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A note on random walk in random scenery

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Abstract

We consider a random walk in random scenery $\{X_n = \eta(S_0) + \cdots + \eta(S_n), n \in \mathbb{N}\}$, where a centered walk $\{S_n, n \in \mathbb{N}\}$ is independent of the scenery $\{\eta(x), x \in \mathbb{Z}^d\}$, consisting of symmetric i.i.d. with tail distribution $P(\eta(x) > t) \sim \exp(-c_\alpha t^\alpha)$, with $1 \leqslant \alpha < d/2$. We study the probability, when averaged over both randomness, that $\{X_n > ny\}$ for $y > 0$, and n large. In this note, we show that the large deviation estimate is of order $\exp(-c(ny)^a)$, with $a = \alpha/(\alpha+1)$.

Résumé

Soit une marche aléatoire en paysage aléatoire $X_n = \eta(S_0) + \cdots + \eta(S_n)$. La marche $\{S_n\}$ est centrée, et évolue indépendamment d'un paysage formé d'une suite i.i.d $\{\eta(x), x \in \mathbb{Z}^d\}$, caractérisées par une queue de distribution $P(\eta(x) > t) \sim \exp(-c_\alpha t^\alpha)$, avec $1 \leqslant \alpha < d/2$. Nous étudions la probabilité pour que $\{X_n > ny\}$ pour $y > 0$, et n grand.

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1. Introduction

We consider a centered random walk $\{S_k, k \in \mathbb{N}\}$ on $\mathbb{Z}^d$. When $S_0 = x$, we denote the law of the walk by $\mathbb{P}_x$ and the expectation with respect to this law by $\mathbb{E}_x$. Each site $x \in \mathbb{Z}^d$ is associated with a random variable $\eta(x)$, and we assume that the *scenery* $\{\eta(x), x \in \mathbb{Z}^d\}$ consists of symmetric i.i.d. unbounded random variables, independent of the random walk. We denote the law of the scenery by P_η, and by E_η the expectation with respect to this law.

The random walk in random scenery (RWRS) is the process $\{X_n, n \in \mathbb{N}\}$ defined by

$$X_n := \sum_{k=0}^{n} \eta(S_k) = \sum_{x \in \mathbb{Z}^d} l_n(x)\eta(x), \quad \text{where } l_n(x) = \sum_{k=0}^{n} \mathbb{1}_{S_k = x}. \tag{1.1}$$

RWRS has been introduced by Kesten, Spitzer [9], and Borodin [4,5] as a case-study for sums of dependent random variables, in order to exhibit new scaling and new self-similar limiting laws. Indeed, the convergence in law of X_n,

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studied for $d \neq 2$ in [9,4,5], and for $d = 2$ by Bolthausen [3], needs a *super-diffusive* scaling in dimensions 1 and 2. In terms of the mean square, the dominant orders are the following.

$$\mathbb{E}_0 \otimes E_\eta\big[X_n^2\big] \simeq \begin{cases} n^{3/2} & \text{for } d = 1, \\ n\log(n) & \text{for } d = 2, \\ n & \text{for } d \geqslant 3. \end{cases} \tag{1.2}$$

Recently, the moderate and large deviations for X_n have been studied in [7,1,2,6] for the Brownian motion in various sceneries, and in [10,11] in the original random walk setting.

For our estimates, the only important feature of the scenery variables is their tail decay. Thus, we make the hypothesis that there is $\alpha > 0$, and a positive constant c_α such that

$$\lim_{t\to\infty} \frac{\log P_\eta(\eta(0) > t)}{t^\alpha} = -c_\alpha. \tag{1.3}$$

- When $\alpha < 1$, $\eta(x)$ has no exponential moments, and is called a heavy-tail variable. In a recent paper [11], van der Hofstad, Gantert and König deal with $X_n = \sum l_n(x)\eta(x)$, by conditioning on the local times $\{l_n(x), x \in \mathbb{Z}^d\}$, thus obtaining a weighted sum of i.i.d. heavy-tail variables. Using classical heavy-tail estimates (as those of [13]), they show that $\{X_n > ny\}$ is realized as only one term of the series reaches level ny. Thus, in terms of logarithmic equivalence ($\approx$),

  $$\mathbb{P}_0 \otimes P_\eta[X_n \geqslant ny] \approx \mathbb{P}_0 \otimes P_\eta\big[l_n(0)\eta(0) \geqslant ny\big].$$

  Now, recall that for a time $k \ll n$, the local time at site 0, $l_n(0)$, satisfies the following property

  $$\mathbb{P}_0\big(l_n(0) = k\big) \approx \exp(-\kappa_0 k), \quad \text{with } \kappa_0 := \log\bigg(\frac{1}{\mathbb{P}_0(H_0 < \infty)}\bigg), \tag{1.4}$$

  where $H_0 = \inf\{n \geqslant 1, X_n = 0\}$. Thus, for $y > 0$, [11] shows that for an explicit $J > 0$

  $$\begin{aligned} \mathbb{P}_0 \otimes P_\eta\big(l_n(0)\eta(0) > ny\big) &\approx \sup_{k=1,2,\ldots} \mathbb{P}_0\big(l_n(0) = k\big) P_\eta\bigg(\eta(0) > \frac{ny}{k}\bigg) \\ &\approx \exp\bigg(-\inf_{k\geqslant 1}\bigg(\kappa_0 k + c_\alpha\bigg(\frac{ny}{k}\bigg)^\alpha\bigg)\bigg) \\ &\approx \exp\big(-J(ny)^{\alpha/(\alpha+1)}\big). \end{aligned} \tag{1.5}$$

  In the optimal strategy $l_n(0)$ is of order $(ny)^{\alpha/(\alpha+1)}$.
- When $\alpha > \max(d/2, 1)$, a different behavior holds: for all $y > 0$

  $$\mathbb{P}_0 \otimes P_\eta[X_n \geqslant ny] \approx \exp\big(-n^{d/(d+2)}J(y)\big) \quad \text{(with a } J(y) > 0 \text{ known explicitly)}. \tag{1.6}$$

  This result is proved in [2] for Brownian motion in a bounded scenery (i.e. $\alpha = \infty$), in [6] for a Gaussian scenery ($\alpha = 2$ and $d \leqslant 3$), and in [10] for a random walk in a general scenery. The best strategy to realize $\{X_n > ny\}$ is the following.
  - Force the random walk to spend all its time in a ball of radius r_n with $1 \ll r_n \ll \sqrt{n}$, in such a way that for x in this ball, $l_n(x)$ is of order n/r_n^d. This has a cost of order $\exp(-n/r_n^2)$.
  - Require the scenery to satisfy $\sum_{\|x\|\leqslant r_n} \eta(x) \geqslant r_n^d y$. This has a cost of order $\exp(-r_n^d)$.

  The exponent $d/(d+2)$ appears as one sets equal n/r_n^2 and r_n^d. Thus, in the optimal strategy, the walk spends a time $n^{2/(d+2)}$ on each site of a ball of about $n^{d/(d+2)}$ sites.
  Observe that when $\eta(x)$ satisfies (1.3), then $l_n(x)\eta(x)$ has a heavy tail (see (1.5)). Also $X_n = \sum \eta(x)l_n(x)$ is a sum of about n-terms (in dimensions $d \geqslant 3$). However, the variables $\{l_n(x)\eta(x), x \in \mathbb{Z}^d\}\}$ are not independent, and the extreme value of the sum does not dominate.
- The regime $1 \leqslant \alpha < d/2$ is the purpose of this note.

Our main result is the following.

Proposition 1.1. *Let $\{S_n, n \in \mathbb{N}\}$ be a walk with centered independent increments with finite exponential moments. Assume that $\{\eta(x), x \in \mathbb{Z}^d\}$ are symmetric i.i.d. variables with tail parameter α with $1 \leqslant \alpha < d/2$, and whose law has a density decreasing on $\mathbb{R}^+$. There are $c_1, c_2 > 0$, such that when n is large enough and $y > 0$*

$$\exp\bigl(-c_1(ny)^{\alpha/(\alpha+1)}\bigr) \leqslant \mathbb{P}_0 \otimes P_\eta[X_n \geqslant ny] \leqslant \exp\bigl(-c_2(ny)^{\alpha/(\alpha+1)}\bigr). \tag{1.7}$$

In the course of deriving the upper bound, we rely on a localization lemma of independent interest.

Lemma 1.2. *Assume $d \geqslant 3$. There is a constant $\kappa_d > 0$ such that for any $\Lambda \subset \mathbb{Z}^d$, and any $t > 0$*

$$\mathbb{P}_0\bigl(l_\infty(\Lambda) > t\bigr) \leqslant \exp\left(-\kappa_d \frac{t}{|\Lambda|^{2/d}}\right), \tag{1.8}$$

where $l_\infty(\Lambda)$ is the total sojourn time of the walk in the region Λ.

Note that the recent paper [8] gives a representation of $\mathbb{P}_0(l_\infty(\Lambda) > t)$, in terms of the eigenvalues and eigenvectors of the matrix whose entries are the Green function restricted to Λ. It is not clear to us how to deduce Lemma 1.2 from the type of representation of [8].

This note is organized as follows. We specify the model in Section 2. In Section 3, we deal with the lower bound. In Section 4, we deal with the upper bound. Finally, we have gathered in the Appendix the proof of Lemma 1.2, and the proof of some technical facts.

2. Model

Assumptions on the random walk. We assume that the increments of the walk are centered, with finite exponential moments, i.e.

$$S_k = \sum_{j=1}^{k} \xi_j, \quad \xi_j \text{ i.i.d}, \quad E[\xi_1] = 0, \quad E\bigl[\exp(\lambda \xi_1)\bigr] < \infty \quad \text{for all } \lambda \in \mathbb{R}^d. \tag{2.1}$$

It is then easy to see that there exist constants $C, c > 0$, such that for all n,

$$\mathbb{P}_0\Bigl[\max_{k \leqslant n} \|S_k\| \geqslant n\Bigr] \leqslant C \exp(-cn). \tag{2.2}$$

Assumptions on the scenery. Besides our basic tail assumption (1.3), we make assumptions on the law of the scenery whose goal is to simplify the technical parts. Thus, we say that a random variable with value in $\mathbb{R}$ is *bell-shaped* (usually called symmetric unimodal distribution), if its law has a density with respect to Lebesgue which is even, and decreasing on $\mathbb{R}^+$. Throughout the paper, we will assume that $\{\eta(x); x \in \mathbb{Z}^d\}$ are i.i.d and bell-shaped, with the following handy consequence, proved in Appendix A.

Lemma 2.1. *When $\{\eta(x), x \in \mathbb{Z}^d\}$ have independent bell-shaped densities, then for any Λ finite subset of $\mathbb{Z}^d$, and any $y > 0$*

$$P\left(\sum_{x \in \Lambda} \alpha_x \eta(x) > y\right) \leqslant P\left(\sum_{x \in \Lambda} \beta_x \eta(x) > y\right), \quad \textit{if } 0 \leqslant \alpha_x \leqslant \beta_x \textit{ for all } x \in \Lambda. \tag{2.3}$$

A typical use of Lemma 2.1 is the following bound

$$P\left(\sum_{\Lambda} \eta(x) > \frac{y}{\min \alpha_x}\right) \leqslant P\left(\sum_{\Lambda} \alpha_x \eta(x) > y\right) \leqslant P\left(\sum_{\Lambda} \eta(x) > \frac{y}{\max \alpha_x}\right). \tag{2.4}$$

Some notations. Throughout the paper, we set $a := \alpha/(\alpha+1)$ and $b := 1/(\alpha+1)$, and for $x \in \mathbb{Z}^d$, $\|x\| := \max_{i=1,\dots,d} |x_i|$. Finally, when considering the variables $\{\eta(x), x \in \Lambda\}$ for a finite region Λ of cardinality L, we will sometimes use the notation $\{\eta_j, 1 \leqslant j \leqslant L\}$.

3. Lower bound

We show in this section the following simple estimate.

Lemma 3.1. *There is a constant $c_1 > 0$ such that, for any $y > 0$, and n large*

$$P\Bigl(\sum_{x\in\mathbb{Z}^d} l_n(x)\eta(x) > ny\Bigr) \geqslant \exp\bigl(-c_1(ny)^a\bigr). \tag{3.1}$$

Proof. The bound (3.1) is obtained by using Lemma 2.1. Thus,

$$\begin{aligned} P\Bigl(\sum_{x} l_n(x)\eta(x) > ny\Bigr) &\geqslant P\bigl(l_n(0)\eta(0) > ny\bigr) = \sum_{k>0} \mathbb{P}_0\bigl(l_n(0) = k\bigr) P_\eta\Bigl(\eta(0) > \frac{ny}{k}\Bigr) \\ &\geqslant \mathbb{P}_0\bigl(l_n(0) = k\bigr) P_\eta\Bigl(\eta(0) > \frac{ny}{k}\Bigr) \quad \text{for any } k. \end{aligned} \tag{3.2}$$

We choose an n-depending k, for instance $k_n = [(ny)^a]$. Since $k_n/(ny)^a \to 1$ as n tends to infinity, we have for any $\epsilon > 0$ and n large that

$$\begin{aligned} P_\eta\Bigl(\eta(0) > \frac{ny}{k_n}\Bigr) &\geqslant \exp\Bigl(-c_\alpha(1+\epsilon)\Bigl(\frac{ny}{k_n}\Bigr)^\alpha\Bigr) \\ &\geqslant \exp\bigl(-c_\alpha(1+2\epsilon)(ny)^a\bigr). \end{aligned}$$

Now, if we set $\kappa_0 = \log(1/\mathbb{P}_0(H_0 < \infty))$, then

$$\begin{aligned} \mathbb{P}_0\bigl(l_n(0) = k_n\bigr) &\geqslant \mathbb{P}_0\Bigl(H_0 \leqslant \frac{n}{k_n}\Bigr)^{k_n} \\ &\geqslant \Bigl(\mathrm{e}^{-\kappa_0} - \mathbb{P}_0\Bigl(\frac{n}{k_n} < H_0 < \infty\Bigr)\Bigr)^{k_n}. \end{aligned} \tag{3.3}$$

Thus, for any $\kappa > \kappa_0$, we have for n large enough

$$P\Bigl(\sum_{x} l_n(x)\eta(x) > ny\Bigr) \geqslant \exp\bigl(-\bigl(\kappa + (1+2\epsilon)c_\alpha\bigr)(ny)^a\bigr). \tag{3.4}$$

This concludes the proof. □

4. Upper bound

The case $\alpha = 1$ is special and much simpler than $\alpha > 1$. Thus, we will treat the former specifically in Remark 4.4. Henceforth, we assume that $\alpha > 1$ and we recall that $a := \alpha/(\alpha+1)$, and $b = 1 - a < a$. We consider a subdivision of $[b, a]$, $b_1 = b < b_2 < \cdots < b_{N+1} = a$, and positive constants $\{y_\downarrow, y_0, \dots, y_N, y_\uparrow\}$ satisfying $y_\downarrow + y_0 + \cdots + y_N \leqslant y$. We will specify N and $\{b_i, y_i, i = 1, \dots, N\}$ after we partition the range of the walk $\mathcal{R}_n$, into $N+3$ sets. For $1 \leqslant i \leqslant N$, we set

$$\mathcal{D}_i := \bigl\{x \in \mathcal{R}_n\colon\ y^a n^{b_i} \leqslant l_n(x) < y^a n^{b_{i+1}}\bigr\}, \tag{4.1}$$

and for a *small* constant z to be chosen later

$$\mathcal{D}_0 := \bigl\{x \in \mathcal{R}_n\colon\ zn^b \leqslant l_n(x) < y^a n^b\bigr\}, \tag{4.2}$$

and lastly, for the two sets at the extremities

$$\mathcal{D}_\downarrow := \bigl\{x \in \mathcal{R}_n\colon\ l_n(x) < zn^b\bigr\}, \quad \text{and} \quad \mathcal{D}_\uparrow := \bigl\{x \in \mathcal{R}_n\colon\ l_n(x) \geqslant (yn)^a\bigr\}. \tag{4.3}$$

Thus,

$$\Bigl\{\sum_{x\in\mathbb{Z}^d} l_n(x)\eta(x) > ny\Bigr\} \subset \bigcup_{i=0}^{N} \Bigl\{\sum_{x\in\mathcal{D}_i} l_n(x)\eta(x) > ny_i\Bigr\} \cup \Bigl\{\sum_{x\in\mathcal{D}_\downarrow} l_n(x)\eta(x) > ny_\downarrow\Bigr\} \cup \{\mathcal{D}_\uparrow \neq \emptyset\}. \tag{4.4}$$

Thus, if we define $\mathcal{A} := \{\max_{k\leqslant n} \|S_k\| < n\}$, and recall that $\mathbb{P}_0(\mathcal{A}^c)$ is negligible compared to $\exp(-n^a)$ by (2.2), then

$$P\Big(\sum_{x\in\mathbb{Z}^d} l_n(x)\eta(x) > ny\Big) \leqslant \mathbb{P}_0\big(\mathcal{A}^c\big) + \sum_{i=0}^{N} P\Big(\mathcal{A}, \sum_{x\in\mathcal{D}_i} l_n(x)\eta(x) > ny_i\Big)$$
$$+ P\Big(\mathcal{A}, \sum_{x\in\mathcal{D}_\downarrow} l_n(x)\eta(x) > ny_\downarrow\Big) + \mathbb{P}_0(\mathcal{D}_\uparrow \neq \emptyset). \tag{4.5}$$

We will now estimate each terms separately in the next section. However, in the course of obtaining an upper bound, we will fall on the following requirement: we will need a positive β, independent of n, such that for $i \leqslant N$

$$\beta y \leqslant y_i n^{(a-b_{i+1})-(1-\delta_0)(a-b_i)} \quad \text{with the positive constant } \delta_0 := \frac{1/\alpha - 2/d}{1-2/d} < 1. \tag{4.6}$$

Thus, a simple choice of $\{b_i, y_i\}$ which fulfills (4.6) is $y_N := \beta y n^{(1-\delta_0)(a-b_N)}$, and

$$\forall i < N, \quad y_i = \beta y n^{-\epsilon_0(a-b_{i+1})}, \quad \text{and} \quad (a-b_i) = (1+\epsilon_0)(a-b_{i+1}), \quad \text{with } \epsilon_0 = \frac{\delta_0/2}{1-\delta_0/2}. \tag{4.7}$$

To explicit further the choices in (4.7), we introduce more notations:

$$z_1 = a - b_N, \quad z_2 = b_N - b_{N-1}, \ldots, z_N = b_2 - b_1. \tag{4.8}$$

Thus, (4.7) is fulfilled when $z_2 = \epsilon_0 z_1$ and for $i > 2$

$$z_i = \epsilon_0(z_1 + \cdots + z_{i-1}) = (1+\epsilon_0)z_{i-1} = (1+\epsilon_0)^{i-2} z_2 = (1+\epsilon_0)^{i-2}\epsilon_0 z_1. \tag{4.9}$$

Note that for $i < N$

$$a - b_{i+1} = z_1 + \cdots + z_{N-i} = \frac{z_{N-i+1}}{\epsilon_0} = (1+\epsilon_0)^{N-i-1} z_1.$$

The condition on y_i in (4.7) will be fulfilled if we choose $z_1 = \chi/\log(n)$, for a constant χ to be tuned later. Indeed, we obtain $y_N = \beta y \exp(\chi(1-\delta_0))$, and

$$\forall i < N, \quad y_i = \beta y \exp\big(-\log(n) z_1 \epsilon_0 (1+\epsilon_0)^{N-i-1}\big) = \beta y \exp\big(-\chi\epsilon_0(1+\epsilon_0)^{N-i-1}\big). \tag{4.10}$$

Thus, since

$$\sum_{i=1}^{\infty}(1+\epsilon_0)^i = \infty, \quad \text{and} \quad \sum_{i=1}^{\infty}\exp\big(-\chi\epsilon_0(1+\epsilon_0)^i\big) < \infty, \tag{4.11}$$

one can find N finite, of order $\log(\log(n))$, $\chi > 0$ and $\beta > 0$ independent of n (or rather χ_n and β_n can be chosen to converge to positive constants, and we omit the subscript n) such that

$$a - b = \sum_{i=1}^{N} z_i = \frac{\chi}{\log(n)}(1+\epsilon_0)^{N-1}, \quad \text{and} \quad \sum_{i=0}^{N} y_i + y_\downarrow + y_\uparrow = y. \tag{4.12}$$

Actually, the choice of $y_\downarrow, y_\uparrow$ is arbitrary since we are not after the exact constant in front of the speed n^a. For instance, we choose $y_\downarrow = y_\uparrow = y/3$.

4.1. Contribution of $\mathcal{D}_\downarrow$

Lemma 4.1. *We set for any $z > 0$, $\mathcal{D}_\downarrow(z) = \{x:\ l_n(x) \leqslant z n^b\}$. Then, for any $y_\downarrow > 0$, we have*

$$\overline{\lim_{z\to 0}}\ \overline{\lim_{n\to\infty}} \frac{1}{n^a} \log P\Big(\sum_{x\in\mathcal{D}_\downarrow(z)} l_n(x)\eta(x) > ny_\downarrow\Big) = -\infty. \tag{4.13}$$

Proof. We fix $z > 0$ and $\{l_n(x)\colon x \in \mathcal{D}_\downarrow(z)\}$ and integrate over the η, to obtain for $\lambda \geqslant 0$

$$P_\eta\bigg(\sum_{x\in\mathcal{D}_\downarrow(z)} l_n(x)\eta(x) > ny_\downarrow\bigg) \leqslant \exp\bigg(-ny_\downarrow\frac{\lambda}{zn^b}\bigg)\prod_{x\in\mathcal{D}_\downarrow(z)} E_\eta\bigg[\exp\bigg(\lambda\eta(x)\frac{l_n(x)}{zn^b}\bigg)\bigg]. \tag{4.14}$$

Note that by hypothesis (1.3), there is $\delta > 0$ such that

$$\nu(\delta) = E_\eta\big[\eta^2(x)\exp\big(\delta\big|\eta(x)\big|\big)\big] < \infty. \tag{4.15}$$

Also, it is an obvious fact that for $0 \leqslant \theta \leqslant \delta$

$$\exp\big(\theta\eta(x)\big) \leqslant 1 + \theta\eta(x) + \frac{\theta^2\eta(x)^2}{2}\,\mathrm{e}^{\delta|\eta(x)|}. \tag{4.16}$$

Thus, after taking expectation in (4.16)

$$E_\eta\big[\exp\big(\theta\eta(x)\big)\big] \leqslant 1 + \frac{\theta^2}{2}\nu(\delta) \leqslant \mathrm{e}^{\theta^2\nu(\delta)/2}. \tag{4.17}$$

Back to estimating (4.14), we choose $\lambda \leqslant \delta$ and use (4.17) to obtain

$$\prod_{x\in\mathcal{D}_\downarrow(z)} E_\eta\bigg[\exp\bigg(\lambda\eta(x)\frac{l_n(x)}{zn^b}\bigg)\bigg] \leqslant \exp\bigg(\frac{\lambda^2}{2}\nu(\delta)\frac{\sum_{x\in\mathcal{D}_\downarrow(z)} l_n(x)^2}{(zn^b)^2}\bigg). \tag{4.18}$$

Now, $\sum_{x\in\mathcal{D}_\downarrow(z)} l_n(x)^2 \leqslant zn^{1+b}$. Thus,

$$P\bigg(\sum_{x\in\mathcal{D}_\downarrow(z)} l_n(x)\eta(x) > ny_\downarrow\bigg) \leqslant \exp\bigg(-\frac{n^{1-b}}{z}\sup_{0\leqslant\lambda\leqslant\delta}\bigg\{y_\downarrow\lambda - \frac{\nu(\delta)\lambda^2}{2}\bigg\}\bigg) \tag{4.19}$$

The result follows since for any $y_\downarrow > 0$, the supremum is positive, and z can be sent to zero. □

4.2. Contributions of $\mathcal{D}_\uparrow$

Lemma 4.2. *For $\mathcal{D}_\uparrow$ given in* (4.3)*, there is $C_d > 0$ such that for n large*

$$\mathbb{P}_0(\mathcal{D}_\uparrow \neq \emptyset) \leqslant C_d n\,\mathrm{e}^{-\kappa_0(yn)^a}. \tag{4.20}$$

Proof. First, note that

$$\mathbb{P}_0(\mathcal{D}_\uparrow \neq \emptyset) = \mathbb{P}_0\big(l_n(x) > (yn)^a \text{ for some } x \in \mathcal{R}_n\big) \leqslant \sum_{x\in\mathbb{Z}^d}\mathbb{P}_0(x\in\mathcal{R}_n)\mathbb{P}_0\big(l_\infty(0) \geqslant (yn)^a\big). \tag{4.21}$$

Now, for $\kappa_0 := 1/\log(\mathbb{P}_0(H_0 < \infty))$, it is clear that

$$\mathbb{P}_0\big(l_\infty(0) \geqslant (yn)^a\big) \leqslant \mathrm{e}^{-\kappa_0(yn)^a}. \tag{4.22}$$

Thus, we conclude by recalling that $\mathbb{E}_0[|\mathcal{R}_n|] \leqslant n+1$. □

4.3. Contributions of $\mathcal{D}_i$ for $i = 0,\dots,N$

Lemma 4.3. *Fix $i = 0,\dots,N$. We have a constant $c_2 > 0$ such that for $y > 0$, and n large*

$$P\bigg(\mathcal{A}, \sum_{x\in\mathcal{D}_i} l_n(x)\eta(x) > ny_i\bigg) \leqslant \exp\big(-c_2(ny)^a\big).$$

Proof. We first treat the case $i > 0$. Note that $|\mathcal{D}_i| \leqslant y^{-a}n^{1-b_i}$, and on $\mathcal{A}$ there are at most $\binom{n^d}{|\mathcal{D}_i|}$ possible choices for $\mathcal{D}_i$ since the walk does not exit a region of radius n. Thus, using Lemma 2.1 (and (2.4))

$$P\Big(\mathcal{A},\sum_{x\in\mathcal{D}_i} l_n(x)\eta(x)>ny_i\Big)\leqslant\sum_{L=1}^{y^{-a}n^{1-b_i}}\mathbb{P}_0\big(\mathcal{A},|\mathcal{D}_i|=L\big)P_\eta\Bigg(\sum_{j=1}^{L}\eta_j>\frac{ny_i}{\max\{l_n(x)\colon x\in\mathcal{D}_i\}}\Bigg)$$
$$\leqslant\sum_{L=1}^{y^{-a}n^{1-b_i}}\binom{n^d}{L}\sup_{\Lambda\colon|\Lambda|=L}\mathbb{P}_0(\mathcal{D}_i=\Lambda)P_\eta\Bigg(\sum_{j=1}^{L}\eta_j>\frac{ny_i}{n^{b_{i+1}}y^a}\Bigg)$$
$$\leqslant\sum_{L=1}^{y^{-a}n^{1-b_i}}\big(n^d\big)^L\sup_{\Lambda\colon|\Lambda|=L}\mathbb{P}_0\big(l_\infty(\Lambda)>Ly^an^{b_i}\big)P_\eta\Bigg(\sum_{j=1}^{L}\eta_j>\frac{ny_i}{n^{b_{i+1}}y^a}\Bigg).\tag{4.23}$$

By using Lemma 1.2, we have

$$\big(n^d\big)^L\sup_{\Lambda\colon|\Lambda|=L}\mathbb{P}_0\big(l_\infty(\Lambda)>Ly^an^{b_i}\big)\leqslant\exp\big(-\kappa_d y^a n^{b_i}L^{1-2/d}+L\log\big(n^d\big)\big),\tag{4.24}$$

and the combinatorial factor n^{dL} is negligible when

$$n^{b_i}L^{1-2/d}\gg L\log(n).\tag{4.25}$$

Since $L\leqslant y^{-a}n^{1-b_i}$ and $b_i\geqslant b=1/(\alpha+1)$, (4.25) requires n large and

$$\frac{2}{d}(1-b_i)<b_i\Longleftarrow\alpha<\frac{d}{2}.\tag{4.26}$$

Thus, the combinatorial factor is always innocuous when $\alpha<d/2$.

Let $B>0$ be a fixed large constant. We say that L is *large* when $n^{b_i}L^{1-2/d}>Bn^a$, and this case poses obviously no problem since the term $\mathbb{P}_0(l_\infty(\Lambda)>Ln^{b_i}y^a)$ suffices to obtain the right speed. Thus, we assume that L is *small*, that is:

$$n^{b_i}L^{1-2/d}\leqslant Bn^a.\tag{4.27}$$

Thus, we consider for a fixed $i=1,\dots,N$

$$L\leqslant An^{\gamma_i},\quad\text{with }\gamma_i:=\frac{a-b_i}{1-2/d},\ \text{and }A=B^{1/(1-2/d)}.\tag{4.28}$$

We want to evaluate $P_\eta(\sum_{\mathcal{D}_i}\eta(x)>n^{1-b_{i+1}}y_iy^{-a},\ |\mathcal{D}_i|\leqslant L)$ when L is as in (4.28). First, note that $n^{\gamma_i}\ll n^{1-b_{i+1}}y_i$, when n is large enough. Indeed, first rewrite

$$1-b_{i+1}-\gamma_i=\left(1-a-\frac{\gamma_i}{\alpha}\right)+\big(a-b_{i+1}-(1-\delta_0)(a-b_i)\big).$$

Then, by noting that $1-a-\gamma_i/\alpha\geqslant1-a-\gamma_1/\alpha\geqslant b\delta_0$, and using (4.6),

$$n^{1-b_{i+1}-\gamma_i}y_i=n^{1-a-\gamma_i/\alpha}n^{a-b_{i+1}-(1-\delta_0)(a-b_i)}y_i\geqslant n^{1-a-\gamma_1/\alpha}\beta y=n^{b\delta_0}\beta y.\tag{4.29}$$

Hence for L satisfying (4.28), $n^{1-b_{i+1}}y_i\gg L$, and using standard Large Deviations estimates (see Lemma A.4 in Appendix A), for all $\epsilon>0$ and n sufficiently large,

$$P_\eta\Bigg(\sum_{j=1}^{L}\eta_j>n^{1-b_{i+1}}\frac{y_i}{y^a}\Bigg)\leqslant\exp\bigg(-c_\alpha(1-\epsilon)L\bigg(\frac{n^{1-b_{i+1}}y_i}{Ly^a}\bigg)^\alpha\bigg)$$
$$\leqslant\exp\bigg(-c_\alpha(1-\epsilon)A\bigg(\frac{n^{1-b_{i+1}-\gamma_i(1-1/\alpha)}y_i}{Ay^a}\bigg)^\alpha\bigg)$$
$$\leqslant\exp\bigg(-c_\alpha(1-\epsilon)\frac{\beta^\alpha}{A^{\alpha-1}}(ny)^a\bigg),\quad\text{using (4.6)}.\tag{4.30}$$

By the same arguments, we can treat the case $\mathcal{D}_0$. Indeed, note that $\gamma_0=(a-b)(1-2/d)^{-1}<a$, and for L *small*, we have

$$P_\eta\Bigg(\sum_{j=1}^{L}\eta_j>n^{1-b}\frac{y_0}{y^a}\Bigg)\leqslant\exp\bigg(-c_\alpha(1-\epsilon)\frac{n^{a\alpha}(y_0/y^a)^\alpha}{L^{\alpha-1}}\bigg),\tag{4.31}$$

which is negligible since $a\alpha - (\alpha - 1)\gamma_0 > a$. □

Remark 4.4. When $\alpha = 1$, then $b = a = 1/2$. Thus, the range of the walk is divided into three sets: $\mathcal{D}_\downarrow, \mathcal{D}_\uparrow$ and $\mathcal{D}_0$ as in (4.3) and (4.2) respectively. Also, we can choose $y_\downarrow = y_\uparrow = y_0 = y/3$. Now, our treatment for $\mathcal{D}_\downarrow, \mathcal{D}_\uparrow$ only assumed small exponential moments for the walk, which hold in this case. To treat $\mathcal{D}_0$ note that only the case L *small* may pose problem. However, since $\gamma_0 = 0$, L small means $L \leqslant A$ for a large constant. It is easy to see that those terms are of the correct order since, there is a constant $\bar{c}$ such that

$$P_\eta\left(\sum_{j=1}^{L} \eta_j > n^{1-a}\frac{y_0}{y^a}\right) \leqslant L P_\eta\left(\eta_1 > \frac{(ny)^{1/2}}{3L}\right) \leqslant L \exp\bigl(-\bar{c}(ny)^{1/2}\bigr). \tag{4.32}$$

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Appendix A

A.1. On bell-shaped densities

We recall that a density f is bell-shaped if it is even and decreasing on $\mathbb{R}^+$. A first observation, which seems to go back to A.Wintner [14], is the following.

Lemma A.1. *If f, g are two bell-shaped densities, so is their convolution $f * g$.*

Proof. First, it is obvious that $f * g$ is even. Indeed, by the evenness of both f and g

$$f * g(-t) = \int_{\mathbb{R}} f(-t-s)g(s)\,\mathrm{d}s = \int_{\mathbb{R}} f(t+s)g(-s)\,\mathrm{d}s = \int_{\mathbb{R}} f(t-s)g(s)\,\mathrm{d}s = f * g(t).$$

Now, assume that f is differentiable. Then,

$$(f * g)'(t) = \int_{\mathbb{R}} f'(t-s)g(s)\,\mathrm{d}s = \int_{\mathbb{R}} f'(s)g(t-s) = \int_0^\infty f'(s)\bigl(g\bigl(|t-s|\bigr) - g(t+s)\bigr)\,\mathrm{d}s, \tag{A.1}$$

where we used the oddness of f' and the evenness of g. Now, for $t, s \geqslant 0$, we have $g(|t-s|) - g(t+s) \geqslant 0$, and $f'(s) \leqslant 0$ implying that $(f * g)'(t) \leqslant 0$.

Now let $\{\varphi_\epsilon, \epsilon > 0\}$ be a differentiable *bell-shaped* approximate identity. By what we just saw, $\varphi_\epsilon * f$ is a bell-shaped differentiable density. So is in turn $(\varphi_\epsilon * f) * g$. Thus, for any $0 \leqslant t \leqslant T$, we have $(\varphi_\epsilon * f) * g(t) \geqslant (\varphi_\epsilon * f) * g(T)$. By pointwise convergence, as ϵ tends to 0, we obtain that $f * g(t) \geqslant f * g(T)$. □

By induction, using Lemma 2.1, we obtain the following corollary.

Corollary A.2. *If $\{\eta_i, i = 1, \ldots, n\}$ are independent bell-shaped variables and $S = \alpha_1\eta_1 + \cdots + \alpha_n\eta_n$, with positive $\{\alpha_i\}$, then S has a bell-shaped density.*

Finally, the useful result is the following.

Lemma A.3. *Let $\{\eta_i, i = 1, \ldots, n\}$ be independent bell-shaped variables and $0 \leqslant \alpha_i \leqslant \beta_i$ for $i = 1, \ldots, n$. Then for any $y > 0$, we have* (2.3).

Proof. We prove the lemma by induction on the number of β_i larger than α_i. Thus, it is enough to show that for any $y > 0$, $x \mapsto P(S + x\eta_n > y)$ is increasing on $\mathbb{R}^+$ when S is a bell-shaped variable independent of η_n.

First note that for symmetric independent ξ, η, we have for $y > 0$

$$P(\xi + \eta > y) = P(\xi > y) + \int_0^\infty P(\eta > z)\big(f_\xi\big(|y - z|\big) - f_\xi(y + z)\big)\,\mathrm{d}z. \tag{A.2}$$

The proof is concluded as we apply (A.2) to $\xi = S$ and $\eta = x\eta_n$, and as we note that $f_S(|y - z|) - f_S(y + z) \geqslant 0$ and $x \mapsto P(\eta_n > z/x)$ is increasing. $\square$

A.2. On a localization result

We first prove Lemma 1.2.

First Step: We show that $\mathbb{E}_0[l_\infty(\Lambda)] \leqslant C_d|\Lambda|^{2/d}$.

The following Green function estimates is standard (see for instance [12] Theorem 10.1): there is C_d such that for any $y \in \mathbb{Z}^d$

$$G(0, y) = \mathbb{E}_0\big[l_\infty(y)\big] \leqslant \frac{C_d}{1 + \|y\|^{d-2}}. \tag{A.3}$$

Now, $l_\infty(\Lambda) = \sum_{y\in\Lambda} l_\infty(y)$ and

$$\mathbb{E}_0\big[l_\infty(\Lambda)\big] = \sum_{y\in\Lambda} \mathbb{E}_0\big[l_\infty(y)\big] \leqslant \sum_{y\in\Lambda} \frac{C_d}{1 + \|y\|^{d-2}}. \tag{A.4}$$

We establish now an upper bound on the right-hand side of (A.4). Let φ be an ordering of the sites of $\mathbb{Z}^d$ in increasing distance from the origin. In other words, $\varphi : \mathbb{N} \to \mathbb{Z}^d$ is a one to one, onto map so that $\|\varphi(i)\| \leqslant \|\varphi(i + 1)\|$, for all $i \in \mathbb{N}$. Let $\psi : \{0, \ldots, |\Lambda| - 1\} \to \varphi^{-1}(\Lambda)$ be an ordering of $\varphi^{-1}(\Lambda)$ (so that $\psi(0) < \cdots < \psi(|\Lambda| - 1)$) and note that

$$k \leqslant \psi(k), \quad \text{and} \quad \big\|\varphi(k)\big\| \leqslant \big\|\varphi\big(\psi(k)\big)\big\|. \tag{A.5}$$

Thus, $g := \varphi\psi^{-1}\varphi^{-1} : \Lambda \to \mathbb{Z}^d$ is a rearrangement of Λ inside a "ball" of radius proportional to $|\Lambda|^{1/d}$. Thus, it is a trivial fact that there is a constant c'_d and $\sup_\Lambda \|g(x)\| \leqslant c'_d|\Lambda|^{1/d}$. Let $r := c'_d|\Lambda|^{1/d}$, and note that

$$\begin{aligned}\sum_{x\in\Lambda} \frac{1}{1 + \|x\|^{d-2}} &\leqslant \sum_{x\in\Lambda} \frac{1}{1 + \|g(x)\|^{d-2}} \leqslant \sum_{\|y\|\leqslant r} \frac{1}{1 + \|y\|^{d-2}} \\ &\leqslant \frac{1}{2} + \int_0^r \frac{s^{d-1}}{1 + s^{d-2}}\,\mathrm{d}s \leqslant \frac{1}{2} + \int_0^r s\,\mathrm{d}s \\ &\leqslant \frac{1}{2} + \frac{r^2}{2} \leqslant r^2.\end{aligned} \tag{A.6}$$

The first step concludes easily. By Chebychev's inequality we have

$$\sup_{x,\Lambda} \mathbb{P}_x\big(l_\infty(\Lambda) > 2C_d r^2\big) < \frac{1}{2}. \tag{A.7}$$

Indeed, the starting point of the walk can very well be any site $x \in \mathbb{Z}^d$ since the transition kernel is translation invariant, and Λ is arbitrary.

Second Step: We show that

$$\mathbb{P}_0\big(l_\infty(\Lambda) > t\big) \leqslant \left(\frac{1}{2}\right)^{t/(2C_d r^2)}$$

Define a sequence of stopping times for $k = 1, 2, \ldots$

$$\sigma_k = \inf\big\{n \geqslant 0\colon\ l_n(\Lambda) > 2kC_d r^2\big\}, \tag{A.8}$$

and note that $\sigma_k = \sigma_{k-1} + \sigma_1 \circ \theta_{\sigma_{k-1}}$. We have used the notation θ_k for the time translation by k-units. Now, the bound (A.7) can be expressed in term of σ_1 as

$$\mathbb{P}_0(\sigma_1 < \infty) < \frac{1}{2}.$$

We express now the total sojourn time in Λ in terms of $\{\sigma_k, k \in \mathbb{N}\}$

$$\mathbb{P}_0(l_\infty(\Lambda) > 2kC_d r^2) = \mathbb{P}_0(\sigma_k < \infty), \tag{A.9}$$

and by the Strong Markov property

$$\begin{aligned} \mathbb{P}_0\big(l_\infty(\Lambda) > 2kC_d r^2\big) &= \mathbb{E}_0\big[\mathbb{1}_{\sigma_{k-1}<\infty}\mathbb{P}_0(\sigma_{k-1} + \sigma_1 \circ \theta_{\sigma_{k-1}} < \infty | \mathcal{F}_{\sigma_{k-1}})\big] \\ &= \mathbb{E}_0\big[\mathbb{1}_{\sigma_{k-1}<\infty}\mathbb{P}_{S_{\sigma_{k-1}}}(\sigma_1 < \infty)\big] \\ &\leqslant \frac{1}{2}\mathbb{P}_0(\sigma_{k-1} < \infty). \end{aligned} \tag{A.10}$$

By induction the bound (1.8) follows readily.

A.3. On a large deviation estimate

To be self-contained, we give an obvious estimate, for which a reference could not be found. We assume that $\alpha > 1$.

Lemma A.4. *For all $\epsilon > 0$, L a positive integer, and for t/L large enough,*

$$P_\eta\left[\sum_{j=1}^{L} \eta_j \geqslant t\right] \leqslant \exp\left(-c_\alpha(1-\epsilon)\frac{t^\alpha}{L^{\alpha-1}}\right).$$

Proof. For $\lambda \in \mathbb{R}$, set $\Lambda(\lambda) := \log E_\eta[e^{\lambda\eta_1}]$.

$$P_\eta\left[\sum_{j=1}^{L} \eta_j \geqslant t\right] \leqslant \exp\left(-\alpha c_\alpha\left(\frac{t}{L}\right)^{\alpha-1} t\right)\exp\left(L\Lambda\left(\alpha c_\alpha\left(\frac{t}{L}\right)^{\alpha-1}\right)\right).$$

By Kasahara's Tauberian theorem, for large x

$$\Lambda(x) \simeq \frac{1}{\bar\alpha(\alpha c_\alpha)^{\bar\alpha-1}}x^{\bar\alpha}, \quad \text{where } \frac{1}{\alpha} + \frac{1}{\bar\alpha} = 1.$$

Hence for all $\epsilon > 0$ and t/L large enough,

$$\begin{aligned} P_\eta\left[\sum_{j=1}^{L} \eta_j \geqslant t\right] &\leqslant \exp\left(-\alpha c_\alpha\frac{t^\alpha}{L^{\alpha-1}}\right)\exp\left((1+\epsilon)\frac{\alpha c_\alpha}{\bar\alpha}\frac{t^\alpha}{L^{\alpha-1}}\right) \\ &\leqslant \exp\left(-c_\alpha\frac{t^\alpha}{L^{\alpha-1}}\left(1 - \frac{\epsilon\alpha}{\bar\alpha}\right)\right). \qquad \square \end{aligned}$$

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