frac{\xi_{T_{b(b^{\leftarrow}(r)-\epsilon)}}}{b(b^{\leftarrow}(r)-\epsilon)}\frac{b(b^{\leftarrow}(r)-\epsilon)}{b(b^{\leftarrow}(r))}\frac{b(b^{\leftarrow}(r))}{r} \leqslant x+1\right).\end{aligned}$$

Making r tend to infinity and using that $b(b^{\leftarrow}(r) - \epsilon)/b(b^{\leftarrow}(r)) \to 1$ as $r \to \infty$, we get that the left and right-hand sides of the previous inequality tend to $\mathbf{P}(\gamma_1 + 1 \leqslant x + 1)$ and so that for any $x > 0$

$$\mathbf{P}\left(\frac{\xi_{T_r} - r}{r} \leqslant x\right) \xrightarrow[r\to\infty]{} \mathbf{P}(\gamma_1 \leqslant x).$$

Furthermore, it is well known that in the case $\alpha\rho \in]0, 1[$ the law of γ_1 is the generalized arc-sine law with parameter $\alpha\mathbf{P}(X_1 > 0) = \alpha\rho$, that is

$$\mathbf{P}(\gamma_1 \in \mathrm{d}x) = \frac{\sin(\alpha\rho\pi)}{\pi}x^{-\alpha\rho}(1+x)^{\alpha\rho-1}\mathrm{d}x, \quad x > 0.$$

In the case $\alpha\rho = 0$ the random variable γ_1 is degenerate at infinity and in the case $\alpha\rho = 1$ it is degenerate at 0. Thus, in any case the Sinaĭ index of ξ is $\alpha\rho$. Which finish the proof of Theorem 2.

For shake of completeness in the following lemma we provide necessary and sufficient conditions on the tail behavior of the Lévy measure of ξ in order that the hypotheses of Theorem 2 be satisfied. This result concerns only the case $t \to \infty$ in (i) of Theorem 2 and $\alpha \in]0,1[$. The triple (a, q^2, Π) denotes the characteristics of the Lévy process ξ, that is, its linear and Gaussian term, a, q and Lévy measure Π and are such that

$$\Psi(\lambda) = \mathrm{i}a\lambda - \frac{\lambda^2 q^2}{2} + \int\limits_{\mathbb{R}\setminus\{0\}} \left(\mathrm{e}^{\mathrm{i}\lambda x} - 1 - \mathrm{i}\lambda x 1_{\{|x|<1\}}\right)\Pi(\mathrm{d}x), \quad \lambda \in \mathbb{R}.$$

By $\overline{\Pi}^+$ and $\overline{\Pi}^-$ we denote the right- and left-hand tails of the Lévy measure Π respectively, i.e. $\overline{\Pi}^+(x) = \Pi]x,\infty[$ and $\overline{\Pi}^-(x) = \Pi]-\infty,-x[$, for $x > 0$.

Lemma 5. *Let $\alpha \in]0,1[$ and $\delta \in [-1,1]$. The following are equivalent*

(DA) *There exists a function $b :]0,\infty[\to]0,\infty[$ which is regularly varying at infinity with index $\beta = 1/\alpha$ and such that the limit in Eq.* (2) *holds as $t \to \infty$.*

(TB) *The function $\overline{\Pi}^+(\cdot) + \overline{\Pi}^-(\cdot)$ is regularly varying at infinity with index $-\alpha$ and*

$$\frac{\overline{\Pi}^+(x)}{\overline{\Pi}^+(x) + \overline{\Pi}^-(x)} \to p, \quad \frac{\overline{\Pi}^-(x)}{\overline{\Pi}^+(x) + \overline{\Pi}^-(x)} \to q, \quad \text{as } x \to \infty; \quad p + q = 1,\ p - q = \delta.$$

Proof of Lemma 5. It is plain, that for any $t > 0$ the function $\Psi^{(t)}(\lambda) := t\Psi(\lambda/b(t))$ is the characteristic exponent of the infinitely divisible random variable $X^{(t)} := \xi_t/b(t)$, which by the hypothesis DA(α) converges to a stable law $X(1)$ whose characteristic exponent is given by Eq. (2). The characteristic exponent $\Psi^{(t)}$ can be written as

$$\Psi^{(t)}(\lambda) = \mathrm{i}\lambda a^{(t)} - \lambda^2\left(q^{(t)}\right)^2/2 + \int\limits_{\mathbb{R}\setminus\{0\}} \left(\mathrm{e}^{i\lambda z} - 1 - \mathrm{i}\lambda h(z)\right)\Pi^{(t)}(\mathrm{d}z),$$

where $h(z) = z 1_{\{|z|\leqslant 1\}} + z^{-1} 1_{\{|z|>1\}}$ and $(a^{(t)}, q^{(t)}, \Pi^{(t)})$ are given by

$$a^{(t)} = \frac{ta}{b(t)} + \frac{t}{b(t)} \int x 1_{\{1<|x|\leqslant b(t)\}} \Pi(\mathrm{d}x) + t b(t) \int x^{-1} 1_{\{|x|>b(t)\}} \Pi(\mathrm{d}x),$$
$$q^{(t)} = q\left(\frac{t}{b(t)^2}\right)^{1/2}, \qquad \Pi^{(t)}(\mathrm{d}z) = t\Pi\left(b(t)\,\mathrm{d}z\right).$$

According to a well known result on the convergence of infinite divisible laws, see e.g. Sato [21], the convergence in law of $X^{(t)}$ to X as $t \to \infty$, is equivalent to the convergence of the triplet $(a^{(t)}, q^{(t)}, \Pi^{(t)})$ to $(l, 0, \Pi_S)$ as $t \to \infty$, with

$$\Pi_S(\mathrm{d}x) = \left(c_+ x^{-1-\alpha} 1_{\{x>0\}} + c_- |x|^{-1-\alpha} 1_{\{x<0\}}\right)\mathrm{d}x, \quad c_+, c_- \in \mathbb{R}^+,$$

and

$$l = \frac{2(c_+ - c_-)}{1 - \alpha^2},$$

and they are such that for every $\lambda \in \mathbb{R}$

$$-c|\lambda|^\alpha\left(1 - \mathrm{i}\delta \operatorname{sgn}(\lambda) \tan\left(\frac{\pi\alpha}{2}\right)\right) = \int\limits_{\mathbb{R}\setminus\{0\}} \left(\mathrm{e}^{i\lambda x} - 1\right)\Pi_S(\mathrm{d}x) = \mathrm{i}l\lambda + \int\limits_{\mathbb{R}\setminus\{0\}} \left(\mathrm{e}^{\mathrm{i}\lambda x} - 1 - \mathrm{i}\lambda h(x)\right)\Pi_S(\mathrm{d}x),$$

with $c > 0$ convenably chosen. The term δ in the previous equation is determined by $\delta = p - q = (c_+ - c_-)/(c_+ + c_-)$.

That the hypotheses on the tail behavior of Π are equivalent to the convergence of Π^t to Π_S is a quite standard fact in the theory of domains of attraction and so we refer to [4] Section 8.3.2, for a proof. This implies in particular that for any $x > 0$,

$$t\overline{\Pi}^+\left(b(t)x\right) \to c_+ x^{-\alpha}, \quad \text{and} \quad t\overline{\Pi}^-\left(b(t)x\right) \to c_- x^{-\alpha} \quad \text{as } t \to \infty.$$

The only technical detail that requires a proof is that $a^{(t)} \to l$ as $t \to \infty$. Indeed, under the conditions (ii) of Theorem 2 and $0 < p < 1$ the functions $\overline{\Pi}^+(\cdot)$ and $\overline{\Pi}^-(\cdot)$ are regularly varying at infinity with index $0 < \alpha < 1$, this implies that $at/b(t) \to 0$ as $t \to \infty$. Moreover, it is justified by making an integration by parts that

$$\int x 1_{\{1<x\leqslant b(t)\}}\Pi(\mathrm{d}x) \sim \overline{\Pi}^+(1) - b(t)\overline{\Pi}^+\big(b(t)\big) + \int_1^{b(t)} \overline{\Pi}^+(z)\,\mathrm{d}z$$
$$\sim \overline{\Pi}^+(1) - b(t)\overline{\Pi}^+\big(b(t)\big) + \frac{b(t)\overline{\Pi}^+(b(t))}{1-\alpha},$$

as $t\to\infty$. Multiplying by $t/b(t)$ we get that

$$\frac{t}{b(t)}\int x 1_{\{1<x\leqslant b(t)\}}\Pi(\mathrm{d}x) \sim -t\overline{\Pi}^+\big(b(t)\big) + \frac{t\overline{\Pi}^+(b(t))}{1-\alpha} \to -c_+ + \frac{c_+}{1-\alpha}$$

as $t\to\infty$. Similarly, it is proved that

$$\frac{t}{b(t)}\int x 1_{\{1<-x\leqslant b(t)\}}\Pi(\mathrm{d}x) \underset{t\to\infty}{\longrightarrow} c_- - \frac{c_-}{1-\alpha}.$$

Concerning the term $\int x^{-1}1_{\{|x|>b(t)\}}\Pi(\mathrm{d}x)$, an integration by parts and Karamata's theorem yield

$$\int x^{-1} 1_{\{x>b(t)\}}\Pi(\mathrm{d}x) \sim \big(b(t)\big)^{-1}\overline{\Pi}^+\big(b(t)\big) + \int_{b(t)}^{\infty} z^{-2}\overline{\Pi}^+(z)\,\mathrm{d}z$$
$$\sim \big(b(t)\big)^{-1}\overline{\Pi}^+\big(b(t)\big) + \frac{\overline{\Pi}^+(b(t))}{b(t)(1+\alpha)},$$

as $t\to\infty$ and therefore

$$tb(t)\int x^{-1} 1_{\{x>b(t)\}}\Pi(\mathrm{d}x) \sim t\overline{\Pi}^+\big(b(t)\big) + \frac{t\overline{\Pi}^+(b(t))}{1+\alpha} \to c_+ + \frac{c_+}{1+\alpha}.$$

Analogously, we prove

$$tb(t)\int x^{-1} 1_{\{x<-b(t)\}}\Pi(\mathrm{d}x) \underset{n\to\infty}{\longrightarrow} -c_- - \frac{c_-}{1+\alpha}.$$

Finally, adding up these four terms it follows that

$$\lim_{t\to\infty} a^{(t)} = \frac{c_+ - c_-}{1-\alpha} + \frac{c_+ - c_-}{1+\alpha} = l.$$

The proof that $a^{(t)}\to l$ in the case $p=1$, respectively $p=0$, is quite similar but uses that $\overline{\Pi}^- = \mathrm{o}(\overline{\Pi}^+)$, respectively $\overline{\Pi}^+ = \mathrm{o}(\overline{\Pi}^-)$. □

Remark 8. The proof of Theorem 2 is a reworking of its analogous for random walks, which was established by Rogozin [20] Theorem 9.

Remark 9. The result in Lemma 5 holds also true for $\alpha\in\,]0,2[$ if the Lévy process is assumed to be symmetric (the proof of Lemma 5 can be easily extended to this case). Furthermore, there is also an analogue of this result when $t\to 0$ in (i) of Theorem 2 in the cases $1<\alpha<2$ or $0<\alpha<2$ and ξ is assumed to be symmetric. Its proof is quite similar to that of Lemma 5, see e.g. the recent work of De Weert [6].

4. Proofs of Theorem 3 and Propositions 1 and 2

Proof of (a) in Theorem 3. To prove that (a-1) is equivalent to (a-2) we will prove that either of this conditions implies that

$$\overline{po}(x) = \int_0^\infty \overline{\Pi}^+(x+y)\widehat{V}(\mathrm{d}y) \sim \frac{1}{\mu}\int_x^\infty \overline{\Pi}^+(z)\,\mathrm{d}z \quad \text{as } x\to\infty, \tag{15}$$

with $\mu := \mathbf{E}(\widehat{H}_1)$, from where the result follows. (Observe that the assumption that $\widehat{H}$ has a finite mean implies that $\int^{\infty} \overline{\Pi}^{+}(z)\,\mathrm{d}z < \infty$.)

Assume that (a-1) holds. Indeed, by the renewal theorem for subordinators we have that for any $h > 0$,

$$\lim_{t\to\infty} \widehat{V}]t, t+h] = \frac{h}{\mu}.$$

Thus, for any $h > 0$ given and any $\epsilon > 0$ there exists a $t_0(h, \epsilon) > 0$ such that

$$(1-\epsilon)\frac{h}{\mu} < \widehat{V}]t, t+h] < (1+\epsilon)\frac{h}{\mu}, \quad \forall t > t_0,$$

and as a consequence, if N_0 is an integer such that $N_0 h > t_0$, we have the following inequalities

$$\begin{aligned}
\int_0^{\infty} \overline{\Pi}^{+}(x+y)\widehat{V}(\mathrm{d}y) &\leqslant \overline{\Pi}^{+}(x)\widehat{V}[0, N_0 h] + \sum_{k=N_0}^{\infty} \overline{\Pi}^{+}(kh+x)\widehat{V}]kh, kh+h] \\
&\leqslant \overline{\Pi}^{+}(x)\widehat{V}[0, N_0 h] + (1+\epsilon)\sum_{k=N_0}^{\infty} \overline{\Pi}^{+}(kh+x)\frac{h}{\mu} \\
&\leqslant \overline{\Pi}^{+}(x)\widehat{V}[0, N_0 h] + \frac{1+\epsilon}{\mu}\int_{(N_0-1)h}^{\infty} \overline{\Pi}^{+}(x+z)\,\mathrm{d}z \\
&\leqslant \overline{\Pi}^{+}(x)\widehat{V}[0, N_0 h] + \frac{1+\epsilon}{\mu}\int_x^{\infty} \overline{\Pi}^{+}(z)\,\mathrm{d}z.
\end{aligned}$$

It follows from the previous inequalities and the fact that

$$\overline{\Pi}^{+}(x) \Big/ \int_x^{\infty} \overline{\Pi}^{+}(z)\,\mathrm{d}z \to 0 \quad \text{as } x \to \infty,$$

since $\overline{\Pi}^{+}_I \in \mathcal{L}^0$ and $\overline{\Pi}^{+}$ is decreasing, that

$$\limsup_{x\to\infty} \frac{\int_0^{\infty} \overline{\Pi}^{+}(x+y)\widehat{V}(\mathrm{d}y)}{\frac{1}{\mu}\int_x^{\infty} \overline{\Pi}^{+}(z)\,\mathrm{d}z} \leqslant 1.$$

Analogously, we prove that

$$\int_0^{\infty} \overline{\Pi}^{+}(x+y)\widehat{V}(\mathrm{d}y) \geqslant \frac{1-\epsilon}{\mu}\int_x^{\infty} \overline{\Pi}^{+}(z)\,\mathrm{d}z - \frac{1-\epsilon}{\mu}\overline{\Pi}^{+}(x)(N_0+1)h, \quad x > 0.$$

Therefore,

$$\liminf_{x\to\infty} \frac{\int_0^{\infty} \overline{\Pi}^{+}(x+y)\widehat{V}(\mathrm{d}y)}{\frac{1}{\mu}\int_x^{\infty} \overline{\Pi}^{+}(z)\,\mathrm{d}z} \geqslant 1.$$

Which ends the proof of the claim (15).

We assume now that (a-2) holds and we will prove that the estimate in (15) holds. On the one hand, we know that for every $z > 0$

$$\overline{\Pi}^{+}(z) = \int_z^{\infty} po(\mathrm{d}y)\overline{ne}(y-z) + \hat{d}\,\bar{p}(z),$$

since $\hat{k}_0 = 0$, because under our assumptions the Lévy process does not drift to ∞. Integrating this relation between x and ∞ and using Fubini's theorem we obtain that for any $x > 0$

$$\int_x^\infty \mathrm{d}z\,\overline{\Pi}^+(z) = \int_x^\infty po(\mathrm{d}y)\int_0^{y-x}\mathrm{d}z\,\overline{ne}(z) + \hat{d}\,\overline{po}(x) \leqslant \overline{po}(x)\left(\int_0^\infty \mathrm{d}z\,\overline{ne}(z) + \hat{d}\right) = \overline{po}(x)\mu < \infty.$$

Thus,

$$\limsup_{x\to\infty}\frac{\frac{1}{\mu}\int_x^\infty \mathrm{d}z\,\overline{\Pi}^+(z)}{\overline{po}(x)} \leqslant 1.$$

On the other hand, to prove that

$$\liminf_{x\to\infty}\frac{\frac{1}{\mu}\int_x^\infty \mathrm{d}z\,\overline{\Pi}^+(z)}{\overline{po}(x)} \geqslant 1,$$

we will use an argument based on some facts of renewal theory. To that end we recall that it was proved by Bertoin and Doney [3] that the potential measure of a subordinator $\widehat{H}$ is the delayed renewal measure associated to the law $F(x) = \mathbf{P}(\widehat{H}_\vartheta \leqslant x)$ with ϑ an exponential random variable independent of $\widehat{H}$, that is

$$\widehat{V}(\mathrm{d}y) = \sum_{n=1}^\infty F^{*n}(\mathrm{d}y).$$

We have by hypothesis that $\int_0^\infty (1 - F(x))\,\mathrm{d}x = \mathbf{E}(\widehat{H}_1) = \mu < \infty$ and thus the measure $\widetilde{G}_F(\mathrm{d}y)$ on $]0,\infty[$, with density $G_F(z) := (1 - F(z))/\mu$, $z > 0$, is a probability measure. By standard facts of renewal theory we know that the following equality between measures holds

$$\frac{\mathrm{d}y}{\mu} = \widetilde{G}_F(\mathrm{d}y) + \widetilde{G}_F * \widehat{V}(\mathrm{d}y), \quad y > 0,$$

where $*$ denotes the standard convolution between measures. Using this identity and Eq. (EAI) we have that for any $x > 0$,

$$\begin{aligned}\frac{1}{\mu}\int_0^\infty \mathrm{d}y\,\overline{\Pi}^+(x+y) &= \int_0^\infty \mathrm{d}y\,G_F(y)\overline{\Pi}^+(x+y) + \int_0^\infty \mathrm{d}z\,G_F(z)\int_0^\infty \widehat{V}(\mathrm{d}r)\overline{\Pi}^+(x+z+r)\\ &= \int_0^\infty \mathrm{d}y\,G_F(y)\overline{\Pi}^+(x+y) + \int_0^\infty \mathrm{d}z\,G_F(z)\overline{po}(x+z),\end{aligned}$$

and by Fatou's lemma we get that

$$\liminf_{x\to\infty}\frac{\frac{1}{\mu}\int_0^\infty \mathrm{d}y\,\overline{\Pi}^+(x+y)}{\overline{po}(x)} \geqslant \int_0^\infty \mathrm{d}z\,G_F(z)\liminf_{x\to\infty}\frac{\overline{po}(x+z)}{\overline{po}(x)} = 1.$$

So we have proved that (a-1) and (a-2) are equivalent and imply (a-3). To finish the proof, we will prove that (a-3) implies (a-2). To that end it suffices with proving that

$$\lim_{x\to\infty}\frac{\overline{po}(x) - \overline{po}(x+y)}{\overline{po}(x)} = 0, \quad \text{for any } y > 0.$$

Indeed, using Eq. (EA) we have that for any $y > 0$,

$$\begin{aligned}\mu - \frac{\int_0^\infty \mathrm{d}z\,\overline{\Pi}^+(z+x)}{\overline{po}(x)} &= \frac{\int_0^\infty \mathrm{d}z\,\overline{ne}(z)\overline{po}(x) + \hat{d}\,\overline{po}(x) - \int_x^\infty po(\mathrm{d}y)\int_0^{y-x}\mathrm{d}z\,ne(z) - \hat{d}\,\overline{po}(x)}{\overline{po}(x)}\\ &= \int_0^\infty \mathrm{d}z\,\overline{ne}(z)\frac{\overline{po}(x) - \overline{po}(x+z)}{\overline{po}(x)} \geqslant \int_y^\infty \mathrm{d}z\,\overline{ne}(z)\frac{\overline{po}(x) - \overline{po}(x+y)}{\overline{po}(x)} \geqslant 0,\end{aligned}$$

and the assertion follows making $x \to \infty$ in the latter equation since by assumption its left-hand term tends to 0 as $x \to \infty$. $\square$

Proof of (b) in Theorem 3. The assumption that $\hat{k}_0 > 0$, implies that the renewal measure $\widehat{V}(\mathrm{d}y)$ is a finite measure and $\widehat{V}[0, \infty[= 1/\hat{k}_0$. Thus if $\overline{\Pi}^+(x) \in \mathcal{L}^0$ we have by Eq. (EA) and the dominated convergence theorem that

$$\lim_{x\to\infty} \frac{\overline{po}(x)}{\overline{\Pi}^+(x)} = \lim_{x\to\infty} \int_0^\infty \widehat{V}(\mathrm{d}y) \frac{\overline{\Pi}^+(x+y)}{\overline{\Pi}^+(x)} = \int_0^\infty \widehat{V}(\mathrm{d}y) \lim_{x\to\infty} \frac{\overline{\Pi}^+(x+y)}{\overline{\Pi}^+(x)} = \frac{1}{\hat{k}_0}.$$

Now, that (3) holds is a straightforward consequence of the following identity, for any $x > 0$

$$\begin{aligned} \overline{\Pi}^+(x) &= \int_0^\infty ne(\mathrm{d}y)\big(\overline{\mathrm{po}}(x) - \overline{po}(x+y)\big) + \hat{k}_0\overline{po}(x) + \hat{d}\,\bar{p}(x) \\ &= \overline{po}(x) \int_0^1 ne(\mathrm{d}y)\left(\frac{\overline{po}(x) - \overline{po}(x+y)}{\overline{po}(x)}\right) + \int_1^\infty ne(\mathrm{d}y)\big(\overline{po}(x) - \overline{po}(x+y)\big) + \hat{k}_0\overline{po}(x) + \hat{d}\,\bar{p}(x), \end{aligned} \tag{16}$$

which is obtained using Eq. (EAI) and Fubini's theorem. We have so proved that (b-1) implies (b-2) and (b-3). Next, to prove that (b-2) implies (b-1) and (b-3) we assume that $po \in \mathcal{L}^0$ and (3) holds. Under this assumptions we claim that

$$\overline{\Pi}^+(x) \sim \hat{k}_0\overline{po}(x) + \hat{d}\,\bar{p}(x) \quad \text{as } x \to \infty.$$

Indeed, this can be deduced from Eq. (16), using that $\int_{-\infty}^0 ne(\mathrm{d}y)\min\{|y|, 1\} < \infty$, that $\lim_{x\to\infty} \overline{po}(x+y)/\overline{po}(x) = 1$ for any $y > 0$, and the dominated convergence theorem. Furthermore, we have by hypothesis that $\hat{d}\,\bar{p}(x)/\overline{po}(x) \to 0$ as $x \to \infty$, which implies that

$$\overline{\Pi}^+(x) \sim \hat{k}_0\overline{po}(x) \quad \text{as } x \to \infty.$$

To finish we next prove that (b-3) implies (b-2). Indeed, using Eq. (EAI) and the hypothesis (b-3) we get that

$$\frac{\overline{\Pi}^+(x)}{\overline{po}(x)} - \hat{k}_0 = \frac{\int_x^\infty po(\mathrm{d}z)\overline{ne}(z-x) + \hat{d}\,\bar{p}(x)}{\overline{po}(x)} = \int_0^\infty ne(\mathrm{d}z) \frac{(\overline{po}(x) - \overline{po}(x+z))}{\overline{po}(x)} + \frac{\hat{d}\,\bar{p}(x)}{\overline{po}(x)} \to 0 \quad \text{as } x \to \infty.$$

We deduce therefrom that (3) holds and that $po \in \mathcal{L}^0$ since for any $y > 0$,

$$\int_0^\infty ne(\mathrm{d}z)\left(\frac{\overline{po}(x) - \overline{po}(x+z)}{\overline{po}(x)}\right) \geqslant \overline{ne}(y)\left(\frac{\overline{po}(x) - \overline{po}(x+y)}{\overline{po}(x)}\right) \geqslant 0. \qquad \square$$

Proof of (a) in Proposition 1. According to Theorem 6.3.2 in [23] under these assumptions the measure po has infinite total mass if and only if $\lim_{x\to 0+} \int_x^1 \overline{\Pi}^+(z)\,\mathrm{d}z = \infty$ and in this case

$$\overline{po}(x) \sim \frac{1}{\hat{d}} \int_x^1 \overline{\Pi}^+(z)\,\mathrm{d}z \quad \text{as } x \to 0+.$$

Thus the assertion in (a) Proposition 1 is a consequence of this fact and the monotone density theorem for regularly varying functions. $\square$

Proof of (b) in Proposition 1. According to Theorem 6.3.1 in [23] under these assumptions, if we suppose $\lim_{x\to 0+} \overline{po}(x) = \infty$, then

$$\overline{po}(x) \sim \frac{1}{ne]0, \infty[+ \hat{k}_0} \overline{\Pi}^+(x) \quad \text{as } x \to 0+.$$

The result follows. $\square$

Sketch of proof of Proposition 2. The proof of the assertion (i) implies (ii) is a reworking of the proof of Lemma 3 in Asmussen et al. [1], this can be done in our setting since the only hypothesis needed in that proof is that the dual ladder height has a finite mean.

To show that (i) implies (ii) in Proposition 2 we first prove that under the assumption $\mathbf{E}(\widehat{H}_1) = \mu < \infty$, the condition $\overline{\Pi}^+ \in \mathcal{L}^0$, implies that for any $z > 0$,

$$\text{(BRT)} \qquad po]x, x+z[\sim \frac{z}{\mu}\overline{\Pi}^+(x), \quad x \to \infty.$$

The latter estimate and the fact that $\Pi^+ \in \mathcal{L}^0$ implies that for any $a \geqslant 0$,

$$po]x+a, x+a+z[\sim \frac{z}{\mu}\overline{\Pi}^+(x), \quad x \to \infty. \tag{17}$$

To prove that (BRT) holds, we may simply repeat the argument in the proof of Lemma 3 in Asmussen et al. [1] using instead of Eq. (12) therein, the equation

$$po]x, x+z[= \int_x^\infty \Pi(dy)\widehat{V}]y-x-z, y-x[, \quad z > 0,$$

which is an elementary consequence of Eq. (EAI) and Fubini's theorem.

The result in (ii) in Proposition 2 follows from (BRT) in the same way that the Key renewal theorem is obtained from Blackwell's renewal theorem using the estimate in (17) and the bounds

$$\frac{po]x, x+z[}{\overline{\Pi}^+(x)} \leqslant \widehat{V}(z), \quad x > 0,\ z > 0,$$

which are a simple consequence of the former equation and the fact that $\widehat{V}$ is a renewal measure and so that for any $0 < z < y$, $\widehat{V}(y) - \widehat{V}(y-z) \leqslant \widehat{V}(z)$.

To show that (ii) implies (i) we have to verify that for any $a > 0$

$$\lim_{x\to\infty} \frac{\overline{\Pi}^+(x+a)}{\overline{\Pi}^+(x)} = 1.$$

This is indeed true since using (ii) in Proposition 2 it is straightforward that for any $z > 0$ the assertion in (BRT) holds and a further application of (ii) in Proposition 2 to the function $g_a(\cdot) = 1_{\{]a,a+1[\}}(\cdot)$, $a > 0$, gives that for any $a > 0$,

$$\lim_{x\to\infty} \frac{po]x+a, x+a+1[}{\overline{\Pi}(x)} = \frac{1}{\mu},$$

and therefore, for any $a > 0$,

$$\lim_{x\to\infty} \frac{\overline{\Pi}^+(x+a)}{\overline{\Pi}^+(x)} = \lim_{x\to\infty} \frac{\overline{\Pi}^+(x+a)}{po]x+a, x+a+1[} \frac{po]x+a, x+a+1[}{\overline{\Pi}^+(x)} = 1. \qquad \square$$

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