

## Coarsening, nucleation, and the marked Brownian web

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### Abstract

Coarsening on a one-dimensional lattice is described by the voter model or equivalently by coalescing (or annihilating) random walks representing the evolving boundaries between regions of constant color and by backward (in time) coalescing random walks corresponding to color genealogies. Asymptotics for large time and space on the lattice are described via a continuum space–time voter model whose boundary motion is expressed by the *Brownian web* (BW) of coalescing forward Brownian motions. In this paper, we study how small noise in the voter model, corresponding to the nucleation of randomly colored regions, can be treated in the continuum limit. We present a full construction of the continuum noisy voter model (CNVM) as a random *quasicoloring* of two-dimensional space time and derive some of its properties. Our construction is based on a Poisson marking of the *backward* BW within the *double* (i.e., forward and backward) BW.

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### Résumé

Le coarsening sur un réseau uni-dimensionnel est décrit par le modèle du votant ou de manière équivalente par des marches aléatoires coalescentes (ou annihilantes), qui modélisent l'évolution des frontières séparant les régions de différentes couleurs, et par des marches aléatoires coalescentes backward (en temps) qui correspondent aux généalogies des couleurs. Les limites en espace et en temps sur le réseau sont décrites par le *Brownian web* (BW) associé à des mouvements browniens coalescents forward. Dans cet article, nous étudions comment un faible bruit dans le modèle du votant, correspondant à la nucléation de régions colorées aléatoires, peut être traité dans la limite continue. Nous présentons une entière construction du modèle du votant avec bruit continu (CNVM), vu comme une *quasi-coloration* aléatoire bidimensionnelle en espace et en temps, et nous décrivons certaines de ses propriétés. Notre construction est fondée sur un marquage de Poisson des BW *backward* dans le *double* (i.e., forward et backward) BW.

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## 1. Introduction

In this paper we construct the one-dimensional continuum noisy voter model (CNVM) with  $q$  colors (opinions), including the case  $q = \infty$ . This model can be thought of as the scaling limit of the noisy voter model (NVM) on the one-dimensional lattice  $\mathbb{Z}$ , extending some of what has been done for the voter model without noise [1,24,7,8]. In the present paper we will not address weak convergence issues about the continuum limit procedure (in the spirit of [7,8]) but will rather focus primarily on the continuum model itself (in the spirit of [24]). We will however prove convergence of finite dimensional distributions.

The CNVM is basically a random coloring of the continuum space–time plane in which colors, generated at special nucleation points, propagate to space–time regions. The stochastic geometric properties of this coloring (see Theorems 1.1, 1.3, 1.5) are fairly direct consequences of a general picture (see the discussions following Remark 1.6) based on the Brownian web, introduced by Arratia [1], its fine properties analyzed by Tóth and Werner [24], and a Poissonian marking process of Brownian web double points (see Section 2).

Returning to the discrete voter model, in the non-noisy case, the color at each site is updated after a random exponentially distributed waiting time (with mean one) by taking on the color of a randomly chosen neighbor [19, 20]. For updates in the noisy voter model [13], with probability  $1 - p$  the site takes on the color of a randomly chosen neighbor and with probability  $p$  it takes on a color chosen uniformly at random from all possible colors or, in the case  $q = \infty$ , by taking on a completely novel color.

When  $q = 2$ , and the two colors are chosen to be  $+1$  and  $-1$ , the noisy voter model is exactly the stochastic one-dimensional (nearest-neighbor) Ising model with Glauber (heat-bath) dynamics, where the noise parameter  $p$  is related to the temperature (with the non-noisy case corresponding to zero temperature), see, e.g., [13,10,11] and Subsection 4.7. For more background on the non-noisy case in the contexts of zero-temperature Ising/Potts models and diffusion limited reactions, see [4,3,6].

The scaling limits for voter models, with or without noise, expressed in terms of Brownian webs with or without marks, should be the same scaling limits one gets for certain stochastic PDE models that arise in a variety of physical settings, e.g., those that describe nucleation, diffusion and annihilation of coherent structures (*kinks*) in a regime where they can be regarded as pointwise objects—see, e.g., [15] (also [14]) and references therein. This happens in the stochastic Ginzburg–Landau equation in the limit of small noise and large damping [5] or in a classical  $(1 + 1)$ -dimensional  $\phi^4$  field theory at finite (low) temperature [15].

We remark that there is also interest (see, e.g., [10]) in stochastic Ising models where the temperature/noise parameter is not constant in time (and/or space). These type of voter models are inhomogeneous in time Markov processes. Although we will not explicitly consider those types of models in this paper, it is clear how to extend the marking constructions given in Section 2 by using inhomogeneous in time (and/or space) Poisson marking processes. Of course, some of the simple formulas for two-point functions given later in this section become more complicated. We now return to models that are space–time homogeneous.

In the voter model without noise one naturally expects to see large blocks of the same color and this indeed happens. With noise, new colors appear within color blocks. This introduces two new boundaries (or one, when the new color appears exactly at an already existing boundary of two blocks of different colors). Most of these new colors and associated boundaries on the (microscopic) lattice will not survive for very long, but occasionally some will survive for long enough to nucleate a macroscopic region of that new color. In this paper, we introduce a continuum model which describes the long time behavior of the lattice voter model with small noise on the appropriate spatial scale.

In the voter model without noise, the boundaries evolve according to coalescing (or annihilating, if the colors on either side of the coalesced boundary are the same) random walks, which in the scaling limit converge to coalescing Brownian motions. The collection of *all* coalescing walks converges to the Brownian web (BW) [1,24,7,8]. In the BW, space–time (double) points where two distinct paths start correspond in the continuum voter model to two *possible* boundaries which start at a microscopic distance and survive for a macroscopic time. In the continuum version of the noisy voter model, if a new color appears at such a double point, then it will survive for a nonzero (macroscopic) time and that double point is a nucleation point for the new color. All other newly created colors can be observed at the microscopic level only, so they do not appear in the continuum limit. Even though most space–time points are not double points and most double points are not nucleation points, we shall see that the nucleation points form a dense (countable) subset of the space–time plane.

Roughly speaking, the CNVM is a coloring of the continuum space–time plane using the boundaries emerging from the nucleation points of the newly created colors. Two things need to be explained—how the nucleation points are chosen from among all the double points of the Brownian web, and how a new color taken on at a nucleation point propagates to a region of space–time. We will discuss both of these in the rest of this section, and then with more detail in Section 2 and Subsection 4.2 respectively.

We begin by stating a theorem that describes the nature of the  $q = \infty$  CNVM, followed by one describing the finite  $q$  CNVM. As mentioned previously, the detailed properties given in these theorems are fairly direct consequences of a general picture based on the marking of the Brownian web, which we begin to describe later in this section of the paper. Except near the end of this section, we restrict attention, in both the lattice and continuum settings, to the stationary noisy voter model with time  $t \in (-\infty, \infty)$ . A natural object we will focus upon is the mapping from space–time points to the one or finitely many nucleation points whose color is eventually inherited by that space–time point. In the lattice case, one can easily define things so that the mapping is to a single nucleation point; in the continuum limit it is more natural to map onto finitely many points. For the lattice case, let  $\Theta_1^p = (X_1^p, T_1^p)$  denote that mapping on  $\mathbb{Z} \times \mathbb{R}$  for the  $q = \infty$  voter model on  $\mathbb{Z}$  with noise parameter  $p$ . Using the diffusive scalings  $\delta^{-1}x$  and  $\delta^{-2}t$ , we define, for  $\delta > 0$ , the rescaled mapping on  $(\delta\mathbb{Z}) \times (\delta^2\mathbb{R})$ ,

$$\Theta_\delta^p(x, t) = (\delta X_1^p(\delta^{-1}x, \delta^{-2}t), \delta^2 T_1^p(\delta^{-1}x, \delta^{-2}t)). \quad (1.1)$$

**Theorem 1.1.** *For each  $\lambda > 0$ , there is a space–time translation invariant random mapping  $\Theta$  from  $\mathbb{R}^2$  to finite subsets of  $\mathbb{R}^2$  with the following properties:*

- (1) *For deterministic  $(x, t)$ ,  $\Theta(x, t)$  is almost surely a singleton and distributed as  $(x + B(R_\lambda), t - R_\lambda)$  where  $B(\cdot)$  is a standard Brownian motion and  $R_\lambda$  is exponentially distributed with mean  $1/\lambda$  and is independent of  $B(\cdot)$ .*
- (2) *For deterministic  $(x_1, t_1), \dots, (x_n, t_n)$ :  $\Theta$  evaluated at  $(x_1, t_1), \dots, (x_n, t_n)$  is the limit in distribution as  $\delta \rightarrow 0$  of  $\Theta_\delta^{\delta^2\lambda}$  at  $(x_1^\delta, t_1^\delta), \dots, (x_n^\delta, t_n^\delta)$  if  $(x_i^\delta, t_i^\delta) \rightarrow (x_i, t_i)$  for each  $i$ . In addition,*

$$\mathbb{P}(\Theta(x_1, t_1) = \dots = \Theta(x_n, t_n)) \quad (1.2)$$

$$= \lim_{\delta \rightarrow 0} \mathbb{P}(\Theta_\delta^{\delta^2\lambda}(x_1^\delta, t_1^\delta) = \dots = \Theta_\delta^{\delta^2\lambda}(x_n^\delta, t_n^\delta)). \quad (1.3)$$

- (3) *Almost surely, the set of all nucleation points,*

$$\mathcal{N} \equiv \bigcup_{(x,t) \in \mathbb{R}^2} \Theta(x, t), \quad (1.4)$$

*is a dense countable subset of  $\mathbb{R}^2$ , and for each nucleation point  $(x', t') \in \mathcal{N}$ , its color region,*

$$C_{(x',t')} \equiv \{(x, t) \in \mathbb{R}^2: \Theta(x, t) \ni (x', t')\}, \quad (1.5)$$

*is a compact, perfect subset of  $\mathbb{R}^2$  with an empty interior.*

- (4) Almost surely, the cardinalities  $|\Theta(x, t)|$  as  $(x, t)$  varies over  $\mathbb{R}^2$  take on only the values 1, 2 and 3. Almost surely, the set of all  $(x, t)$  with  $|\Theta(x, t)| \geq 2$  has Hausdorff dimension  $3/2$  (and hence zero two-dimensional Lebesgue measure).
- (5)  $\psi(x, t)$ , defined as  $\mathbb{P}(\Theta(0, 0) = \Theta(x, t))$ , is expressible as

$$\psi(x, t) = e^{-\lambda|t|} \mathbb{E}(e^{-\sqrt{2\lambda}|x+B(|t|)|}), \tag{1.6}$$

where  $B$  is standard Brownian motion. In particular the autocorrelation function is

$$\psi(0, t) = 1 - \frac{1}{\sqrt{\pi}} \int_{-\sqrt{\lambda|t|}}^{\sqrt{\lambda|t|}} \exp(-v^2) dv, \tag{1.7}$$

and the equal time correlation function is

$$\psi(x, 0) = e^{-\sqrt{2\lambda}|x|}. \tag{1.8}$$

- (6) For any deterministic time  $t$ , almost surely: for every  $K < \infty$ , the cardinalities  $|\Theta(x, t)|$  as  $x$  varies over  $[-K, K]$  are all one except for finitely many  $x$ 's where the cardinality is two; for each nucleation point  $(x', t')$ , the intersection of its color region,  $C_{(x', t')}$ , with the horizontal line  $\mathbb{R} \times \{t\}$  is either empty or consists of finitely many closed intervals with nonempty interiors.
- (7) For any deterministic  $x$ , almost surely: the set of  $t$ 's such that  $\Theta(x, t)$  is a singleton has full (one-dimensional) Lebesgue measure and  $\Theta(x, t)$  is a doubleton for all other  $t$ 's; for every nonempty open interval  $I$  of  $t$ 's, there are infinitely many distinct color regions,  $C_{(x', t')}$ , that intersect the vertical line segment  $\{x\} \times I$  (including both singleton and doubleton points).
- (8) For  $(x', t') \in \mathcal{N}$ , let  $C_{(x', t')}^u$  denote the unique-color subset of  $C_{(x', t')}$ :

$$C_{(x', t')}^u = \{(x, t): \Theta(x, t) = (x', t')\}. \tag{1.9}$$

Almost surely, for every  $(x', t') \in \mathcal{N}$ ,  $C_{(x', t')}^u$  has strictly positive (two-dimensional) Lebesgue measure.

**Remark 1.2.** Notice the contrast of the (a.s.) natures of the space time color regions  $C_{(x', t')}$  on one hand, and the fixed deterministic time color regions  $C_{(x', t')} \cap [\mathbb{R} \times \{t\}]$ , on the other hand. The former are Cantor-like sets, while the latter are (possibly empty) finite disjoint unions of closed intervals with nonempty interiors.

To construct the CNVM when  $q$  is finite, i.i.d. uniformly distributed random variables taking values in  $\{1, \dots, q\}$  are assigned to each of the countably many nucleation points (of the  $q = \infty$  model of Theorem 1.1). Composing this random color assignment with the random mapping  $\Theta$  results in a stationary random *quasicoloring*  $\Phi(x, t)$  mapping  $\mathbb{R}^2$  into finite subsets of  $\{1, \dots, q\}$  (with cardinality 1 or 2 or 3). Similarly one constructs the corresponding random quasicoloring  $\Phi_\delta^p$  on  $(\delta\mathbb{Z}) \times (\delta^2\mathbb{R})$  for the lattice noisy voter model. As a corollary of Theorem 1.1, we have the following.

**Theorem 1.3.** For  $\lambda \in (0, \infty)$  and  $q \in \{2, 3, \dots\}$ , there exists a space–time translation invariant random  $\{1, 2, \dots, q\}$ -quasicoloring  $\Phi$  of the plane such that its finite dimensional distributions are the limits as  $\delta \rightarrow 0$  of those of  $\Phi_\delta^{\delta^2\lambda}$ , the diffusively rescaled stationary one-dimensional voter model with  $q$  colors and noise parameter  $\delta^2\lambda$ . In particular, for deterministic  $(x, t)$ , and  $i, j \in \{1, \dots, q\}$ ,

$$\mathbb{P}(\Phi(x, t) = \{i\}) = q^{-1} \tag{1.10}$$

and

$$\mathbb{P}(\Phi(0, 0) = \{i\}, \Phi(x, t) = \{j\}) = q^{-1} \psi(x, t) \delta_{i, j} + q^{-2} (1 - \psi(x, t)), \tag{1.11}$$

where  $\psi(x, t)$  is  $\mathbb{P}(\Theta(0, 0) = \Theta(x, t))$ , as given in Theorem 1.1. For any deterministic time  $t$ , almost surely,  $\Phi(\cdot, t)$  partitions the line into open single-color intervals and a locally finite set of points where  $\Phi(x, t)$  is a doubleton  $\{i, j\}$  separating two intervals of different colors. For any deterministic  $x$ , there is zero probability that  $\Phi(x, \cdot)$  is constant on any nonempty open interval of  $t$ -values.

**Remark 1.4.** We note that (1.10) and (1.11) are immediate consequences of the coloring procedure in which i.i.d. uniformly random colors are assigned to each nucleation point. The convergence of the finite dimensional color distributions follows from the convergence (given in Theorem 1.1) to  $\mathbb{P}(\Theta(x_1) = \dots = \Theta(x_n))$  and the standard fact that such probabilities (like the connectivities in lattice percolation models) determine algebraically the probabilities of all events involving partitions of  $x_1, \dots, x_m$  by distinct color values.

The single time color distribution of the (stationary) CNVM (the model whose existence is established in Theorem 1.3) is (of course) an invariant distribution for the CNVM viewed as a (continuum) spin dynamics. The above theorems give partial results about the nature of this distribution. In the case where  $q = 2$ , it can be fully characterized from the above and the following facts. In this case, the noisy voter model on the lattice corresponds to a stochastic Ising model (as we will discuss in more detail in Subsection 4.7). Hence, its invariant distribution is the Gibbs distribution for a nearest neighbor Ising model on  $\mathbb{Z}$ , which is simply a stationary two-state (spatial) Markov chain. It is natural to expect the analogous facts to be valid in the continuum limit and the following theorem states that this is indeed the case.

**Theorem 1.5.** *In the special case when  $q = 2$ , for  $\lambda \in (0, \infty)$  and deterministic  $t$ ,  $\Phi(\cdot, t)$  partitions the line into single color intervals whose lengths are independent and exponentially distributed with mean  $\sqrt{2/\lambda}$ . I.e.,  $\Phi(x, t)$  as a random function of the continuous variable  $x$ , for deterministic  $t$ , is the stationary two-state Markov chain with transition rate  $\sqrt{\lambda/2}$  from each state to the other.*

**Remark 1.6.** For  $2 < q < \infty$ , the continuum limit of a nearest neighbor  $q$ -state Potts model on  $\mathbb{Z}$  is a stationary  $q$ -state (spatial) Markov chain  $\Psi(x)$  for  $x \in \mathbb{R}$  with transition rate  $r/(q - 1)$  from each state to any of the other  $q - 1$  states. This process does not appear to agree with the fixed time  $t$  CNVM coloring process  $\Phi(x, t)$  for any  $q > 2$ ; that claim can be verified at least for large  $q$  as follows. To have agreement of the two-point functions, one must take  $r = \sqrt{2\lambda}(q - 1)/q$ , but then it can be shown that as  $\varepsilon \rightarrow 0$ ,

$$\mathbb{P}(\Psi(-\varepsilon) = \Psi(+\varepsilon) \neq \Psi(0)) \approx \frac{r^2}{q(q - 1)^2} \varepsilon^2 = \frac{2\lambda}{q^3} \varepsilon^2, \tag{1.12}$$

while

$$\mathbb{P}(\Phi(-\varepsilon, 0) = \Phi(+\varepsilon, 0) \neq \Phi(0, 0)) \tag{1.13}$$

$$\geq \frac{q - 1}{q} \mathbb{P}(\Theta(-\varepsilon, 0) = \Theta(+\varepsilon, 0) \neq \Theta(0, 0)) \tag{1.14}$$

$$\approx \frac{q - 1}{q} C\lambda\varepsilon^2, \tag{1.15}$$

for a universal constant  $C > 0$ . These two formulas do not agree for sufficiently large  $q$ .

To give even a preliminary construction of the continuum nucleation mapping  $\Theta$  (and hence of the set  $\mathcal{N}$  of nucleation points and their color regions  $C_{(x', t')}$ ), we need to review some of the properties of the Brownian web and its associated dual web, primarily due to Tóth and Werner [24] (see also [7–9]). The BW is a random collection of paths with specified starting points in space–time. For deterministic starting points, there are almost surely unique paths starting from those points and they are distributed as coalescing standard (except for their starting point)

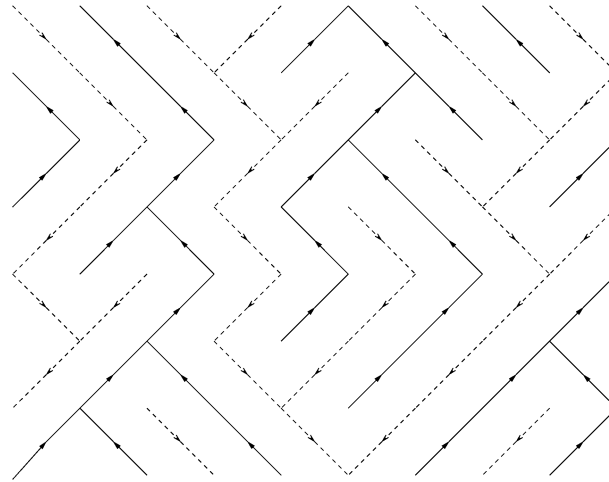


Fig. 1. Forward coalescing random walks (full lines) in discrete time and their dual backward walks (dashed lines).

Brownian motions. But there are random (realization-dependent) double and triple points from which respectively two and three paths start.

Coexisting with the Brownian web paths are backward paths; Fig. 1 shows the corresponding forward and backward walks in discrete space and discrete time (vertical coordinate). The forward and backward Brownian web paths have the following properties [24,8,9]: (1) from a deterministic point  $(x, t)$  with unique forward path  $W_{(x,t)}$ , the backward path is the locus of space–time points separating starting points whose forward path coalesces with  $W_{(x,t)}$  from the left vis-a-vis starting points whose forward path coalesces with  $W_{(x,t)}$  from the right, (2) every point which is *passed through* by a backwards path (i.e., is not merely a starting point for that backwards path) is either a double point or a triple point (for forward paths), and (3) the distribution of all backward paths is exactly that of a time-reversed BW.

Now we can begin to explain the nucleation mapping  $\Theta$ . A natural procedure is to have Poisson processes along paths of the *dual* Brownian web of backward paths, which we for now think of as a BW with paths going backward in time and “reflecting” on the (forward) BW paths. As already mentioned, the paths in the dual BW are the loci of the double points of the forward web. Follow a (Brownian) path in the dual web and mark it according to a Poisson process in time with intensity  $\lambda$ . We do this for every path in the dual BW in such a way that on every path segment the markings have intensity  $\lambda$ . This selects a random countable dense set of double points (of the forward BW). Almost surely, marks will only appear on points *passed through* by a dual path and hence only on (some of the) double points of the forward web. We will give a more precise construction in the next section of the paper.  $\Theta(x, t)$  is now defined by taking the first marked point backwards in time along every path of the dual BW starting from  $(x, t)$ . (In order to satisfy the claim of property (3) in Theorem 1.1 that the color regions are closed, we additionally include  $(x, t)$  itself if it is a nucleation point.) In the remaining sections of the paper, we will explore this construction in more detail.

We conclude this section by briefly turning to the case of non-stationary voter models. In this out of equilibrium setting, a fairly arbitrary assignment of colors to the points on the horizontal line  $\mathbb{R} \times t_0$  (with  $t_0$  and the color assignment deterministic) will lead to a random quasi-coloring of  $\mathbb{R} \times (t_0, \infty)$ . As we have seen, the nucleation points in  $\Theta(x, t)$  are simply the locations of the first marks (going backwards in time) on the one or more paths in the backward BW starting from  $(x, t)$ . In the non-stationary setting, the corresponding  $\Theta_{t_0}(x, t)$  includes those nucleation marked points  $(x', t')$  only if  $t' > t_0$ ; if the first mark  $(x', t')$  along a backward path has  $t' < t_0$  (we leave out  $t' = t_0$  since that is a zero probability event), then  $(x', t')$  is replaced in  $\Theta_{t_0}(x, t)$  by the location  $(x'', t_0)$  where the backward path crosses the horizontal line at  $t_0$ . Whether or not  $q$  is finite, the colors assigned to these  $(x'', t_0)$

points are simply those given by the deterministic initial conditions; the colors assigned to the nucleation points  $(x', t')$  with  $t' > t_0$  are random, as in the stationary setting. We remark that almost surely, only countably many such  $(x'', t_0)$ 's arise as  $(x, t)$  varies over  $\mathbb{R} \times (t_0, \infty)$ .

The rest of the paper is organized as follows. Section 2 reviews some aspects of the BW and then presents in detail the Poisson marking process of the BW. Section 3 gives background (and some new results) about the double BW (i.e., the forward BW jointly with the backward BW). Section 4 presents the proofs of Theorems 1.1, 1.3, and 1.5, including (in Subsection 4.2) providing more details about the color regions  $C_{(x', t')}$  of the CNVM. There is also an appendix, which is used for one part of Section 3 (the analysis of the Hausdorff dimension of the set of type (1, 2) points of the BW).

## 2. Marked Brownian web

In this section we construct the *marked Brownian web* (MBW) as a collection of coalescing marked Brownian paths. In our application to the continuum voter model to be discussed later on, we actually mark the backward (in time) paths.

Before explaining the markings, we review some features of the (unmarked) Brownian web. As in [7,8], we use three metric spaces:  $(\mathbb{R}^2, \rho)$ ,  $(\Pi, d)$  and  $(\mathcal{H}, d_{\mathcal{H}})$ . The elements of the three spaces are respectively: points in space–time, paths with specified starting points in space–time and collections of paths with specified starting points. The BW will be an  $(\mathcal{H}, \mathcal{F}_{\mathcal{H}})$ -valued random variable, where  $\mathcal{F}_{\mathcal{H}}$  is the Borel  $\sigma$ -field associated to the metric  $d_{\mathcal{H}}$ . Complete definitions of the three metric spaces are given at the end of this section. The next theorem, taken from [8], gives some of the key properties of the BW.

**Theorem 2.1.** *There is an  $(\mathcal{H}, \mathcal{F}_{\mathcal{H}})$ -valued random variable  $\bar{W}$  whose distribution is uniquely determined by the following three properties.*

- (o) *from any deterministic point  $(x, t)$  in  $\mathbb{R}^2$ , there is almost surely a unique path  $W_{x,t}$  starting from  $(x, t)$ .*
- (i) *for any deterministic  $n, (x_1, t_1), \dots, (x_n, t_n)$ , the joint distribution of  $W_{x_1, t_1}, \dots, W_{x_n, t_n}$  is that of coalescing Brownian motions (with unit diffusion constant), and*
- (ii) *for any deterministic, dense countable subset  $\mathcal{D}$  of  $\mathbb{R}^2$ , almost surely,  $\bar{W}$  is the closure in  $(\mathcal{H}, d_{\mathcal{H}})$  of  $\{W_{x,t} : (x, t) \in \mathcal{D}\}$ .*

In our marking procedure the only points in the plane that will be marked are those points  $(x, t)$  such that a BW path from some time  $t' < t$  passes through  $(x, t)$ . As previously noted, throughout this section we will be marking the forward BW, but later when we deal with the noisy voter model, we will then work with the marked dual (backward in time) BW.

For each point  $(x, t)$ , we define the *age*  $\tau(x, t)$  of that point as the supremum of the set

$$\{s : \text{there exists a path passing through } (x, t) \text{ from time } t - s\}.$$

All marked points will have *strictly* positive age. We proceed with the presentation of four different but (distributionally) equivalent constructions of the MBW.

### 2.1. Construction via age-truncation

We start by defining the  $\varepsilon$ -age-truncation of the BW for any  $\varepsilon > 0$  as follows. For each realization of the BW, consider the set  $\mathcal{T}_\varepsilon$  of all points  $(x, t)$  in the plane with age  $\tau(x, t) > \varepsilon$ . Next shorten every path in the web by removing (if necessary) the initial segment consisting of those points of age  $\tau \leq \varepsilon$ .  $\mathcal{T}_\varepsilon$  is the union of the graphs of all these age-truncated BW paths and it is almost surely “locally sparse,” in the sense that for every bounded set  $U$ ,

the intersection  $\mathcal{T}_\varepsilon \cap U$  equals the intersection of  $U$  with the union of *finitely* many continuous path segments (which may be chosen to be disjoint). The locally sparse property can be verified as follows: it is known, see [1], that for any  $t$ , the intersection  $\mathcal{T}_\varepsilon \cap (\mathbb{R} \times \{t\})$  is (almost surely) locally finite for all  $\varepsilon > 0$ . By intersecting  $\mathcal{T}_\varepsilon$  with horizontal strips of height (in the time variable)  $\varepsilon/2$ , one sees that there are only locally finitely many paths passing through the strip.

We now mark each (disjoint) path segment in  $\mathcal{T}_\varepsilon$  according to a Poisson process in time with rate  $\lambda$ . Consider now a sequence of  $\varepsilon$ 's decreasing to 0. The marking procedures described above can be carried out for each  $\varepsilon$  and can be coupled in an obvious way so that the marking for the whole sequence of positive  $\varepsilon$ 's can be realized on the same probability space. Taking the union over all positive  $\varepsilon$ 's of these markings gives our first construction of the marked BW.

Given any  $\varepsilon > 0$ , we denote by  $\mathcal{M}_\varepsilon$  the set of all marked points with age greater than  $\varepsilon$ . *Conditional* on the BW realization, and hence on the set  $\mathcal{T}_\varepsilon$  (the “trace” of the  $\varepsilon$ -age-truncated BW),  $\mathcal{M}_\varepsilon$  is a spatial Poisson process on the plane with intensity measure  $\lambda\mu_\varepsilon$ , where  $\mu_\varepsilon$  is the locally finite measure that assigns to each age-truncated path segment in  $\mathcal{T}_\varepsilon$  its  $t$ -coordinate Lebesgue measure. The main drawback of this construction is that for any (bounded) subset  $U$  of the plane with nonempty interior,  $\mu_\varepsilon(U) \rightarrow \infty$  as  $\varepsilon \rightarrow 0$  (this is proved in Subsection 4.2) so that  $\lim_{\varepsilon \rightarrow 0} \mu_\varepsilon$  is unpleasant to deal with as a measure on  $\mathbb{R}^2$ . Our next construction remedies that feature by using the age as a third coordinate.

## 2.2. Construction via 3D embedding

The set  $\mathcal{T}_\varepsilon$  of all  $(x, t)$  with age  $\tau(x, t) > \varepsilon$  is a tree graph embedded continuously in  $\mathbb{R}^2$ . In our second construction, we lift  $\mathcal{T}_\varepsilon$  into  $\mathbb{R}^3$  (or more accurately, into  $\mathbb{R}^2 \times (0, \infty)$ ) so that we may let  $\varepsilon \rightarrow 0$  and still have a locally sparse set. However, the resulting 3D set,

$$\mathcal{T}^3 = \{(x, t, \tau) : \tau = \tau(x, t) > 0\}, \quad (2.1)$$

is no longer a connected tree graph, but rather consists of disconnected segments of curves. The projection of each segment onto the  $(x, t)$ -plane is a segment of a path in the BW. Notice that each segment in 3D ends when its 2D projection coalesces with another segment that has an earlier starting point, so that the age of the point of coalescence is strictly greater than the limit of the age as the segment (that is about to stop) approaches the coalescence point. (This age-based priority rule for stopping or continuing at points of coalescence underlies our next construction.) We remark that it is natural to regard the 3D curve segments as being relatively open, i.e., they do not include either the starting ( $\tau = 0$ ) or ending point.

We may now define a measure  $\mu^3$  on  $\mathbb{R}^2 \times (0, \infty)$  which is supported on  $\mathcal{T}^3$  and assigns to each curve segment its  $t$ -coordinate Lebesgue measure. We also define  $\mathcal{M}^3$  as the spatial Poisson process on  $\mathbb{R}^2 \times (0, \infty)$  whose intensity measure is  $\lambda\mu^3$ . Note that  $\mu^3(A) < \infty$  (almost surely with respect to the realization of the BW) for any bounded *closed*  $A$  contained in  $\mathbb{R}^2 \times (0, \infty)$ . The projection of  $\mathcal{M}^3$  onto the  $(x, t)$ -plane is our random collection of marked points which is a.s. countable and dense in  $\mathbb{R}^2$ . Note that every point  $(x', t')$  in  $\mathcal{M}^3$  has  $\tau(x, t) > 0$ .

In our next construction, we explain how the 2D projection of the connected components of  $\mathcal{T}^3$  may be defined directly in  $\mathbb{R}^2$  without recourse to a 3D embedding.

## 2.3. Construction via tip-path correspondence

To follow the construction we are about to present, some working knowledge of the double (forward jointly with dual backward) BW is needed; this may be obtained by first skimming Section 3 and Subsection 4.2. We are going to mark the paths of the forward BW, taking advantage of the backward BW, using properties that hold almost surely.

Each coalescence (type  $(2, 1)$ ) point of the *forward* web is the starting tip of two *backward* bubbles (with disjoint interiors)—see Subsection 4.2 and Fig. 2. For each such point, associate the subpath of the forward web



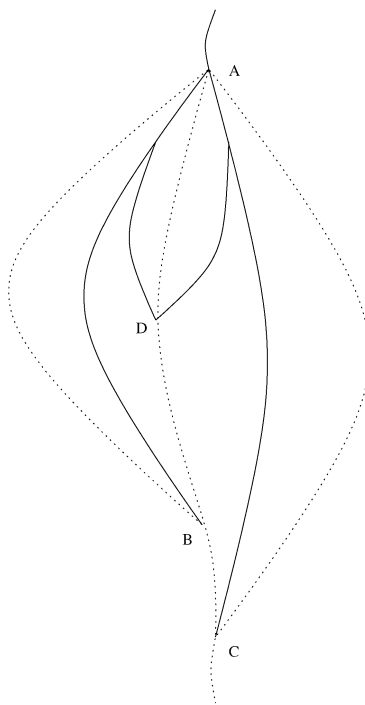


Fig. 2. Forward (upward) paths are in full lines; backward (downward) ones are dotted. A is a coalescence point of the forward web, and a starting tip of the backward bubbles formed by the four (overlapping) backward path segments, namely two from A to B, and two from A to C. The forward path associated to A is the forward one from B to A. D is a double point of the forward web, and a starting tip of a forward bubble (see Section 4.2), for which A is an ending tip.

starting at the highest ending tip of the two ending tips (one for each of the two backward bubbles), staying within the respective bubble, and ending at the coalescence point/starting tip of the respective bubble—see Fig. 2. To be consistent with the 3D embedding construction, this subpath should be taken relatively open at both ends.

This one-to-one association of subpaths to coalescence points yields a countable family (because the set of all coalescence points (of the forward web) is countable) of disjoint BW path segments. Every subpath is the initial segment of a path of the BW belonging to the countable family of all the middle paths starting at all the triple (type  $(0, 3)$ ) points (each ending tip of a backward bubble is a triple point, and the chosen subpath starting there, since it is required to stay within the bubble, is an initial segment of the middle path from that triple point). Every triple point will play this role with the initial segment of its middle path ending when that path coalesces at a point of larger age. The marking may now be done by using independent rate- $\lambda$  Poisson processes in time, one for each of the countably many segments.

The three constructions we have presented thus far all use the notion of the age  $\tau(x, t)$  either explicitly, or implicitly in the tip-path correspondence construction where an age-based precedence relation between coalescing paths of the BW determines which segment continues past the coalescence point.

In the tip-path correspondence construction, one chooses a particular “skeleton” (as in (ii) of Theorem 2.1) in which the initial points of the skeleton are not from a *deterministic* dense countable set  $\mathcal{D}$ , but rather are the triple points of the BW realization. We proceed to present a construction in which one can use any deterministic  $\mathcal{D}$ . The main awkward feature of that construction is that it is not a priori clear that the resulting MBW has a distribution not depending on the choice of  $\mathcal{D}$ .

### 2.4. Sequential construction

This construction begins with independent Brownian paths starting from any deterministic dense countable subset  $\mathcal{D}$  of  $\mathbb{R}^2$ . Mark each Brownian path with marks from a Poisson process of rate  $\lambda$ . One way to do this marking is to consider the set of Brownian paths  $\mathcal{W} := \{W_{x,t}, (x, t) \in \mathcal{D}\}$ , where  $W_{x,t}$  denotes the path starting at  $(x, t)$ , and an independent i.i.d. family  $\mathcal{N} := \{N_{x,t}, (x, t) \in \mathcal{D}\}$  of Poisson process (in the time coordinate) of rate  $\lambda$ . Now mark the path  $W_{x,t} = (f(s), s)_{s \geq t}$  at the points  $(f(S_i), S_i)_{i \geq 1}$ , where  $S_1, S_2, \dots$  are the successive event times of  $N_{x,t}$  after  $t$ . Let us denote the marked path thus obtained  $W_{x,t}^*$  and the set of marked paths  $\mathcal{W}^* := \{W_{x,t}^*, (x, t) \in \mathcal{D}\}$ . Now introduce the set of coalescing marked paths  $\tilde{\mathcal{W}}^* := \{\tilde{W}_{x,t}^*, (x, t) \in \mathcal{D}\}$ , as in [8], by imposing a precedence relation on the set of marked paths (note that this is *not* the precedence relation based on age used previously but a simpler one just based on some initial deterministic ordering of  $\mathcal{D}$ . The first coalescing marked path of  $\tilde{\mathcal{W}}^*$  is the first marked path of  $\mathcal{W}^*$ . The  $(n + 1)$ -st coalescing marked path of  $\tilde{\mathcal{W}}^*$  is formed first with the portion of the  $(n + 1)$ -st marked path of  $\mathcal{W}^*$  until it first hits any of the  $n$  first coalescing marked paths of  $\tilde{\mathcal{W}}^*$ ; from then on, it follows that marked path (the one it has first hit).

The Brownian web  $\tilde{\mathcal{W}}$  (as in Theorem 2.1 (ii)) is the closure of the paths in  $\tilde{\mathcal{W}}^*$ . It is important to note however, that *there are no new marks in  $\tilde{\mathcal{W}}$  beyond those already in the marked skeleton  $\tilde{\mathcal{W}}^*$ .*

**Remark 2.2.** When this procedure is used for marking the dual BW then each mark is a double point of the forward web which is a starting point of a bubble. The total time that bubble exists from its initial to its final point is identical to the age of the dual web marked point that coincides with the forward web double point (i.e., the age in the dual web equals the bubble lifetime in the forward web).

We end this section with the precise definition of our three metric spaces.  $(\overline{\mathbb{R}^2}, \rho)$  is the completion (or compactification) of  $\mathbb{R}^2$  under the metric  $\rho$ , where

$$\rho((x_1, t_1), (x_2, t_2)) = \left| \frac{\tanh(x_1)}{1 + |t_1|} - \frac{\tanh(x_2)}{1 + |t_2|} \right| \vee |\tanh(t_1) - \tanh(t_2)|. \tag{2.2}$$

$\overline{\mathbb{R}^2}$  may be thought as the image of  $[-\infty, \infty] \times [-\infty, \infty]$  under the mapping

$$(x, t) \rightsquigarrow (\Phi(x, t), \Psi(t)) \equiv \left( \frac{\tanh(x)}{1 + |t|}, \tanh(t) \right). \tag{2.3}$$

For  $t_0 \in [-\infty, \infty]$ , let  $C[t_0]$  denote the set of functions  $f$  from  $[t_0, \infty]$  to  $[-\infty, \infty]$  such that  $\Phi(f(t), t)$  is continuous. Then define

$$\Pi = \bigcup_{t_0 \in [-\infty, \infty]} C[t_0] \times \{t_0\}, \tag{2.4}$$

where  $(f, t_0) \in \Pi$  represents a path in  $\overline{\mathbb{R}^2}$  starting at  $(f(t_0), t_0)$ . For  $(f, t_0)$  in  $\Pi$ , we denote by  $\hat{f}$  the function that extends  $f$  to all  $[-\infty, \infty]$  by setting it equal to  $f(t_0)$  for  $t < t_0$ . Then we take

$$d((f_1, t_1), (f_2, t_2)) = \left( \sup_t |\Phi(\hat{f}_1(t), t) - \Phi(\hat{f}_2(t), t)| \right) \vee |\Psi(t_1) - \Psi(t_2)|. \tag{2.5}$$

$(\Pi, d)$  is a complete separable metric space.

Let now  $\mathcal{H}$  denote the set of compact subsets of  $(\Pi, d)$ , with  $d_{\mathcal{H}}$  the induced Hausdorff metric, i.e.,

$$d_{\mathcal{H}}(K_1, K_2) = \sup_{g_1 \in K_1} \inf_{g_2 \in K_2} d(g_1, g_2) \vee \sup_{g_2 \in K_2} \inf_{g_1 \in K_1} d(g_1, g_2). \tag{2.6}$$

$(\mathcal{H}, d_{\mathcal{H}})$  is also a complete separable metric space.

Our description of the continuum noisy voter model, of which the continuum stochastic Ising model is a particular case, will involve apart from the MBW, also the dual web to the unmarked BW. In the next section, we describe the dual BW and the joint object, the double BW.

### 3. Dual and double Brownian webs

In this section, we construct and characterize the *double Brownian web*, which combines the Brownian web with a *dual Brownian web* of coalescing Brownian motions moving backwards in time.

In the graphical representation of Harris for the one-dimensional voter model [16], coalescing random walks forward in time and coalescing dual random walks backward in time (with forward and backward walks not crossing each other) are constructed simultaneously (see, e.g., the discussion in [10,11]). Fig. 1 provides an example in discrete time. Note that there is no crossing between forward and backward walks — a property that holds also for the *double Brownian web* (DBW), which can be seen as their scaling limit. The simultaneous construction of forward and (dual) backward Brownian motions was emphasized in [24,22] and their approach and results can be applied to extend both the characterization and convergence results of [8] to the DBW which includes simultaneously the forward BW and its dual backward BW.

Our construction and analysis of its properties will rely on a paper [22] of Soucaliuc, Tóth and Werner together with results from [7,8] on the (forward) Brownian web (see also [9]).

We begin with an (ordered) dense countable set  $\mathcal{D} \subset \mathbb{R}^2$ , and a family of i.i.d. standard B.M.’s  $B_1, B_1^b, B_2, B_2^b, \dots$  and construct forward and backward paths  $W_1, W_1^b, W_2, W_2^b, \dots$  starting from  $(x_j, t_j) \in \mathcal{D}$ :

$$W_j(t) = x_j + B_j(t - t_j), \quad t \geq t_j, \tag{3.1}$$

$$W_j^b(t) = x_j + B_j^b(t_j - t), \quad t \leq t_j. \tag{3.2}$$

Then we construct coalescing and “reflecting” paths  $\tilde{W}_1, \tilde{W}_1^b, \dots$  inductively, as follows.

$$\tilde{W}_1 = W_1; \quad \tilde{W}_1^b = W_1^b; \tag{3.3}$$

$$\tilde{W}_n = CR(W_n; \tilde{W}_1, \tilde{W}_1^b, \dots, \tilde{W}_{n-1}, \tilde{W}_{n-1}^b); \tag{3.4}$$

$$\tilde{W}_n^b = CR(W_n^b; \tilde{W}_1, \tilde{W}_1^b, \dots, \tilde{W}_{n-1}, \tilde{W}_{n-1}^b), \tag{3.5}$$

where the operation  $CR$  is defined in [22], Subsubsection 3.1.4. We proceed to explain  $CR$  for the simplest case, in the definition of  $\tilde{W}_2$ .

As pointed out in [22], the nature of the reflection of a forward Brownian path  $\tilde{W}$  off a backward Brownian path  $\tilde{W}^b$  (or vice-versa) is special. It is actually better described as a push of  $\tilde{W}$  off  $\tilde{W}^b$  (see Subsection 2.1 in [22]). It does not have an explicit formula in general, but in the case of one forward path and one backward path, the form is as follows. Following our notation and construction, we ignore  $\tilde{W}_1$  and consider  $\tilde{W}_1^b$  and  $\tilde{W}_2$  in the time interval  $[t_2, t_1]$  (we suppose  $t_2 < t_1$ ; otherwise,  $\tilde{W}_1^b$  and  $\tilde{W}_2$  are independent). Given  $W_2$  and  $\tilde{W}_1^b$ , for  $t_2 \leq t \leq t_1$ ,

$$\tilde{W}_2(t) = \begin{cases} W_2(t) + \sup_{t_2 \leq s \leq t} (W_2(s) - \tilde{W}_1^b(s))^- , & \text{if } W_2(t_2) > \tilde{W}_1^b(t_2); \\ W_2(t) - \sup_{t_2 \leq s \leq t} (W_2(s) - \tilde{W}_1^b(s))^+ , & \text{if } W_2(t_2) < \tilde{W}_1^b(t_2). \end{cases} \tag{3.6}$$

After  $t_1$ ,  $\tilde{W}_2$  interacts only with  $\tilde{W}_1$ , by coalescence.

We call  $\mathcal{W}_n^D := \{\tilde{W}_1, \tilde{W}_1^b, \dots, \tilde{W}_n, \tilde{W}_n^b\}$  *coalescing/reflecting forward and backward Brownian motions (starting at  $\{(x_1, t_1), \dots, (x_n, t_n)\}$ )*. We will also use the alternative notation  $\mathcal{W}^D(\mathcal{D}_n)$  in place of  $\mathcal{W}_n^D$ , where  $\mathcal{D}_n := \{(x_1, t_1), \dots, (x_n, t_n)\}$ .

**Remark 3.1.** In Theorem 8 of [22], it is proved that the above construction is a.s. well-defined, gives a perfectly coalescing/reflecting system (see Subsubsection 3.1.1 in [22]), and for every  $n \geq 1$ , the distribution of  $\mathcal{W}_n^D$  does not depend on the ordering of  $\mathcal{D}_n$ . It also follows from that result that  $\{\tilde{W}_1, \dots, \tilde{W}_n\}$  and  $\{\tilde{W}_1^b, \dots, \tilde{W}_n^b\}$  are separately forward and backward coalescing Brownian motions, respectively. Thus  $\{\tilde{W}_1, \tilde{W}_2, \dots\}$  and  $\{\tilde{W}_1^b, \tilde{W}_2^b, \dots\}$  are forward and backward Brownian web skeletons, respectively.

**Remark 3.2.** One can alternatively use a set  $\mathcal{D}^b$  of starting points for the backward paths different than  $\mathcal{D}$  rather than our choice above of  $\mathcal{D}^b = \mathcal{D}$ .

We now define dual spaces of paths going backward in time  $(\Pi^b, d^b)$  and a corresponding  $(\mathcal{H}^b, d_{\mathcal{H}^b})$  in an obvious way, so that they are the dual versions of  $(\Pi, d)$  and  $(\mathcal{H}, d_{\mathcal{H}})$ , and then define  $\mathcal{H}^D = \mathcal{H} \times \mathcal{H}^b$  and

$$d_{\mathcal{H}^D}((K_1, K_1^b), (K_2, K_2^b)) = \max(d_{\mathcal{H}}(K_1, K_2), d_{\mathcal{H}^b}(K_1^b, K_2^b)).$$

As in the construction of the (forward) BW, we now define

$$\mathcal{W}_n^D(\mathcal{D}) = \{\tilde{W}_1, \dots, \tilde{W}_n\} \times \{\tilde{W}_1^b, \dots, \tilde{W}_n^b\}, \tag{3.7}$$

$$\mathcal{W}^D(\mathcal{D}) = \{\tilde{W}_1, \tilde{W}_2, \dots\} \times \{\tilde{W}_1^b, \tilde{W}_2^b, \dots\}, \tag{3.8}$$

$$\overline{\mathcal{W}^D}(\mathcal{D}) = \overline{\{\tilde{W}_1, \tilde{W}_2, \dots\}} \times \overline{\{\tilde{W}_1^b, \tilde{W}_2^b, \dots\}}. \tag{3.9}$$

The latter closures are in  $\Pi$  for the first factor and in  $\Pi^b$  for the second one.

From Remark 3.1, we have that

$$\overline{\mathcal{W}} := \overline{\{\tilde{W}_1, \tilde{W}_2, \dots\}} \quad \text{and} \quad \overline{\mathcal{W}^b} := \overline{\{\tilde{W}_1^b, \tilde{W}_2^b, \dots\}}$$

are forward and backward Brownian webs, respectively. We proceed to state three propositions and one theorem; their proofs follow directly from the results and methods of [22,7,8]

**Proposition 3.3.** *Almost surely,  $\overline{\mathcal{W}^D}(\mathcal{D}) \in \mathcal{H}^D$  (i.e.  $\overline{\{\tilde{W}_1, \tilde{W}_2, \dots\}}$  and  $\overline{\{\tilde{W}_1^b, \tilde{W}_2^b, \dots\}}$  are compact).*

**Remark 3.4.** It is immediate from this proposition that

$$\overline{\mathcal{W}^D}(\mathcal{D}) = \lim_{n \rightarrow \infty} \mathcal{W}_n^D(\mathcal{D}),$$

where the limit is in the  $d_{\mathcal{H}^D}$  metric.

**Proposition 3.5.**  $\overline{\mathcal{W}^D}(\mathcal{D})$  satisfies

- (o<sup>D</sup>) *From any deterministic  $(x, t)$  there is almost surely a unique forward path and unique backward path.*
- (i<sup>D</sup>) *For any deterministic  $\mathcal{D}'_n := \{(y_1, s_1), \dots, (y_n, s_n)\}$  the forward and backward paths from  $\mathcal{D}'_n$ , denoted  $\overline{\mathcal{W}^D}(\mathcal{D}, \mathcal{D}'_n)$ , are distributed as coalescing/reflecting forward and backward Brownian motions starting at  $\mathcal{D}'_n$ . In other words,  $\overline{\mathcal{W}^D}(\mathcal{D}, \mathcal{D}'_n)$  has the same distribution as  $\mathcal{W}^D(\mathcal{D}'_n)$ .*

**Proposition 3.6.** *The distribution of  $\overline{\mathcal{W}^D}(\mathcal{D})$  as an  $(\mathcal{H}^D, \mathcal{F}_{\mathcal{H}^D})$ -valued random variable (where  $\mathcal{F}_{\mathcal{H}^D} = \mathcal{F}_{\mathcal{H}} \times \mathcal{F}_{\mathcal{H}^b}$ ), does not depend on  $\mathcal{D}$ . Furthermore,*

(ii<sup>D</sup>) *for any deterministic dense  $\mathcal{D}'$ , almost surely*

$$\overline{\mathcal{W}^D}(\mathcal{D}) = \overline{\{W_{x,t}: (x, t) \in \mathcal{D}'\}} \times \overline{\{W_{x,t}^b: (x, t) \in \mathcal{D}'\}},$$

where  $W_{x,t}, W_{x,t}^b$  are respectively the forward and backward paths in  $\overline{\mathcal{W}^D}(\mathcal{D})$  starting from  $(x, t)$ , and the closures in (ii<sup>D</sup>) are in  $\Pi$  for the first factor and in  $\Pi^b$  for the second one.

**Theorem 3.7.** *The double Brownian web is characterized (in distribution, on  $(\mathcal{H}^D, \mathcal{F}_{\mathcal{H}^D})$ ) by conditions (o<sup>D</sup>), (i<sup>D</sup>) and (ii<sup>D</sup>).*

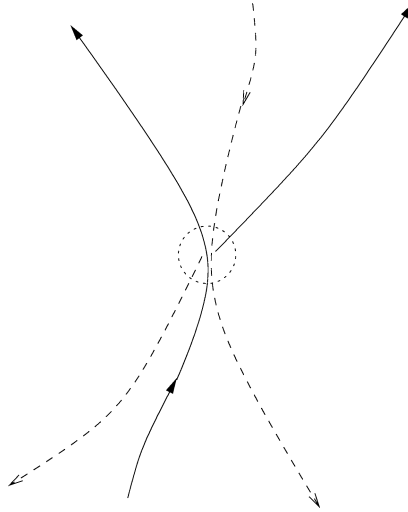


Fig. 3. A schematic diagram of a point  $(x_0, t_0)$  of type  $(m_{\text{in}}, m_{\text{out}}) = (1, 2)$ , with necessarily also  $(m_{\text{in}}^b, m_{\text{out}}^b) = (1, 2)$ . In this example the incoming forward path connects to the leftmost outgoing path (with a corresponding dual connectivity for the backward paths); at some of the other points of type  $(1, 2)$  it will connect to the rightmost path.

We now discuss “types” of points  $(x, t) \in \mathbb{R}^2$ , whether deterministic or not. For the (forward) Brownian web, we define

$$m_{\text{in}}(x_0, t_0) = \lim_{\varepsilon \downarrow 0} \{\text{number of paths in } \mathcal{W} \text{ starting at some } t_0 - \varepsilon \text{ that pass through } (x_0, t_0) \text{ and are disjoint for } t_0 - \varepsilon < t < t_0\}; \tag{3.10}$$

$$m_{\text{out}}(x_0, t_0) = \lim_{\varepsilon \downarrow 0} \{\text{number of paths in } \mathcal{W} \text{ starting at } (x_0, t_0) \text{ that are disjoint for } t_0 < t < t_0 + \varepsilon\}. \tag{3.11}$$

For  $\mathcal{W}^b$ , we similarly define  $m_{\text{in}}^b(x_0, t_0)$  and  $m_{\text{out}}^b(x_0, t_0)$ .

**Definition 3.8.** The type of  $(x_0, t_0)$  is the pair  $(m_{\text{in}}, m_{\text{out}})$ —see Fig. 3. We denote by  $S_{i,j}$  the set of points of  $\mathbb{R}^2$  that are of type  $(i, j)$ , and by  $\bar{S}_{i,j}$  the set of points of  $\mathbb{R}^2$  that are of type  $(k, l)$  with  $k \geq i, l \geq j$ .

**Remark 3.9.** Using the translation and scale invariance properties of the Brownian web distribution, it can be shown that for any  $i, j$ , whenever  $S_{i,j}$  is nonempty, it must be dense in  $\mathbb{R}^2$ . The same can be said of  $S_{i,j} \cap \mathbb{R} \times \{t\}$  for deterministic  $t$ . These denseness properties can also be shown for each  $i, j$  by more direct arguments.

**Proposition 3.10.** For the double Brownian web, almost surely for every  $(x_0, t_0)$  in  $\mathbb{R}^2$ ,  $m_{\text{in}}^b(x_0, t_0) = m_{\text{out}}(x_0, t_0) - 1$  and  $m_{\text{out}}^b(x_0, t_0) = m_{\text{in}}(x_0, t_0) + 1$ . See Fig. 3.

**Proof.** It is enough to prove (i) that for every incoming forward path to a point  $(x, t)$ , there are two locally disjoint backward paths starting at that point with one on either side of the forward path; and (ii) that for every two locally disjoint backward paths starting at a point  $(x, t)$ , there is an incoming forward path to  $(x, t)$  between the two backward paths. (Note that by a  $t \leftrightarrow -t$  time reflection argument, one would then get a similar result for incoming backward paths and pairs of outgoing forward paths.)

Let us start with the first assertion. Let  $\gamma$  be an incoming forward path to  $(x, t)$ . This means that the starting time  $s$  of  $\gamma$  is such that  $s < t$ . By Proposition 4.3 of [8], the portion of  $\gamma$  above time  $s + \varepsilon$  is in the forward skeleton for every  $\varepsilon > 0$ . Now consider a sequence of pairs of backward paths  $(\gamma_k, \gamma'_k)$  starting at  $((x_k, t_k), (y_k, s_k)) \in \mathcal{D} \times \mathcal{D}$  with  $((x_k, t_k), (y_k, s_k)) \rightarrow ((x, t), (x, t))$  as  $k \rightarrow \infty$ ,  $s + \varepsilon < s_k, t_k < t, x_k < \gamma(t_k)$  and  $y_k > \gamma(s_k)$ . From the reflection of the forward and backward skeletons off each other and the fact that two backward paths in the skeleton must coalesce once they meet, it follows that  $\gamma_k(t') < \gamma'_k(t')$  for all  $t' \in [\max\{s_k, t_k\}, t']$ . We then conclude from compactness that there are two locally disjoint limit paths, one for  $(\gamma_k)$  and one for  $(\gamma'_k)$ , both starting from  $(x, t)$ .

We argue (ii) similarly. Given two locally disjoint backward paths  $\gamma, \gamma'$  starting at  $(x, t)$ , there exists  $s < t$  such that either  $\gamma(t') < \gamma'(t')$  for  $s < t' < t$  or  $\gamma'(t') < \gamma(t')$  for  $s < t' < t$ . Suppose it is the first case; otherwise, switch labels. Then choose a point  $(x', s') \in \mathcal{D}$  with  $s < s' < t$  and  $\gamma(s') < x' < \gamma'(s')$ . The fact that the portions of  $\gamma$  and  $\gamma'$  below time  $t - \varepsilon$  is in the backward skeleton for every  $\varepsilon > 0$  and the reflection of the forward and backward skeletons off each other now implies that the forward path starting at  $(x', s')$  is squeezed between  $\gamma$  and  $\gamma'$  and goes to  $(x, t)$ .  $\square$

**Theorem 3.11.** *For the (double) Brownian web, almost surely, every  $(x, t)$  has one of the following types, all of which occur:  $(0, 1), (0, 2), (0, 3), (1, 1), (1, 2), (2, 1)$ .*

**Remark 3.12.** Points of type  $(1, 2)$  are particularly interesting in that the single incident path continues along exactly one of the two outward paths—with the choice determined intrinsically rather than by some convention. See Fig. 3 for a schematic diagram of a “left-handed” continuation. An  $(x_0, t_0)$  is of type  $(1, 2)$  precisely if both a forward and a backward path pass through  $(x_0, t_0)$ . It is either left-handed or right-handed according to whether the forward path is to the left or the right of the backward path near  $(x_0, t_0)$ . Both varieties occur and the proof of Theorem 3.13 below shows that the Hausdorff dimension of 1 applies separately to each of the two varieties.

Tóth and Werner [24] gave a definition of types of points of  $\mathbb{R}^2$  similar to ours, but for a somewhat different process and proved the above theorem with that definition and for that process (see definition at page 385, paragraph of Eq. (2.28) and Proposition 2.4 in [24]). One way then to establish Theorem 3.11 is to show the equivalence of ours and Tóth and Werner’s definition and that their arguments hold for our process. We prefer, for the sake of simplicity and completeness, to give a direct argument, out of which the following complementary results also follow.

**Theorem 3.13.** *Almost surely,  $S_{0,1}$  has full Lebesgue measure in  $\mathbb{R}^2$ ,  $S_{1,1}$  and  $S_{0,2}$  have Hausdorff dimension  $3/2$  each,  $S_{1,2}$  has Hausdorff dimension 1, and  $S_{2,1}$  and  $S_{0,3}$  are both countable and dense in  $\mathbb{R}^2$ .*

**Theorem 3.14.** *Almost surely: for every  $t$*

- (a)  $S_{0,1} \cap \mathbb{R} \times \{t\}$  has full Lebesgue measure in  $\mathbb{R} \times \{t\}$ ;
- (b)  $S_{1,1} \cap \mathbb{R} \times \{t\}$  and  $S_{0,2} \cap \mathbb{R} \times \{t\}$  are both countable and dense in  $\mathbb{R} \times \{t\}$ ;
- (c)  $S_{1,2} \cap \mathbb{R} \times \{t\}, S_{2,1} \cap \mathbb{R} \times \{t\}$  and  $S_{0,3} \cap \mathbb{R} \times \{t\}$  have all cardinality at most 1.

*For every deterministic  $t, S_{1,2} \cap \mathbb{R} \times \{t\}, S_{2,1} \cap \mathbb{R} \times \{t\}$  and  $S_{0,3} \cap \mathbb{R} \times \{t\}$  are almost surely empty.*

**Proof of Theorems 3.11 and 3.13.** We start by ruling out the cases that do not occur almost surely. For  $i, j \geq 0, S_{i,j} = \emptyset$  almost surely if  $j = 0$  or  $i + j \geq 4$ . The first case is trivial. We only need to consider  $\bar{S}_{i,j}$  for the cases  $i = 3, j = 1$  and  $i = 2, j = 2$ , since the other ones are either contained or dual to these. By Proposition 4.3 of [8],  $\bar{S}_{3,1}$  consists of points which are almost surely in the skeleton and where three paths coalesce. But the event that three coalescing Brownian paths starting at distinct points coalesce at the same time is almost surely empty. By Proposition 4.3 of [8],  $\bar{S}_{2,2}$  consists of points (almost surely in the double skeleton) where two different forward

paths coalesce and a backward path passes. Since for any two forward and one backward Brownian paths in the double skeleton, the event that this happens is almost surely empty, by the perfectly coalescing/reflecting property of the paths in the double skeleton (see Subsubsection 3.1.1 and Theorem 8 of [22]) the conclusion follows.

Now, for the types that do occur.

*Type (2, 1).* By the above,  $S_{2,1} = \bar{S}_{2,1}$  almost surely, and  $\bar{S}_{2,1}$  consists almost surely of *points of coalescence*, that is all points where two paths coalesce. By Proposition 4.3 of [8], it is almost surely a subset of the skeleton, and thus is countable (since there is at most one coalescence point for each pair of paths starting from  $\mathcal{D}$  in the skeleton). It is easy to see that it is dense since the paths from a pair of nearby points in  $\mathcal{D}$  also coalesce nearby with probability close to one.

*Type (1, 2).* By the above,  $S_{1,2} = \bar{S}_{1,2}$  almost surely, and  $\bar{S}_{1,2}$  consists almost surely of points where forward paths meet backward paths. Thus, it is a subset of the (union of the traces of all the paths in the) skeleton. It is easy to see that it is almost surely nonempty (and also dense). We need only consider two such paths, say  $W$  and  $W^b$ , the former a forward one starting at  $(0, 0)$  (without loss of generality, by the translation invariance of the law of  $\mathcal{W}^D$ ), and the latter a backward one starting at an arbitrary deterministic  $(x_0, t_0)$ , with  $t_0 > 0$  to avoid a trivial case. It is clear that the random set  $\Lambda$  of space–time points  $(t, W(t))$  for times  $t \in [0, t_0]$  when  $W(t) = W^b(t)$  has a positive, less than one probability of being empty. We will argue next the following claim.

**Claim.**  *$\Lambda$  has Hausdorff dimension 1 for almost every pair of trajectories  $(W, W^b)$  for which it is nonempty.*

By Proposition 3.5, the distribution of  $\{(W(t), W^b(t)) : 0 \leq t \leq t_0\}$  (which is all that matters for this) can be described in terms of two (forward) independent standard Brownian motions  $B, B^b$  as follows (see Eqs. (3.1)–(3.6)). Let  $W^b(t) = x_0 + B^b(t_0 - t)$ ,  $t \leq t_0$ , and  $\tau = \inf\{t \in [0, t_0] : B(t) = W^b(t)\}$ , with  $\inf \emptyset = \infty$ . If  $\tau = \infty$ , then  $W = B$ ; otherwise,  $W(t) = B(t)$  for  $0 \leq t \leq \tau$ , and for  $\tau \leq t \leq t_0$ ,

$$W(t) = \begin{cases} B(t) + \sup_{0 \leq s \leq t} (W^b(s) - B(s)), & \text{if } W^b(0) < 0; \\ B(t) - \sup_{0 \leq s \leq t} (B(s) - W^b(s)), & \text{if } W^b(0) > 0. \end{cases}$$

Rewriting in terms of  $W'(t) := W^b(t) - W^b(0)$ ,  $0 \leq t \leq t_0$ , which is a standard Brownian motion independent of  $B$ , we have (for  $0 \leq t \leq t_0$ )

$$W(t) = \begin{cases} B(t) + \sup_{0 \leq s \leq t} \{W'(s) - B(s)\} - W'(t_0) + x_0, & \text{if } W'(t_0) > x_0; \\ B(t) + \inf_{0 \leq s \leq t} \{W'(s) - B(s)\} - W'(t_0) + x_0, & \text{if } W'(t_0) < x_0, \end{cases}$$

if  $\tau \leq t \leq t_0$ , with  $\tau = \inf\{t \in [0, t_0] : B(t) = W'(t) - W'(t_0) + x_0\}$ ; otherwise,  $W(t) = B(t)$ .

From the above discussion, we conclude that  $\Lambda$  has the same distribution as the random set  $\mathcal{G}$  obtained as follows. Let  $\mathcal{T}^+$  and  $\mathcal{T}^-$  be the sets of positive and negative record times of the standard Brownian motion  $X(t) := (W'(t) - B(t))/\sqrt{2}$ , respectively, i.e.,  $\mathcal{T}^+$  is the set of  $t \geq 0$  such that  $X(t) = \sup_{0 \leq s \leq t} X(s)$  and  $\mathcal{T}^-$  is the same except with  $\inf$  in place of  $\sup$ . Consider also the standard Brownian motion  $Y(t) := (W'(t) + B(t))/\sqrt{2}$ , which is independent of  $X$ . If  $W'(t_0) > x_0$ , then  $\mathcal{G} = \{[(X(t) + Y(t))/\sqrt{2}] - [(X(t_0) + Y(t_0))/\sqrt{2}] + x_0, t) : t \in \mathcal{T}^+ \cap [\tau, t_0]\}$ ; if  $W'(t_0) < x_0$ , then  $\mathcal{G} = \{[(X(t) + Y(t))/\sqrt{2}] - [(X(t_0) + Y(t_0))/\sqrt{2}] + x_0, t) : t \in \mathcal{T}^- \cap [\tau, t_0]\}$ .

It follows from Proposition A.1 in Appendix A that the sets  $\mathcal{G}^\pm := \{(X(t) + Y(t), t) : t \in \mathcal{T}^\pm \cap [0, t_0]\}$  (one for each sign, respectively) both have Hausdorff dimension 1 almost surely. Since the events  $\{W'(t_0) > x_0\}$ ,  $\{W'(t_0) < x_0\}$  and  $\{\tau < t_0\}$  all have positive probability, the claim follows.

*Type (1, 1).*  $\bar{S}_{1,1}$  almost surely consists of *points of continuation* of paths, that is, all points  $(x, t)$  such that there is a path starting earlier than  $t$  that touches  $(x, t)$ . By Proposition 4.3 of [8],  $\bar{S}_{1,1}$  is almost surely a subset of the skeleton. Since the trace of any single path has Hausdorff dimension  $3/2$  [23] and the countable union of such sets has the same dimension, it follows that  $\bar{S}_{1,1}$  has Hausdorff dimension  $3/2$  almost surely. By the previous parts of the proof,  $\bar{S}_{1,1} \setminus S_{1,1}$  has lower dimension and so  $S_{1,1}$  has the same Hausdorff dimension of  $3/2$ .

*Type (0, 1).* We claim that any deterministic point is a.s. of this type, hence (by applying Fubini’s Theorem)  $S_{0,1}$  is a.s. of full Lebesgue measure in the plane. That  $m_{\text{in}}(x_0, t_0) = 0$  a.s. for every deterministic  $(x_0, t_0)$  follows from

Proposition 4.3 of [8], since, if  $m_{\text{in}}(x_0, t_0) \geq 1$ , then there would be a path in the skeleton passing through  $(x_0, t_0)$ , but this event clearly has probability zero. The assertion that  $m_{\text{out}}(x_0, t_0) = 1$  a.s. for every deterministic  $(x_0, t_0)$  is property (o) of Theorem 2.1.

By Proposition 3.10, the remaining types (0, 2) and (0, 3) are dual respectively to (1, 1) and (2, 1), since the other types are dual to these. Since  $\bar{\mathcal{W}}^b$  is distributed like the standard Brownian web (modulo a time reflection), the claimed results for types (0, 2) and (0, 3) follow from those already proved for (1, 1) and (2, 1).  $\square$

### Proof of Theorem 3.14.

*Type (0, 1).*  $\bar{S}_{1,1}$  is almost surely in the skeleton, thus making  $\bar{S}_{1,1} \cap \mathbb{R} \times \{t\}$  countable for all  $t$ . By a duality argument, the same is true for  $\bar{S}_{0,2}$ . Since  $\bar{S}_{0,1} = \mathbb{R}^2$  a.s. by Theorem 3.11, it follows that a.s. for all  $t$ ,  $S_{0,1} \cap \mathbb{R} \times \{t\}$  is of full Lebesgue measure in the line.

Again, of the remaining types, it is enough by duality to consider (1, 1), (2, 1) and (1, 2).

*Type (2, 1).* For any deterministic  $t$  and  $(x_i, t_i)$  with  $t_i < t$ ,  $i = 1, 2$ , the probability that two independent Brownian paths starting at  $(x_i, t_i)$ ,  $i = 1, 2$ , respectively, coalesce exactly at time  $t$  is zero. Since  $S_{2,1}$  is in the skeleton,  $S_{2,1} \cap \mathbb{R} \times \{t\} = \emptyset$  almost surely. Now, for any  $t$ ,  $|S_{2,1} \cap \mathbb{R} \times \{t\}| > 1$  implies that there are four independent Brownian paths starting at different points, and such that the coalescence time of the first two and that of the last two are the same. That this has zero probability implies that a.s. for all  $t$ ,  $|S_{2,1} \cap \mathbb{R} \times \{t\}| \leq 1$ .

*Type (1, 2).* For any deterministic  $t$ ,  $S_{1,2} \cap \mathbb{R} \times \{t\} = \emptyset$  almost surely, since the probability of two fixed paths, one forward, one backward, meeting at a given deterministic time is 0. Indeed, from the analysis of type (1, 2) done above in the proof of Theorem 3.13, this is because the probability that a Brownian motion has a record value at a given deterministic time is 0. For any  $t$ ,  $|S_{1,2} \cap \mathbb{R} \times \{t\}| > 1$  implies that there exist in the double Brownian web skeleton two pairs, each consisting of one forward and one backward path, such that in both pairs the forward and backward paths meet at the same time. We claim that this has zero probability and thus that  $|S_{1,2} \cap \mathbb{R} \times \{t\}| \leq 1$  almost surely. To verify the claim, we again use the analysis of type (1, 2) done for Theorem 3.13, which shows that it suffices to prove that there is zero probability that two independent standard Brownian motions  $B_1, B_2$  have a common strictly positive record time. But, as noted in Appendix A, this is the same as having zero probability for  $B_1, B_2$  to both have a zero at a common strictly positive time. This latter probability is indeed zero because of the well known fact that the two-dimensional Brownian motion  $(B_1, B_2)$  a.s. does not return to  $(0, 0)$ .

*Type (1, 1).* Since points with  $m_{\text{in}} \geq 1$  are a.s. in the skeleton,  $\bar{S}_{1,1} \cap \mathbb{R} \times \{t\}$  is a.s. countable (and easily seen to be dense) for every  $t \in \mathbb{R}$ . Now the previous parts of the proof imply that the same holds for  $S_{1,1} \cap \mathbb{R} \times \{t\}$  for every  $t \in \mathbb{R}$ .  $\square$

## 4. Proofs of Theorems 1.1, 1.3, and 1.5

### 4.1. Proofs of Theorem 1.1 (1) and (2) – convergence

(1)  $\Theta(x, t)$  is defined (see Sections 1 and 2) by considering *all* paths in the *backward* Brownian web from  $(x, t)$  and taking the set of first marked points (i.e. closest in time to  $t$ ) of those paths. The marking is done with rate  $\lambda$  and in the special case where  $(x, t)$  is itself a mark,  $\Theta(x, t)$  includes both  $(x, t)$  and the first mark  $(x', t')$  with  $t' < t$ . Property (1) of Theorem 1.1 follows from Property (o) of Theorem 2.1.

(2) We recall that  $\Theta_\delta^{\delta^2\lambda}(x, t)$  is the value of  $\Theta$  for a rescaled process where time is scaled like  $\delta^2$ , space is scaled like  $\delta$  and the nucleation rate is  $\delta^2\lambda$ . Therefore it follows from well known results that the coalescing random walks starting from  $(x_i^\delta, t_i^\delta)$ ,  $1 \leq i \leq n$  converge in distribution to coalescing Brownian motions starting from  $(x_i, t_i)$ ,  $1 \leq i \leq n$ . Since the rate of the Poisson clocks (nucleation rate) is  $\delta^2\lambda$ , the markings for the  $n$  coalescing random walks for all  $\delta$  and the  $n$  coalescing Brownian motions can be done using  $n$  fixed Poisson processes each of rate  $\lambda$ . That is, the marking part of the process for all  $n$  rescaled walks and the limiting coalescing Brownian motions can be coupled.



We define the coalescing walks and Brownian motions by introducing priorities. When a walk (or a BM) with label  $i$  meets a walk (or BM) with label  $j < i$  it follows the path of the walk (or BM) with label  $j$  after that time. Let  $T_i^\delta, 2 \leq i \leq n$  be the time when the walk starting from  $(x_i^\delta, t_i^\delta)$  meets a walk starting from  $(x_j^\delta, t_j^\delta)$  with  $j < i$  and let  $T_i^\delta = t_i^\delta - T_i'^\delta$  (recall that we are moving backwards in time). Denote by  $T_1^\delta = \min(T_2^\delta, \dots, T_n^\delta)$  the time when all the  $n$  walkers have coalesced and let  $T_1^\delta = t_1^\delta - T_1'^\delta$ . Let  $T_i, 1 \leq i \leq n$ , be the corresponding times for the Brownian motions starting from  $(x_i, t_i), 1 \leq i \leq n$ .

Since the  $T_i^\delta$ 's are functionals of the  $n$  random walks starting from  $(x_i^\delta, t_i^\delta)$ , it follows that not only the walks, but also  $T_i^\delta, 1 \leq i \leq n$ , converge in joint distribution to the continuum paths and  $T_i, 1 \leq i \leq n$ . Property (2) then follows; for example, to prove the second claim of Property (2), we observe that  $\mathbb{P}(\Theta_\delta^{\delta^2\lambda}(x_1^\delta, t_1^\delta) = \dots = \Theta_\delta^{\delta^2\lambda}(x_n^\delta, t_n^\delta)) = \mathbb{E}e^{-\lambda \sum_1^n T_i^\delta}$ . Now since  $T_i^\delta, 1 \leq i \leq n$  converge in distribution to  $T_i, 1 \leq i \leq n$  we have

$$\begin{aligned} \mathbb{P}(\Theta(x_1, t_1) = \dots = \Theta(x_n, t_n)) &= \mathbb{E}(e^{-\lambda \sum_1^n T_i}) = \lim_{\delta \rightarrow 0} \mathbb{E}(e^{-\lambda \sum_1^n T_i^\delta}) \\ &= \lim_{\delta \rightarrow 0} \mathbb{P}(\Theta_\delta^{\delta^2\lambda}(x_1^\delta, t_1^\delta) = \dots = \Theta_\delta^{\delta^2\lambda}(x_n^\delta, t_n^\delta)). \end{aligned}$$

#### 4.2. Proofs of Theorem 1.1 (3) and (4) – nucleation points and color regions

The MBW, in the context of Theorem 1.1, is a dual/backwards dynamics, in the sense that it is the continuum version of marked coalescing random walks, which is dual to the noisy voter model and runs backwards in time. In this way we get an indirect/dual/backwards description of the continuum version of the noisy voter model (CNVM). We can get a direct/forward description of the CNVM by considering not only the forward web, but also simultaneously the dual web. The dual web is needed in order to get the marks placed on the dual paths.

Once the marks are in place, we can focus on the paths of the forward web starting at the marks in  $\mathbb{R}^2$ . We note that, since the marks are on (non-starter points of) dual paths, each one is a double point of the forward web, and thus is the origin of a “bubble” (of the color it was assigned). We note that bubbles will occur inside other bubbles, with the color of the inside one prevailing.

Let us look at these bubbles, each consisting of the closed region of  $\mathbb{R}^2$  bounded by the two paths starting at a mark until they coalesce. In this situation, we call the mark the *starting tip* of the bubble. These are the nucleation points. We will call the space time point where a bubble ends, i.e., the space time point above the starting tip where the bubble boundary paths meet (and coalesce), the *ending tip* of that bubble. See Fig. 4.

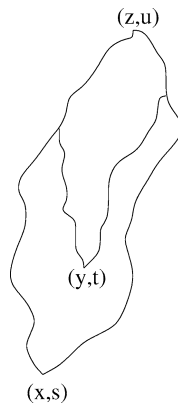


Fig. 4. Two space–time bubbles with starting tips at  $(x, s)$  and  $(y, t)$ , and common ending tip at  $(z, u)$ .

We argue that almost surely there are countably many nucleation points, since almost surely there are countably many marks. This easily follows from any of the constructions of Section 2; for example using the sequential construction, one may make the following two observations:

- (i) The marked points occur only on the skeleton of the (dual) BW and the skeleton is a countable collection of paths (i.e., those starting at the countable set  $\mathcal{D}$ ).
- (ii) On each path of the skeleton the nucleation events (marks) are Poisson events corresponding to a Poisson process with rate  $\lambda$ .

In order to show that almost surely the set of all nucleation points is dense in  $\mathbb{R}^2$  it is sufficient to show that almost surely for all  $\varepsilon > 0$  the square of side  $\varepsilon$  centered at the origin,  $S_0^\varepsilon$ , contains a nucleation point. This can be shown in a variety of ways; we proceed with one of them. For large enough  $n \in \mathbb{N}$ , divide  $S_0^\varepsilon$  into small rectangles of horizontal side length  $n^{-1/4}$  and vertical side length  $n^{-1}$ . There are about  $C_1 \varepsilon^2 n^{5/4}$  such rectangles in  $S_0^\varepsilon$ . For each of those rectangles the probability that there is no path within the rectangle from the midpoint of the top edge to the bottom edge is bounded by the probability of the event that a (backward) Brownian motion starting at the midpoint of the top edge leaves the rectangle through one of the side edges. By standard arguments, this probability is bounded by  $c_1 e^{-c_2 \sqrt{n}}$ . For each of the  $n^{-1/4}$  by  $n^{-1}$  rectangles, the conditional probability, given that there is such a path in the rectangle, that there is no mark on that path in the rectangle is equal to  $e^{-\lambda n^{-1}}$ . Since there are  $C_1 \varepsilon^2 n^{5/4}$  such rectangles we have that the probability of no nucleation point in  $S_0^\varepsilon$  is bounded above by

$$C \varepsilon^2 n^{5/4} e^{-c_2 n^{1/2}} + e^{-\lambda n^{-1}} C_1 \varepsilon^2 n^{5/4} \rightarrow 0 \quad \text{as } n \rightarrow \infty. \tag{4.1}$$

This proves that almost surely the set of all nucleation points is dense.

Now we show that  $C_{(x',t')}$  for a nucleation point  $(x', t')$  is a compact subset of  $\mathbb{R}^2$ . That it is bounded is clear since it is contained in the bubble from  $(x', t')$ , which is a.s. bounded. It is thus enough to show it is closed. We start with the space–time bubble starting at  $(x', t')$ . This is a closed subset of  $\mathbb{R}^2$ . Let  $(x_1, t_1), (x_2, t_2), \dots$  be some ordering of the nucleation points in the interior of the bubble  $B_{(x',t')}$ . Then we claim that

$$C_{(x',t')} = B_{(x',t')} - \lim_{n \rightarrow \infty} \text{rint} \left( \bigcup_{j=1}^n B_{(x_j,t_j)} \right) = \bigcap_{n=1}^{\infty} (B_{(x',t')} - \text{rint} \left( \bigcup_{j=1}^n B_{(x_j,t_j)} \right))$$

where  $\text{rint}$  denotes the relative interior (relative to  $B_{(x',t')}$ ). This would show that  $C_{(x',t')}$  is closed since it would be the intersection of closed sets.

To justify the claim we note first that  $(x_0, t_0) \in B_{(x',t')}$  is also in  $C_{(x',t')}$  if and only if there is a backwards path from  $(x_0, t_0)$  within  $B_{(x',t')}$  which touches none of  $(x_1, t_1), (x_2, t_2), \dots$ . We thus need only show that for all  $n$ , a point  $(x_0, t_0) \in B_{(x',t')}$  does *not* have a backward path from  $(x_0, t_0)$  touching any of  $(x_1, t_1), (x_2, t_2), \dots, (x_n, t_n)$  if and only if  $(x_0, t_0)$  belongs to  $\text{rint}(\bigcup_{j=1}^n B_{(x_j,t_j)})$ .

For a point  $(x_0, t_0)$  in  $\text{rint}(\bigcup_{j=1}^n B_{(x_j,t_j)})$ , it is not hard to see that *every* backward path from  $(x_0, t_0)$  must pass through one of the  $(x_k, t_k)$ ,  $1 \leq k \leq n$ , (before it reaches  $(x', t')$ ). On the other hand, for points in  $B_{(x',t')} - \text{rint}(\bigcup_{j=1}^n B_{(x_j,t_j)})$  every backward path enters (in arbitrarily small time) the interior of  $B_{(x',t')} - \text{rint}(\bigcup_{j=1}^n B_{(x_j,t_j)})$  and then can no longer touch any of the  $(x_k, t_k)$ ,  $1 \leq k \leq n$ . If it did, the mark at that  $(x_k, t_k)$  would be at a point where two backward paths coalesced. There is zero probability of such a mark occurring, as can be seen from, e.g., the sequential construction of the MBW in Section 2.

Perfectness of  $C_{(x',t')}$  follows immediately from the fact that its points belong to nondegenerate (continuous) path segments which are themselves in  $C_{(x',t')}$ . That  $C_{(x',t')}$  has empty interior follows immediately from the denseness of  $\mathcal{N}$ .

Property (4) is an immediate consequence of Theorems 3.11 and 3.13.

4.3. Proof of Theorem 1.1 (5) – two-point functions

The two-point correlation function  $\psi(x, t)$  is defined as  $\mathbb{P}(\Theta(0, 0) = \Theta(x, t))$ . It is the probability that two (almost surely unique) backwards paths starting at  $(0, 0)$  and  $(x, t)$  will not get marks before meeting.

Without loss of generality, we may assume  $t \geq 0$ . The case when  $t < 0$  negative is then easily reduced to this by time translation-invariance. Denote by  $Y$  the position at time 0 of the backward path starting at  $(x, t)$ . Now observe that  $\psi(x, t) = \mathbb{E}[\psi(Y, 0) e^{-\lambda t}]$  and that

$$\mathbb{E}[\psi(Y, 0)] = \mathbb{E}(e^{-2\lambda \widehat{T}}),$$

where  $\widehat{T}$  is the time when two independent BMs starting at  $(0, 0)$  and  $(0, Y)$  first meet.

By elementary properties of BM, this equals  $\mathbb{E}(e^{-\lambda T})$ , where  $T$  is the time standard BM first reaches  $|Y|$ . Now this is simply the Laplace transform of the distribution of a hitting time of BM. By the optional sampling theorem (see, e.g., [2]), it can be proved without calculation (see, e.g., [17]) that

$$\mathbb{E}(e^{-\lambda T}) = e^{-\sqrt{2\lambda}|Y|}.$$

So we have

$$\psi(x, t) = e^{-\lambda t} \mathbb{E}(e^{-\sqrt{2\lambda}|Y|}),$$

where  $Y$  is distributed as  $\mathcal{N}(x, t)$ .

Whether or not  $t \geq 0$ , we thus have

$$\psi(x, t) = e^{-\lambda|t|} \mathbb{E}(e^{-\sqrt{2\lambda}|x+B(|t|)|}),$$

where  $B$  is a standard Brownian motion. When  $t = 0$ ,

$$\psi(x, 0) = e^{-\sqrt{2\lambda}|x|}$$

since  $Y = x$ . When  $x = 0$  so that  $Y$  is centered, we change variables in the integral and get:

$$\mathbb{E}(e^{-\sqrt{2\lambda}|Y|}) = e^{\lambda t} \frac{2}{\sqrt{\pi}} \int_{\sqrt{\lambda|t|}}^{\infty} \exp(-y^2) dy,$$

yielding that

$$\psi(0, t) = \frac{2}{\sqrt{\pi}} \int_{\sqrt{\lambda|t|}}^{\infty} \exp(-y^2) dy.$$

4.4. Proof of Theorem 1.1 (6) – fixed time coloring

The above descriptions raise a natural question as to how the color configurations of the MBW dynamics look at fixed times. By the direct description we know that we have space–time color clusters one inside the other almost surely. It is not difficult to see that each cluster has another cluster inside it, e.g., by concluding from the scaling of the marked random walks that the marks of the MBW are dense in  $\mathbb{R}^2$ . This might suggest that the latter picture occurs also for fixed positive times, i.e., the color clusters at positive times<sup>1</sup> would also be such that each one has another cluster inside it. But this is not the case, as one sees for the case  $q = 2$  from Theorem 1.5. In fact, we argue

<sup>1</sup> By a color cluster at fixed time, say  $t$ , we mean any connected component of the intersection with  $\mathbb{R} \times \{t\}$  of a space–time color cluster, say  $C_0$ , minus the intersections with  $\mathbb{R} \times \{t\}$  of the closures of the space–time color clusters contained in  $C_0$ .

next that, even in the case that  $q = \infty$ , the color configurations of the MBW dynamics at fixed positive times have finitely many clusters in each finite interval almost surely.

It is enough to consider a single-time segment  $[a, b] \times \{t\}$  for deterministic  $a, b, t$  with  $a < b$  and show that the expected total time-length of all the disjoint pieces of all the backward paths of the backward BW starting at  $[a, b] \times \{t\}$  down to time  $t - s$  is almost surely finite for arbitrary  $s > 0$ .

For  $r > 0$ , let  $\eta(t, r; a, b)$  be the number of disjoint points at  $\mathbb{R} \times \{t - r\}$  which are touched by the backward paths starting on  $[a, b] \times \{t\}$ . Then the above mentioned total sum can be expressed as

$$\int_0^s \eta(t, r; a, b) dr. \tag{4.2}$$

We then need to show

$$\mathbb{E} \int_0^s \eta(t, r; a, b) dr = \int_0^s \mathbb{E}[\eta(t, r; a, b)] dr \tag{4.3}$$

is finite. And this follows from the formula

$$\mathbb{E}[\eta(t, r; a, b)] = (b - a) / \sqrt{\pi r}, \tag{4.4}$$

which holds for all  $r > 0$  (see Theorem 1.1 in [7]).

#### 4.5. Proof of Theorem 1.1 (7) – nonpersistence

We will show in this section that persistence, in its usual sense of no (or only finitely many) color changes at fixed spatial locations for strictly positive amounts of (rescaled) time, does not occur in the continuum noisy voter model (in contrast to the non-noisy voter model [10,3]). More precisely, we show for  $q = \infty$  that

almost surely, any deterministic vertical interval with nonzero length has infinitely many colors; the set of points with a unique color has full Lebesgue measure in the interval; all other points have exactly two colors and there are infinitely many of them.

For definiteness, we take  $\{0\} \times [0, 1]$  as the deterministic vertical interval. The above claims will follow from the fact that  $L$ , the total time-length of all the disjoint pieces of all the paths of the backward BW starting at  $\{0\} \times [0, 1]$  down to time 0, is almost surely infinite.

We first write  $L$  as

$$L = \int_0^1 N_s ds, \tag{4.5}$$

where, for  $0 \leq s \leq 1$ ,  $N_s$  denotes the number of distinct points in  $\mathbb{R} \times \{1 - s\}$  touched by paths starting on  $\{0\} \times [1 - s, 1]$ .

We now show that, for  $0 < s \leq 1$ ,  $N_s = \infty$  almost surely. This implies the above claim. By rescaling, if  $0 < s \leq 1$ , then  $N_s$  has the same distribution as  $\widehat{N}_t$ , the number of distinct points in  $\mathbb{R} \times \{0\}$  touched by paths starting on  $\{0\} \times [0, t]$ , for any  $t > 0$ . Now  $\widehat{N}_t$  is nondecreasing in  $t$ . Let  $\widehat{N}_\infty := \lim_{t \rightarrow \infty} \widehat{N}_t$ , the number of distinct points in  $\mathbb{R} \times \{0\}$  touched by paths starting on  $\{0\} \times [0, \infty)$ . Then  $N_s$  has the same distribution as  $\widehat{N}_\infty$  and it is thus enough to argue that  $\widehat{N}_\infty = \infty$  almost surely.

One straightforward way of arguing the latter point (there are other slicker arguments that use the Double Brownian Web) is to show that the event that there exists a sequence of paths starting on  $\{0\} \times [0, \infty)$  which are

disjoint down to time 0 has probability 1. For that, it is enough to exhibit for every  $\delta > 0$ , a sequence  $t_1 < t_2 < \dots$  such that the event  $A$  that the backward paths starting at  $(0, t_1), (0, t_2), \dots$  are disjoint down to time 0 has probability at least  $1 - \delta$ . The paths from  $(0, t_1), (0, t_2), \dots$  can be taken as independent Brownian paths.

To choose  $t_1, t_2, \dots$ , we start with a sequence  $p_1, p_2, \dots$  such that  $p_i > 0$  for all  $i \geq 1$  and  $\prod_{i=1}^{\infty} p_i \geq 1 - \delta$ . We take  $t_0 = 0$  and  $M_0 = 0$  and proceed inductively as follows. Having defined  $t_0, M_0, \dots, t_{n-1}, M_{n-1}$ , let  $t_n$  be such that the probability of the event  $\hat{A}_n$  that the path from  $(0, t_n)$  does not touch the rectangle  $[-M_{n-1}, M_{n-1}] \times [0, t_{n-1}]$  by time 0 is at least  $(1 + p_n)/2$ . That there exists such  $t_n$  follows from the fact that for any  $t, M > 0$ , the probability that the path from  $(0, t')$  does not touch the rectangle  $[-M, M] \times [0, t]$  by time 0 goes to 1 as  $t' \rightarrow \infty$ . With such a  $t_n$  picked, choose  $M_n$  such that the probability of the event  $\hat{A}_n$  that the path from  $(0, t_n)$  does not touch the vertical sides of the rectangle  $[-M_n, M_n] \times [0, t_n]$  by time 0 is at least  $(1 + p_n)/2$ . That there exists such  $M_n$  follows from the fact that for any  $t > 0$ , the probability that the path from  $(0, t)$  does not touch the vertical sides of the rectangle  $[-M, M] \times [0, t]$  by time 0 goes to 1 as  $M \rightarrow \infty$ .

Now let  $A_n = \hat{A}_n \cap \hat{A}_n$ . Then  $A \supset \bigcap_{n=1}^{\infty} A_n$ , and

$$\mathbb{P}(A) \geq \prod_{n=1}^{\infty} \mathbb{P}(A_n) \geq \prod_{n=1}^{\infty} [(1 + p_n)/2]^2 \geq \prod_{n=1}^{\infty} p_n \geq 1 - \delta, \tag{4.6}$$

as desired.

We have thus far showed that  $L = \infty$  almost surely. Take now an ordered countable dense deterministic subset  $\{\theta_n\}_{n \geq 1}$  of  $\{0\} \times [0, 1]$  and let  $\{\gamma_n\}_{n \geq 1}$  be defined inductively as follows.  $\gamma_1$  is the subpath of the path from  $\theta_1$  down to time 0; for  $n \geq 2$ ,  $\gamma_n$  is the subpath of the path from  $\theta_n$  down to time either 0 or when the latter path meets any of the  $\gamma_i, 1 \leq i \leq n - 1$ , whichever time is greater. Then  $\{\gamma_n\}_{n \geq 1}$  is a disjoint family, and the sum of the length of the  $\gamma_n$ 's, which equals  $L$ , is almost surely infinite. This implies that there almost surely are infinitely many marks in the union of the traces of the  $\gamma_n$ 's. Since each  $\gamma_n$  is finite, each one has almost surely finitely many marks. This and the previous statement imply that there almost surely are infinitely many marked  $\gamma_n$ 's and hence infinitely many *distinct* nucleation points from among the  $\Theta(\theta_n)$ 's. Thus there are infinitely many colors for the  $\theta_n$ 's in the  $q = \infty$  case.

By the last part of property (3) of Theorem 1.1 it follows that between every two  $\theta_n$ 's of different color, there must occur at least one point on the interval with (at least) those two colors, and so there are infinitely many points with at least two colors. To see that these have zero Lebesgue measure in the interval and that there are no points with three (or more) colors, note that all such points must be double points (i.e., two backward paths going down from that point) or (if they have more than two colors) triple points of the backward BW. Double points of the backward BW at  $x = 0$  correspond to (non-starting point) zeros of the paths of the forward BW, which have Hausdorff dimension 1/2 and zero Lebesgue measure, while triple points of the backward BW correspond to places where two paths of the forward BW coalesce, which has zero probability of occurring at a deterministic value  $x = 0$ . This completes the proof of Property (7).

#### 4.6. Proof of Theorem 1.1 (8) – color region Lebesgue measure

By the sequential construction of Section 2, every nucleation point  $(x', t')$  is the  $j$ th marked point for some  $j$  along the backward Brownian web path starting from some point  $(\bar{x}_i, \bar{t}_i)$  in a deterministic dense countable set  $\mathcal{D}$  of  $\mathbb{R}^2$ . Since the MBW distribution does not depend on the ordering of  $\mathcal{D}$ , we will consider the  $j$ 'th mark on the path from the first point  $(\bar{x}_1, \bar{t}_1)$ . Furthermore since our arguments do not depend on the value of  $(\bar{x}_1, \bar{t}_1)$ , we will take it to be the origin  $(0, 0)$ .

Let  $(x_k, \tau_k)$  denote the  $k$ th marked point along the backward web path starting at  $(0, 0)$ . Our object is to prove that for  $k \geq 1$ , the unique-color region,

$$C_{(x_k, \tau_k)}^u = \{(x, t) : \Theta(x, t) = (x_k, \tau_k)\},$$

has Lebesgue measure  $\mathcal{L}(C_{(x_k, \tau_k)}^u) > 0$  a.s.

We first consider  $k = 1$ . Let  $B_\varepsilon(x, t) = (x - \varepsilon, x + \varepsilon) \times (t - \varepsilon, t)$  and denote  $B_\varepsilon(0, 0)$  by  $B_\varepsilon$ . We also define  $Y_\varepsilon = \mathcal{L}(C_{(x_1, \tau_1)}^u \cap B_\varepsilon)$  and  $X_\varepsilon = (2\varepsilon^2)^{-1} Y_\varepsilon$  so that  $0 \leq X_\varepsilon \leq 1$ . Since  $Y_\varepsilon$  is decreasing in  $\varepsilon$ , to prove  $\mathbb{P}(Y_1 > 0) = 1$ , we will use that for  $\varepsilon < 1$ ,

$$\mathbb{P}(Y_1 > 0) \geq \mathbb{P}(Y_\varepsilon > 0) \geq \mathbb{P}(Y_\varepsilon \geq \varepsilon^2) = \mathbb{P}(X_\varepsilon \geq 1/2),$$

and argue that  $X_\varepsilon \rightarrow 1$  in probability as  $\varepsilon \rightarrow 0$ . These imply that  $\mathbb{P}(Y_1 > 0) \geq 1 - \delta$  for every  $\delta > 0$  and thus that  $\mathbb{P}(Y_1 > 0) = 1$  as desired. That  $X_\varepsilon \rightarrow 1$  will be a consequence of showing that  $E(X_\varepsilon) \rightarrow 1$  as  $\varepsilon \rightarrow 0$  which we proceed to do now.

Using Fubini’s Theorem, we have that

$$\mathbb{E}(X_\varepsilon) = \frac{1}{2\varepsilon^2} \int_{B_\varepsilon} \mathbb{P}(\Theta(x, t) = \Theta(0, 0)) \, dx \, dt.$$

Since  $\psi(x, t) = \mathbb{P}(\Theta(x, t) = \Theta(0, 0))$  is equal to 1 at  $(x, t) = (0, 0)$ , to see that  $\mathbb{E}(X_\varepsilon) \rightarrow 1$ , it suffices to show that  $\psi$  is continuous at  $(0, 0)$ . This can be seen easily from (1.6), or can be shown directly by considering two marked Brownian paths starting at  $(0, 0)$  and  $(x, t)$ .

To extend the argument to  $k \geq 2$ , we note that the same reasoning shows that it suffices to show that the expression

$$\frac{1}{2\varepsilon^2} \int_{\mathbb{R}^2} \mathbb{P}[1\{(x, t) \in B_\varepsilon(x_{k-1}, \tau_{k-1})\} 1\{(\Theta(x, t) = \widehat{\Theta}(x_{k-1}, \tau_{k-1}))\}] \, dx \, dt \tag{4.7}$$

tends to 1 as  $\varepsilon \rightarrow 0$ , where  $\widehat{\Theta}(x_{k-1}, \tau_{k-1}) = (x_k, \tau_k)$  is the first mark strictly after  $(x_{k-1}, \tau_{k-1})$  along the backward path from  $(0, 0)$ .

In the above integral,  $(x, t)$  is deterministic and by the sequential construction of MBW,  $\Theta(x, t)$  is simply the first mark along the backward path from  $(x, t)$ . If we denote the marked Brownian web (backward) path starting from  $(0, 0)$  by  $\widehat{B}_{(0,0)}(s)$  for  $s \leq 0$ , then by the strong Markov property for (a single) marked Brownian motion we have that  $\widehat{B}_{k-1}(s) := \widehat{B}_{(0,0)}(s + \tau_{k-1}) - \widehat{B}_{(0,0)}(\tau_{k-1})$ ,  $s \leq 0$  is a standard (reversed) Brownian motion and  $\widehat{B}_{k-1}(s)$ ,  $s \leq 0$  is independent of  $\widehat{B}_{(0,0)}(s)$ ,  $s \geq \tau_{k-1}$ . Now taking conditional expectation with respect to the value  $(y, u)$  of  $(x_{k-1}, \tau_{k-1})$ , expression (4.7) becomes

$$\begin{aligned} & \frac{1}{2\varepsilon^2} \int_{\mathbb{R}^2} \mathbb{E}_{(y,u)}(\mathbb{P}[1\{(x, t) \in B_\varepsilon(y, u)\} 1\{(\Theta(x, t) = \widehat{\Theta}(y, u))\}]) \, dx \, dt \\ &= \frac{1}{2\varepsilon^2} \mathbb{E}_{(y,u)} \int_{\mathbb{R}^2} \mathbb{P}[1\{(x, t) \in B_\varepsilon(y, u)\} 1\{(\Theta(x, t) = \widehat{\Theta}(y, u))\}] \, dx \, dt \\ &= \frac{1}{2\varepsilon^2} \int_{B_\varepsilon} \mathbb{P}(\Theta(x, t) = \Theta(0, 0)) \, dx \, dt \end{aligned}$$

where we have used the independence of  $\widehat{B}_{k-1}(s)$ ,  $s \leq 0$ , and  $\widehat{B}_{(0,0)}(s)$ ,  $s \geq \tau_{k-1}$ , in the second line. This reduces the argument to the  $k = 1$  case, thus proving the theorem.

#### 4.7. Proof of Theorem 1.5

We start by giving the relationship of the noisy voter model (NVM) on  $\mathbb{Z}$  and the stochastic Ising model (alluded to just before the statement of Theorem 1.5).

The stochastic Ising model at inverse temperature  $\beta$  is an interacting particle system with state space  $\{-1, +1\}^{\mathbb{Z}}$  whose flip rate at  $x \in \mathbb{Z}$  for a state  $\sigma$  is given by [19,21]

$$c(x, \sigma) = \frac{1}{1 + \exp\{\beta\sigma(x)[\sigma(x - 1) + \sigma(x + 1)]\}}. \tag{4.8}$$

One readily checks that this is equivalent to the NVM with  $p = 2/(1 + e^{2\beta})$ .

The invariant measure is the Ising model Gibbs measure for the formal Hamiltonian [19]

$$H(\sigma) = -\frac{1}{2} \sum_{x \in \mathbb{Z}} \sigma(x)\sigma(x + 1) \tag{4.9}$$

at inverse temperature  $\beta$ . This is a stationary (spatial) Markov chain with state space  $\{-1, +1\}$ , transition matrix [12]

$$\frac{1}{1 + e^\beta} \begin{pmatrix} e^\beta & 1 \\ 1 & e^\beta \end{pmatrix}, \tag{4.10}$$

and with single site uniform distribution on  $\{-1, +1\}$ . For this chain, runs of  $+1$ 's and  $-1$ 's have i.i.d. lengths with a common geometric distribution of mean  $e^\beta$ . Under the rescaling in the statement of Theorem 1.3, the block lengths are i.i.d. geometrics with mean  $\delta^{-1}\sqrt{2/\lambda - \delta^2}$  multiplied by  $\delta$ . They thus converge to i.i.d. exponentials with mean  $\sqrt{2/\lambda}$ . The limiting color configuration can be then described as a stationary (spatial) Markov jump process with state space  $\{-1, +1\}$  and uniform jump rate  $\sqrt{\lambda/2}$ .

We want now to identify the fixed time color configuration (as a function of  $x$ ) of the two-color CNVM with the above jump process. For that, we first note, by property (6) of Theorem 1.1, that the configuration can be described as a  $\{-1, +1\}$ -valued jump process. To characterize this process, it is thus enough to describe its finite dimensional distributions. But, by Theorem 1.3, these are limits of the scaled two-color NVM finite dimensional distributions, which in turn, by the above paragraph, are the finite dimensional distributions of the Markov jump process described there.

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### Appendix A. Hausdorff dimension of the graph of the sum of two Brownian motions

**Proposition A.1.** *Let  $X, Y$  be two independent standard Brownian motions and let  $\mathcal{T}^+$  denote the set of record times of  $X$ , i.e.,  $\mathcal{T}^+ = \{t \geq 0 : X(t) = M(t)\}$ , where  $M(t) := \sup_{0 \leq s \leq t} X(s)$  is the maximum of  $X$  up to time  $t$ . Then, for  $t_0 > 0$  and  $a, b \in \mathbb{R}$  with  $|a| + |b| > 0$ , the set  $\mathcal{G}^+ := \{(aX(t) + bY(t), t) : t \in \mathcal{T}^+ \cap [0, t_0]\}$ , the projection of  $\mathcal{T}^+ \cap [0, t_0]$  onto the graph of  $aX + bY$ , has Hausdorff dimension 1 almost surely.*

**Proof.** An upper bound of 1 for the Hausdorff dimension follows readily from the fact that  $\mathcal{G}^+$  is the image of a set,  $\mathcal{T}^+ \cap [0, t_0]$ , of Hausdorff dimension  $1/2$  a.s. (since  $\mathcal{T}^+ \cap [0, t_0]$  has the same distribution as the set of zeros of  $X$ ,  $\{t \in [0, t_0] : X(t) = 0\}$ ; this follows from  $(M(t) - X(t) : 0 \leq t \leq t_0)$  having the same distribution as  $(|X(t)| : 0 \leq t \leq t_0)$  [18])—see [23]—under a map which is a.s. (uniformly) Hölder continuous of exponent  $\alpha$  for

every  $\alpha < 1/2$ , namely, the map  $t \rightarrow (aX(t) + bY(t), t)$ , where we use the well known Hölder continuity properties of Brownian motion.

The desired lower bound is obtained by noting that the Hausdorff dimension of  $\mathcal{G}^+$  is bounded below by the Hausdorff dimension of the image of  $\mathcal{T}^+ \cap [0, t_0]$  under  $aX + bY$ , or equivalently under  $aM + bY$ , namely  $\{aM(t) + bY(t) : t \in \mathcal{T}^+ \cap [0, t_0]\}$ . Notice that the latter set equals  $\{as + bY(T(s)) : s \in [0, M(t_0)]\}$ , where  $T$  is the hitting time process associated to  $X$ , defined as  $T(x) := \inf\{t \geq 0 : X(t) = x\}$ . It suffices to show that the Hausdorff dimension of  $\{as + bY(T(s)) : s \in [0, L]\}$  is a.s. greater than or equal to 1 for every deterministic  $L > 0$ . But that follows from known results as well.  $Z(t) := at + bY(T(t))$  is a self similar process of exponent 1 with stationary increments and satisfies also the following condition of Theorem 3.3 in [25], from which the dimension bound follows. The condition is that there exists a constant  $K$  such that  $\mathbb{P}(|Z(1)| \leq x) \leq Kx$  for every  $x \geq 0$ . This property is readily obtained from the distributions of  $Y$  and the hitting time variable  $T(1)$ .  $\square$

## References

- [1] R. Arratia, Coalescing Brownian motions and the voter model on  $\mathbb{Z}$ , Unpublished partial manuscript (circa 1981), available from rarratia@math.usc.edu.
- [2] L. Breiman, Probability, reprint ed., SIAM, 1992.
- [3] B. Derrida, V. Hakim, V. Pasquier, Exact exponent for the number of persistent spins in the zero-temperature dynamics of the one-dimensional Potts model, *J. Statist. Phys.* 85 (1996) 763–797.
- [4] B. Derrida, R. Zeitak, Distribution of domain sizes in the zero temperature Glauber dynamics of the one-dimensional Potts model, *Phys. Rev. E* 54 (1996) 2513–2525.
- [5] I. Fatkullin, E. Vanden-Eijnden, Coarsening by diffusion-annihilation in a bistable system driven by noise, Preprint, 2003.
- [6] I. Fatkullin, E. Vanden-Eijnden, Statistical description of contact-interacting Brownian particles on the line, *J. Statist. Phys.* 112 (2003) 155–163.
- [7] L.R.G. Fontes, M. Isopi, C.M. Newman, K. Ravishankar, The Brownian web, *Proc. Nat. Acad. Sci.* 99 (2002) 15888–15893.
- [8] L.R.G. Fontes, M. Isopi, C.M. Newman, K. Ravishankar, The Brownian web: characterization and convergence, *Ann. Probab.* 32 (2004) 2857–2883.
- [9] L.R.G. Fontes, M. Isopi, C.M. Newman, K. Ravishankar, The Brownian web: characterization and convergence, Long version, math.PR/0304119.
- [10] L.R.G. Fontes, M. Isopi, C.M. Newman, D.L. Stein, Aging in 1D discrete spin models and equivalent systems, *Phys. Rev. Lett.* 87 (2001) 110201.
- [11] L.R.G. Fontes, M. Isopi, C.M. Newman, D.L. Stein, 1D aging, in preparation.
- [12] H.-O. Georgii, Gibbs Measures and Phase Transitions, Walter de Gruyter, 1988.
- [13] B. Granovsky, N. Madras, The noisy voter model, *Stochastic Process. Appl.* 55 (1995) 23–43.
- [14] S. Habib, K. Lindenberg, G. Lythe, C. Molina-Paris, Diffusion-limited reaction in one dimension: paired and unpaired nucleation, *J. Chem. Phys.* 115 (2001) 73–89.
- [15] S. Habib, G.D. Lythe, Dynamics of kinks: nucleation, diffusion and annihilation, *Phys. Rev. Lett.* 84 (2000) 1070–1073.
- [16] T.E. Harris, Additive set-valued Markov processes and graphical methods, *Ann. Probab.* 6 (1978) 355–378.
- [17] S. Karlin, H. Taylor, A First Course in Stochastic Processes, second ed., Academic Press, New York, 1975.
- [18] P. Lévy, Processus Stochastiques et Mouvement Brownien, Gauthier-Villars, Paris, 1948.
- [19] T.M. Liggett, Interacting Particle Systems, Springer-Verlag, New York, 1985.
- [20] T.M. Liggett, Stochastic Interacting Systems: Contact, Voter and Exclusion Processes, Springer-Verlag, New York, 1999.
- [21] E.J. Neves, R.H. Schonmann, Behavior of droplets for a class of Glauber dynamics at very low temperature, *Probab. Theory Related Fields* 91 (1992) 331–354.
- [22] F. Soucaliuc, B. Tóth, W. Werner, Reflection and coalescence between independent one-dimensional Brownian paths, *Ann. Inst. H. Poincaré Probab. Statist.* 36 (2000) 509–545.
- [23] S.J. Taylor, The  $\alpha$ -dimensional measure of the graph and set of zeros of a Brownian path, *Proc. Cambridge Philos. Soc.* 51 (1955) 265–274.
- [24] B. Tóth, W. Werner, The true self-repelling motion, *Probab. Theory Related Fields* 111 (1998) 375–452.
- [25] Y. Xiao, H. Lin, Dimension properties of sample paths of self-similar processes, *Acta Math. Sinica (N.S.)* 10 (1994) 289–300.