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Stochastic integral of divergence type with respect to fractional Brownian motion with Hurst parameter $H \in (0, \frac{1}{2})$

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Abstract

We define a stochastic integral with respect to fractional Brownian motion B^H with Hurst parameter $H \in (0, \frac{1}{2})$ that extends the divergence integral from Malliavin calculus. For this extended divergence integral we prove a Fubini theorem and establish versions of the formulas of Itô and Tanaka that hold for all $H \in (0, \frac{1}{2})$. Then we use the extended divergence integral to show that for every $H \in (\frac{1}{6}, \frac{1}{2})$ and all $g \in C^3(\mathbb{R})$, the Russo–Vallois symmetric integral $\int_a^b g(B_t^H)\, \mathrm{d}^0 B_t^H$ exists and is equal to $G(B_b^H) - G(B_a^H)$, where $G' = g$, while for $H \in (0, \frac{1}{6}]$, $\int_a^b (B_t^H)^2\, \mathrm{d}^0 B_t^H$ does not exist.

Résumé

Nous définissons une intégrale stochastique par rapport au mouvement brownien fractionnaire B^H avec paramètre de Hurst $H \in (0, \frac{1}{2})$ qui généralise l'intégrale du type divergence du calcul de Malliavin. Pour cette intégrale de divergence généralisée nous montrons un théorème de Fubini et nous établissons des versions des formules d'Itô et Tanaka pour tout $H \in (0, \frac{1}{2})$. Ensuite nous utilisons l'intégrale de divergence généralisée pour démontrer que pour $H \in (\frac{1}{6}, \frac{1}{2})$ et $g \in C^3(\mathbb{R})$, l'intégrale symétrique de Russo–Vallois $\int_a^b g(B_t^H)\, \mathrm{d}^0 B_t^H$ existe et vaut $G(B_b^H) - G(B_a^H)$, où $G' = g$, alors que pour $H \in (0, \frac{1}{6}]$, $\int_a^b (B_t^H)^2\, \mathrm{d}^0 B_t^H$ n'existe pas.

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1. Introduction

A fractional Brownian motion (fBm) $B^H = \{B_t^H, t \in \mathbb{R}\}$ with Hurst parameter $H \in (0,1)$ is a continuous Gaussian process with zero mean and covariance function

$$\mathbb{E}[B_t^H B_s^H] = \frac{1}{2}\big(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\big). \tag{1.1}$$

If $H = \frac{1}{2}$, then B^H is a two-sided Brownian motion, but for $H \neq \frac{1}{2}$, $\{B_t^H, \ t \geqslant 0\}$ is not a semimartingale (for a proof in the case $H \in (\frac{1}{2}, 1)$ see Example 4.9.2 in Liptser and Shiryaev [18], for a general proof see Maheswaran and Sims [19] or Rogers [31]).

It can easily be seen from (1.1) that

$$\mathbb{E}\big[|B_t^H - B_s^H|^2\big] = |t-s|^{2H}.$$

Hence, it follows from Kolmogorov's continuity criterion (see e.g. Theorem I.2.1 in Revuz and Yor [27]) that on any finite interval, almost all paths of B^H are β-Hölder continuous for all $\beta < H$. Therefore, if u is a stochastic process with Hölder continuous trajectories of order $\gamma > 1 - H$, then, by Young's theorem on Stieltjes integrability (see [33]), the path-wise Riemann–Stieltjes integral $\int_0^T u_t(\omega)\,\mathrm{d}B_t^H(\omega)$ exists for all $T \geqslant 0$. In particular, if $H > \frac{1}{2}$, the path-wise integral $\int_0^T f'(B_t^H)\,\mathrm{d}B_t^H$ exists for all $f \in C^2(\mathbb{R})$, and

$$f(B_T^H) - f(0) = \int_0^T f'(B_t^H)\,\mathrm{d}B_t^H$$

(more about path-wise integration with respect to fBm can be found in Lin [17], Mikosch and Norvaiša [21], Zähle [34] or Coutin and Qian [8]).

If $H \leqslant \frac{1}{2}$, the path-wise Riemann–Stieltjes integral $\int_0^T f'(B_t^H)\,\mathrm{d}B_t^H$ does not exist. For $H = \frac{1}{2}$, the stochastic integral introduced by Itô [16] has proven to be a very fruitful approach and has led to the development of classical stochastic calculus. Gaveau and Trauber [11] and Nualart and Pardoux [23] proved that the Itô stochastic integral coincides with the divergence operator on the Wiener space. Later, several authors have used the divergence operator to define stochastic integrals with respect to fBm with arbitrary $H \in (0,1)$. See for instance, Decreusefond and Üstünel [9], Carmona, Coutin and Montseny [6], Alòs, Mazet and Nualart [2,3], Coutin, Nualart and Tudor [7]. In [3] it is shown that if $H \in (\frac{1}{4}, 1)$, then for all functions $f \in C^2(\mathbb{R})$ such that f'' does not grow too fast, the divergence of the process $\{f'(B_t^H), \ t \in [0,T]\}$ exists and

$$f(B_T^H) - f(0) = \int_0^T f'(B_t^H)\delta B_t^H + H\int_0^T f''(B_t^H)t^{2H-1}\,\mathrm{d}t. \tag{1.2}$$

In [7] it is proved that for all $H \in (\frac{1}{3}, 1)$, the process $\{\mathrm{sign}(B_t^H), \ t \in [0,T]\}$ is in the domain of the divergence operator, and a fractional version of the Tanaka formula is derived. Privault [26] defined an extended Skorohod integral for a class of processes that satisfy a certain smoothness condition and showed that for this integral, formula (1.2) holds for every $H \in (0,1)$ and all $f \in C^2(\mathbb{R})$ such that f, f' and f'' are bounded. However, when using the approach of [26], the integral with respect to fBm with $H \in (0, \frac{1}{2})$ cannot be defined directly but must be constructed by approximating fBm with more regular processes. Similarly to [3], Hu [14] defined a stochastic integral with respect to fBm by transforming integrands and integrating them with respect to a standard Brownian motion. Provided they both exist, the integral with respect to fBm defined in [14] coincides with the one in [3]. Duncan, Hu and Pasik-Duncan [10] introduced a stochastic integral for fBm with $H \in (\frac{1}{2}, 1)$ as the limit of finite sums involving the Wick product. It is shown in Section 7 of [3] that again, this integral is the same as the divergence integral if both exist. Leaving the framework of random variables and working in the space of Hida distributions, Hu and Øksendal [15] as well as Bender [4] developed the integral of [10] further. In [4], for all $H \in (0, 1)$, a fractional Tanaka formula is proved, and an extended version of the formula (1.2) is shown to hold under the assumption that f is a tempered distribution that can also depend on t and satisfies some mild regularity conditions. Gradinaru, Russo and Vallois [12] proved a change of variables formula for fBm with $H \in [\frac{1}{4}, 1)$ that holds for the symmetric integral introduced in Russo and Vallois [28]. If both exist, the Russo–Vallois symmetric integral differs from the divergence integral by a trace term. For more details, see [1] or the introduction of [12].

In this paper we first explore how generally a stochastic integral for fBm can be defined by using the divergence operator from Malliavin calculus, and in particular, whether for the divergence operator, there exist versions of Itô's and Tanaka's formula for fBm with any $H \in (0, \frac{1}{2})$. Then, we study Russo–Vallois symmetric integrals of the form $\int_a^b g(B_t^H)\,\mathrm{d}^0 B_t^H$, for deterministic functions $g : \mathbb{R} \to \mathbb{R}$.

It turns out that the standard divergence integral of fBm with respect to itself does not exist if $H \in (0, \frac{1}{4}]$, the reason being that in this case, the paths of fBm are too irregular. However, in the right setup, the standard divergence operator can be extended by a simple change of the order of integration in the duality relationship that defines the divergence operator as the adjoint of the Malliavin derivative. The definition of this extended divergence operator is simpler than the definitions of the stochastic integrals in [26,14,15] and [4]. Moreover, it can be shown that for the extended divergence operator, a Fubini theorem holds as well as versions of the formulas of Itô and Tanaka for fBm with any $H \in (0, \frac{1}{2})$. By localization, the extended divergence operator can be generalized further, and one can prove that for every $H \in (0, \frac{1}{2})$, formula (1.2) holds for all $f \in C^2(\mathbb{R})$. A similar formula is valid for arbitrary convex functions. Hence, the change of variables formulas that we show for the extended divergence integral in this paper are valid for more general functions f than the change of variable formulas in [26]. On the other hand, our change of variables formulas for the extended divergence integral are neither more nor less general than the ones in [4]. Whereas in [4] f does not need to be a twice continuously differentiable or convex function, it cannot grow to fast at infinity. Another important difference between the divergence integral in this paper and the stochastic integral of [15] and [4] is that the stochastic integral of divergence type in this paper is always a random variable whereas in [15] and [4], the stochastic integral is defined as a Hida distribution. In the last section we use properties of the extended divergence integral to show that for all real numbers a and b such that $-\infty < a < b < \infty$ and every $H \in (\frac{1}{6}, \frac{1}{2})$, the symmetric integral

$$\int_a^b g(B_t^H)\,\mathrm{d}^0 B_t^H \tag{1.3}$$

in the Russo–Vallois sense exists for all $g \in C^3(\mathbb{R})$ and is equal to $G(B_b^H) - G(B_a^H)$, where $G' = g$, while on the other hand, for $H \in (0, \frac{1}{6}]$, the symmetric integral $\int_a^b (B_t^H)^2\,\mathrm{d}^0 B_t^H$ does not exist.

That $H = \frac{1}{6}$ is a barrier for the existence of integrals of the form (1.3) was simultaneously and independently discovered in the paper [13] by Gradinaru, Nourdin, Russo and Vallois. Their method of proof is different from ours and to show that the integral (1.3) exists for all $H > \frac{1}{6}$, they need that $g \in C^5(\mathbb{R})$. On the other hand, their result holds for more general symmetric stochastic integrals than the one considered in this paper.

The structure of the paper is as follows. In Section 2, we collect some facts from the theory of fractional calculus and discuss the first chaos of fBm with Hurst parameter $H \in (0, \frac{1}{2})$. In Section 3, we show that if $H \in (0, \frac{1}{4}]$, then for $-\infty < a < b < \infty$, the process $B_t^H 1_{(a,b]}(t)$ is not in the domain of the standard divergence operator. We then introduce an extended divergence operator and prove a Fubini theorem. Section 4 contains versions of the formulas of Itô and Tanaka for fBm with Hurst parameter $H \in (0, \frac{1}{2})$. In Section 5, we show that for $-\infty < a < b < \infty$, the Russo–Vallois symmetric integral $\int_a^b g(B_t^H)\,\mathrm{d}^0 B_t^H$ exists for all $g \in C^3(\mathbb{R})$ if and only if $H > \frac{1}{6}$, in which case it is equal to $G(B_b^H) - G(B_a^H)$, where $G' = g$.

2. The first chaos of fBm with $H \in (0, \frac{1}{2})$

Let $\{B_t^H,\ t \in \mathbb{R}\}$ be a fBm with Hurst parameter $H \in (0, \frac{1}{2})$ on a probability space $(\Omega, \mathcal{F}, P)$ such that

$$\mathcal{F} = \sigma\{B_t^H,\ t \in \mathbb{R}\}.$$

By $\mathcal{E}_H$ we denote the linear space of step functions

$$\left\{ \sum_{j=1}^{n} a_j 1_{(t_j, t_{j+1}]}:\ n \geqslant 1,\ -\infty < t_1 < t_2 < \cdots < t_{n+1} < \infty,\ a_j \in \mathbb{R} \right\},$$

equipped with the inner product

$$\left\langle \sum_{j=1}^{n} a_j 1_{(t_j, t_{j+1}]}, \sum_{k=1}^{m} b_k 1_{(s_k, s_{k+1}]} \right\rangle_{\mathcal{E}_H} := \mathbb{E}\left[\sum_{j=1}^{n} a_j (B_{t_{j+1}}^H - B_{t_j}^H) \sum_{k=1}^{m} b_k (B_{s_{k+1}}^H - B_{s_k}^H) \right].$$

Obviously, the linear map

$$\sum_{j=1}^{n} a_j 1_{(t_j, t_{j+1}]} \mapsto \sum_{j=1}^{n} a_j (B_{t_{j+1}}^H - B_{t_j}^H) \tag{2.1}$$

is an isometry between the inner product spaces $\mathcal{E}_H$ and

$$\operatorname{span}\{B_t^H,\ t \in \mathbb{R}\} \subset L^2(\Omega),$$

where span denotes the linear span.

There exists a Hilbert space of functions which contains $\mathcal{E}_H$ as a dense subspace. To describe this Hilbert space, we need the following notions of fractional calculus. We refer the reader to Samko, Kilbas and Marichev [32] for a complete presentation of this theory.

Let $\alpha = \frac{1}{2} - H$. The fractional integrals $I_+^\alpha \varphi$ and $I_-^\alpha \varphi$ of a function φ on the whole real axis are given by

$$I_+^\alpha \varphi(t) := \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{t} (t-s)^{\alpha-1} \varphi(s)\, \mathrm{d}s, \quad t \in \mathbb{R},$$

and

$$I_-^\alpha \varphi(t) := \frac{1}{\Gamma(\alpha)} \int_{t}^{\infty} (s-t)^{\alpha-1} \varphi(s)\, \mathrm{d}s, \quad t \in \mathbb{R},$$

respectively (see page 94 of [32]). The Marchaud fractional derivatives $\mathbf{D}_+^\alpha \varphi$ and $\mathbf{D}_-^\alpha \varphi$ of a function φ on the whole real line are defined by

$$\mathbf{D}_\pm^\alpha \varphi(t) := \lim_{\varepsilon \searrow 0} \mathbf{D}_{\pm,\varepsilon}^\alpha \varphi(t),\ t \in \mathbb{R},$$

where

$$\mathbf{D}_{\pm,\varepsilon}^\alpha \varphi(t) := \frac{\alpha}{\Gamma(1-\alpha)} \int_{\varepsilon}^{\infty} \frac{\varphi(t) - \varphi(t \mp s)}{s^{1+\alpha}}\, \mathrm{d}s, \quad t \in \mathbb{R}$$

(compare page 111 of [32]). It follows from Theorem 5.3 of [32] that I_+^α and I_-^α are bounded linear operators from $L^2(\mathbb{R})$ to $L^{1/H}(\mathbb{R})$. Theorem 6.1 of [32] implies that for all $\varphi \in L^2(\mathbb{R})$,

$$\mathbf{D}_+^\alpha I_+^\alpha \varphi = \varphi \quad \text{and} \quad \mathbf{D}_-^\alpha I_-^\alpha \varphi = \varphi. \tag{2.2}$$

In Corollary 1 to Theorem 11.4 of [32] it is shown that

$$I^\alpha\big(L^2(\mathbb{R})\big) := I_+^\alpha\big(L^2(\mathbb{R})\big) = I_-^\alpha\big(L^2(\mathbb{R})\big).$$

Let

$$c_H = \Gamma\left(H + \frac{1}{2}\right)\left(\int_0^\infty \big[(1+s)^{H-1/2} - s^{H-1/2}\big]^2\,\mathrm{d}s + \frac{1}{2H}\right)^{-1/2}.$$

It follows from (2.2) that the space $I^\alpha(L^2(\mathbb{R}))$ equipped with the inner product

$$\langle \varphi, \psi\rangle_{\Lambda_H} := c_H^2 \langle D_-^\alpha \varphi, D_-^\alpha \psi\rangle_{L^2(\mathbb{R})},$$

is a Hilbert space. We denote it by Λ_H. It is shown in Pipiras and Taqqu [24] that for all $\varphi, \psi \in \mathcal{E}_H$,

$$\langle \varphi, \psi\rangle_{\Lambda_H} = \langle \varphi, \psi\rangle_{\mathcal{E}_H}$$

and that $\mathcal{E}_H$ is dense in Λ_H. Therefore, the isometry (2.1) can be extended to an isometry between Λ_H and the first chaos of $\{B_t^H,\ t \in \mathbb{R}\}$,

$$\overline{\mathrm{span}}^{L^2(\Omega)}\{B_t^H,\ t \in \mathbb{R}\}.$$

We will denote this isometry by

$$\varphi \mapsto B^H(\varphi).$$

Remark 2.1. Let $-\infty < a < b < \infty$, and set

$$\Lambda_H^{(a,b]} := \big\{\varphi \in \Lambda_H\colon \varphi = \varphi 1_{(a,b]}(\cdot)\big\}.$$

Let $\varphi \in \Lambda_H \setminus \Lambda_H^{(a,b]}$. Since I_-^α is a bounded linear operator from $L^2(\mathbb{R})$ to $L^{1/H}(\mathbb{R})$, there exists a constant $c > 0$ such that for all $\psi \in \Lambda_H^{(a,b]}$,

$$c\|\varphi - \psi\|_{\Lambda_H} \geqslant \|\varphi - \psi\|_{L^{1/H}(\mathbb{R})} \geqslant \left(\int_{(-\infty,a]\cup(b,\infty)} \big|\varphi(t)\big|^{1/H}\,\mathrm{d}t\right)^H > 0.$$

This shows that $\Lambda_H^{(a,b]}$ is a closed subspace of Λ_H. On the other hand, let

$$\mathcal{E}_H^{(a,b]} := \big\{\varphi \in \mathcal{E}_H\colon \varphi = \varphi 1_{(a,b]}(\cdot)\big\},$$

and denote by $\overline{\mathcal{E}_H^{(a,b]}}$ the closure of $\mathcal{E}_H^{(a,b]}$ in Λ_H. If $\varphi \in \overline{\mathcal{E}_H^{(a,b]}}$, there exists a sequence $\{\varphi_n\}_{n=1}^\infty$ of functions in $\mathcal{E}_H^{(a,b]}$ such that $\varphi_n \to \varphi$ in Λ_H and therefore also in $L^{1/H}(\Omega)$. It follows that $\varphi \in \Lambda_H^{(a,b]}$. This shows that $\overline{\mathcal{E}_H^{(a,b]}} \subset \Lambda_H^{(a,b]}$.

The right-sided fractional integral $I_{b-}^\alpha \varphi$ of a function φ on the interval $(a,b]$ is given by

$$I_{b-}^\alpha \varphi(t) := \frac{1}{\Gamma(\alpha)} \int_t^b (s-t)^{\alpha-1}\varphi(s)\,\mathrm{d}s, \quad t \in (a,b]$$

(see Definition 2.1 of [32]). The right-sided Riemann–Liouville fractional derivative $\mathcal{D}_{b-}^\alpha \varphi$ of a function φ on the interval $(a,b]$ is given by

$$\mathcal{D}_{b-}^\alpha \varphi(t) := -\frac{1}{\Gamma(1-\alpha)} \frac{\mathrm{d}}{\mathrm{d}t} \int_t^b (s-t)^{-\alpha}\varphi(s)\,\mathrm{d}s, \quad t \in (a,b]$$

(see Definition 2.2 in [32]). It is shown in Theorem 2.6 of [32] that I^α_{b-} is a bounded linear operator from $L^1(a,b]$ to $L^1(a,b]$. It follows from Theorem 2.4 and formula (2.19) of [32] that for all $\varphi \in L^1(a,b]$,

$$\mathcal{D}^\alpha_{b-} I^\alpha_{b-}\varphi = \varphi.$$

Clearly, the linear maps

$$M^{-\alpha} : L^2(a,b] \to L^1(a,b], \quad f(t) \mapsto (t-a)^{-\alpha} f(t)$$

and

$$M^{\alpha} : L^1(a,b] \to L^1(a,b], \quad f(t) \mapsto (t-a)^{\alpha} f(t)$$

are bounded and injective. It follows that the map

$$J := M^\alpha \circ I^\alpha_{b-} \circ M^{-\alpha} : L^2(a,b] \to L^1(a,b]$$

is bounded and injective. Therefore, $\lambda^{(a,b]}_H = J(L^2(a,b])$ with the inner product

$$\langle \varphi, \psi\rangle_{\lambda^{(a,b]}_H} := \frac{\pi(2H-1)H}{\Gamma(2-2H)\sin(\pi(H-1/2))} \langle J^{-1}\varphi, J^{-1}\psi\rangle_{L^2(a,b]}$$

is a Hilbert space. In [25], Pipiras and Taqqu have shown that $\mathcal{E}^{(a,b]}_H$ is dense in $\lambda^{(a,b]}_H$. Let $\{\varphi_n\}_{n=1}^\infty$ be a Cauchy-sequence in $\mathcal{E}^{(a,b]}_H$. Then, there exist functions $\varphi \in \overline{\mathcal{E}^{(a,b]}_H}$ and $\psi \in \lambda^{(a,b]}_H$ such that

$$\varphi_n \to \varphi \quad \text{in } \Lambda_H \text{ and therefore also in } L^{1/H}(a,b]$$

and

$$\varphi_n \to \psi \quad \text{in } \lambda^{(a,b]}_H \text{ and therefore also in } L^1(a,b].$$

It follows that $\varphi = \psi$. This shows that

$$\lambda^{(a,b]}_H = \overline{\mathcal{E}^{(a,b]}_H} \subset \Lambda^{(a,b]}_H. \tag{2.3}$$

3. Extension of the divergence operator

In this section we define an extended divergence operator with respect to $\{B^H_t,\ t\in\mathbb{R}\}$ for $H \in (0,\frac{1}{2})$. We briefly recall the basic notions of the stochastic calculus of variations, also called Malliavin calculus. For more details we refer to the books by Nualart [22] and Malliavin [20]. The set of smooth and cylindrical random variables $\mathcal{S}$ consists of all random variables of the form

$$F = f\big(B^H(\varphi_1), \ldots, B^H(\varphi_n)\big), \tag{3.1}$$

where $n \geqslant 1$, $f \in \mathcal{C}^\infty_p(\mathbb{R}^n)$ (f and all its partial derivatives have polynomial growth), and $\varphi_j \in \Lambda_H$. Since $\mathcal{F} = \sigma\{B^H_t,\ t\in\mathbb{R}\}$, $\mathcal{S}$ is dense in $L^p(\Omega)$ for all $p \geqslant 1$. The derivative of a smooth and cylindrical random variable F of the form (3.1) is defined as the Λ_H-valued random variable

$$DF = \sum_{j=1}^n \frac{\partial f}{\partial x_j}\big(B^H(\varphi_1), \ldots, B^H(\varphi_n)\big)\varphi_j.$$

For all $p \geqslant 1$, $F \mapsto DF$ is a closable unbounded linear operator from $L^p(\Omega)$ to $L^p(\Omega, \Lambda_H)$. We denote the closed operator by D and its domain in $L^p(\Omega)$ by $\mathbb{D}^{1,p}$.

The divergence operator δ is defined as the adjoint of the derivative operator. For $p > 1$, let $\tilde{p} = \frac{p}{p-1} > 1$. By δ_p we denote the adjoint of D viewed as operator from $L^{\tilde{p}}(\Omega)$ to $L^{\tilde{p}}(\Omega, \Lambda_H)$, that is, the domain of δ_p, $\operatorname{Dom}\delta_p$, is the space of processes $u \in L^p(\Omega, \Lambda_H)$ such that

$$F \mapsto \mathbb{E}\langle u, DF\rangle_{\Lambda_H}$$

is a bounded linear functional on $(\mathcal{S}, \|\cdot\|_{\tilde{p}})$, and for $u \in \operatorname{Dom}\delta_p$, $\delta_p(u)$ is the unique element in $L^p(\Omega)$ such that

$$\mathbb{E}\langle u, DF\rangle_{\Lambda_H} = \mathbb{E}\big[\delta_p(u)F\big], \tag{3.2}$$

for all $F \in \mathcal{S}$. Obviously, if $u \in \operatorname{Dom}\delta_p \cap \operatorname{Dom}\delta_q$, for different $p, q > 1$, then $\delta_p(u) = \delta_q(u)$. Hence, one can define

$$\operatorname{Dom}\delta := \bigcup_{p>1} \operatorname{Dom}\delta_p,$$

and for $u \in \operatorname{Dom}\delta$,

$$\delta(u) := \delta_p(u), \tag{3.3}$$

for some $p > 1$ such that $u \in \operatorname{Dom}\delta_p$.

Remark 3.1. Let $-\infty < a < b < \infty$ and consider the process $\{B_t^H,\ a < t \leqslant b\}$ on $(\Omega, \mathcal{F}^{(a,b]}, P)$, where

$$\mathcal{F}^{(a,b]} = \sigma\{B_t^H,\ a < t \leqslant b\} = \sigma\{B_t^H,\ a \leqslant t \leqslant b\}.$$

Let $\delta^{(a,b]}$ be the corresponding divergence operator defined analogously to the divergence operator δ in (3.3). By (2.3), a process $u \in \bigcup_{p>1} L^p(\Omega, \lambda_H^{(a,b]})$ can be viewed as a process in $\bigcup_{p>1} L^p(\Omega, \Lambda_H)$. It can easily be checked that if $u \in \bigcup_{p>1} L^p(\Omega, \lambda_H^{(a,b]}) \cap \operatorname{Dom}\delta$, then $u \in \operatorname{Dom}\delta^{(a,b]}$ as well, and $\delta(u) = \delta^{(a,b]}(u)$.

Proposition 3.2. *Let $-\infty < a < b < \infty$, and set*

$$u_t = B_t^H 1_{(a,b]}(t), \quad t \in \mathbb{R}.$$

Then

$$P[u \in \Lambda_H] = 1, \quad \textit{for } H \in \left(\frac{1}{4}, \frac{1}{2}\right),$$

and

$$P[u \in \Lambda_H] = 0, \quad \textit{for } H \in \left(0, \frac{1}{4}\right].$$

Proof. First, let $H \in (\frac{1}{4}, \frac{1}{2})$. It follows from Kolmogorov's continuity criterion (compare e.g. Theorem I.2.1 in Revuz and Yor [27]) that there exists a measurable set $\widetilde{\Omega} \subset \Omega$ with $P[\widetilde{\Omega}] = 1$ such that for all $\omega \in \widetilde{\Omega}$, there exists a constant $\widetilde{C}(\omega)$ such that

$$\sup_{t \in (a,b]} \left|B_t^H(\omega)\right| \leqslant \widetilde{C}(\omega)$$

and

$$\sup_{t,s \in (a,b];\ t \neq s} \frac{|B_t^H(\omega) - B_s^H(\omega)|}{|t-s|^{1/4}} \leqslant \widetilde{C}(\omega).$$

We fix an $\omega \in \widetilde{\Omega}$ and set

$$\varphi(t) := u_t(\omega) = B_t^H(\omega) 1_{(a,b]}(t), \quad t \in \mathbb{R},$$

and

$$\widehat{C} := \frac{\alpha}{\Gamma(1-\alpha)}\widetilde{C}(\omega), \quad \text{where } \alpha = \frac{1}{2} - H.$$

Let $\varepsilon > 0$. For $t \in (-\infty, a]$,

$$\mathbf{D}^{\alpha}_{+,\varepsilon}\varphi(t) = 0.$$

For $t \in (a, b]$,

$$\left|\mathbf{D}^{\alpha}_{+,\varepsilon}\varphi(t)\right| \leqslant \frac{\alpha}{\Gamma(1-\alpha)}\left(1_{\{t-a>\varepsilon\}}\int_{\varepsilon}^{t-a}\left|\frac{\varphi(t)-\varphi(t-s)}{s^{1+\alpha}}\right| \mathrm{d}s + \left|\varphi(t)\right|\int_{(t-a)\vee\varepsilon}^{\infty} s^{-1-\alpha}\,\mathrm{d}s\right)$$
$$\leqslant \widehat{C}\left(1_{\{t-a>\varepsilon\}}\int_{\varepsilon}^{t-a} s^{-3/4-\alpha}\,\mathrm{d}s + \int_{t-a}^{\infty} s^{-1-\alpha}\,\mathrm{d}s\right) \leqslant \widehat{C}\left[\frac{1}{1/4-\alpha}(t-a)^{1/4-\alpha} + \frac{1}{\alpha}(t-a)^{-\alpha}\right].$$

For $t \in (b, \infty)$,

$$\left|\mathbf{D}^{\alpha}_{+,\varepsilon}\varphi(t)\right| \leqslant \frac{\alpha}{\Gamma(1-\alpha)}\int_{t-b}^{t-a}\frac{|\varphi(t-s)|}{s^{1+\alpha}}\,\mathrm{d}s \leqslant \widehat{C}\int_{t-b}^{t-a} s^{-1-\alpha}\,\mathrm{d}s = \widehat{C}\frac{1}{\alpha}\left[(t-b)^{-\alpha} - (t-a)^{-\alpha}\right].$$

Hence, for all $\varepsilon > 0$, for all $t \in \mathbb{R}$, $|\mathbf{D}^{\alpha}_{+,\varepsilon}\varphi(t)| \leqslant \psi(t)$, where

$$\psi(t) = \begin{cases} 0, & \text{if } t \in (-\infty, a],\\ C[(t-a)^{1/4-\alpha} + (t-a)^{-\alpha}], & \text{if } t \in (a, b],\\ C[(t-b)^{-\alpha} - (t-a)^{-\alpha}], & \text{if } t \in (b, \infty)\end{cases}$$

and

$$C = \widehat{C}\left(\frac{1}{1/4-\alpha} \vee \frac{1}{\alpha}\right).$$

It can easily be checked that $\psi \in L^2(\mathbb{R})$. It follows that φ satisfies condition (1) of Theorem 6.2 of [32]. Condition (2) is trivially satisfied. Therefore, Theorem 6.2 of [32] implies that $\varphi \in \Lambda_H$, which proves the first part of the proposition.

Now, let us assume that $H \in (0, \frac{1}{4}]$. The process

$$\widetilde{B}^H_t := B^H_{t+a} - B^H_a, \quad t \in \mathbb{R},$$

is also a fBm with Hurst parameter H. Since it is H-selfsimilar, for all $t \in (0, b-a)$, the random variable

$$t^{-2H}\int_0^{b-a-t}(\widetilde{B}^H_{s+t} - \widetilde{B}^H_s)^2\,\mathrm{d}s$$

has the same distribution as

$$\int_0^{b-a-t}(\widetilde{B}^H_{s/t+1} - \widetilde{B}^H_{s/t})^2\,\mathrm{d}s = t\int_0^{(b-a)/t-1}(\widetilde{B}^H_{x+1} - \widetilde{B}^H_x)^2\,\mathrm{d}x$$
$$= (b-a-t)\frac{1}{(b-a)/t-1}\int_0^{b-a/t-1}(\widetilde{B}^H_{x+1} - \widetilde{B}^H_x)^2\,\mathrm{d}x. \tag{3.4}$$

The process $(\widetilde{B}^H_{x+1} - \widetilde{B}^H_x)_{x\geqslant 0}$ is stationary and mixing. Therefore, it follows from the ergodic theorem that (3.4) converges to

$$(b-a)\mathbb{E}\big[(\widetilde{B}^H_1)^2\big] > 0, \quad \text{in } L^1 \text{ as } t \to 0.$$

Hence,

$$t^{-2H}\int_0^{b-a-t} (\widetilde{B}^H_{s+t} - \widetilde{B}^H_s)^2\,\mathrm{d}s \xrightarrow{L^1} (b-a)\mathbb{E}\big[(\widetilde{B}^H_1)^2\big], \quad \text{as } t \to 0,$$

as well. It follows that there exists a measurable set $\widetilde{\Omega} \subset \Omega$ with $P[\widetilde{\Omega}] = 1$ and a sequence of positive numbers $\{t_k\}_{k=1}^\infty$ that converges to 0 such that for all $\omega \in \widetilde{\Omega}$ and $k \geqslant 1$,

$$\begin{aligned}\int_{\mathbb{R}} \big(u_{s+t_k}(\omega) - u_s(\omega)\big)^2\,\mathrm{d}s &\geqslant \int_a^{b-t_k} \big(B^H_{s+t_k}(\omega) - B^H_s(\omega)\big)^2\,\mathrm{d}s = \int_0^{b-a-t_k} \big(\widetilde{B}^H_{s+t_k}(\omega) - \widetilde{B}^H_s(\omega)\big)^2\,\mathrm{d}s \\ &\geqslant \frac{b-a}{2}\mathbb{E}\big[(\widetilde{B}^H_1)^2\big]t_k^{2H}. \end{aligned} \tag{3.5}$$

Now, assume that there exists an $\omega \in \widetilde{\Omega}$ such that $u(\omega) \in \Lambda_H$. By (6.40) of [32], the function $u(\omega)$ has the property

$$\int_{\mathbb{R}} \big(u_{s+t}(\omega) - u_s(\omega)\big)^2\,\mathrm{d}s = o(t^{2\alpha}) \quad \text{as } t \to 0. \tag{3.6}$$

But $u(\omega)$ can only satisfy (3.5) and (3.6) at the same time if $H > \alpha = \frac{1}{2} - H$, which contradicts $H \leqslant \frac{1}{4}$. Therefore, $u(\omega) \notin \Lambda_H$ for all $\omega \in \widetilde{\Omega}$, and the proposition is proved. □

Since $\mathrm{Dom}\,\delta \subset \bigcup_{p>1} L^p(\Omega, \Lambda_H)$, Proposition 3.2 implies that processes of the form

$$B^H_t 1_{(a,b]}(t),$$

cannot be in $\mathrm{Dom}\,\delta$ if $H \leqslant \frac{1}{4}$. Note that it follows from (2.3) that for $H \leqslant \frac{1}{4}$, almost surely, no path of $\{B^H_t,\ a < t \leqslant b\}$ is in $\lambda_H^{(a,b]}$ either, and therefore, $\{B^H_t,\ a < t \leqslant b\} \notin \mathrm{Dom}\,\delta^{(a,b]}$. In the following definition we extend the divergence δ to an operator whose domain also contains processes with paths that are not in Λ_H.

We set

$$\Lambda^*_H := I^\alpha_-(\mathcal{E}_H).$$

Since $\mathcal{E}_H$ is dense in $L^2(\mathbb{R})$, Λ^*_H is dense in Λ_H. Furthermore, it can easily be checked that for $-\infty < a < b < \infty$,

$$I^\alpha_+\big[(t-a)_+^{H-1/2} - (t-b)_+^{H-1/2}\big] = \Gamma(H+1/2)1_{(a,b]}(t), \quad t \in \mathbb{R}.$$

It follows from (2.2) that

$$\mathbf{D}^\alpha_+ 1_{(a,b]}(t) = \frac{1}{\Gamma(H+1/2)}\big[(t-a)_+^{H-1/2} - (t-b)_+^{H-1/2}\big], \quad t \in \mathbb{R},$$

which shows that

$$\mathbf{D}^\alpha_+\mathbf{D}^\alpha_-(\Lambda^*_H) = \mathbf{D}^\alpha_+(\mathcal{E}_H) \subset L^p(\mathbb{R}), \tag{3.7}$$

for all

$$p \in \left(\frac{1}{3/2-H}, \frac{1}{1/2-H}\right), \quad \text{in particular, for } p = 2.$$

Corollary 2 to Theorem 6.2 of [32] implies that for all $\varphi \in \Lambda_H$ and $\psi \in \mathcal{E}_H$, the following integration by parts formula holds:

$$\int_{\mathbb{R}} \varphi(x)\mathbf{D}_+^\alpha \psi(x)\,\mathrm{d}x = \int_{\mathbb{R}} \mathbf{D}_-^\alpha \varphi(x)\psi(x)\,\mathrm{d}x. \tag{3.8}$$

By H_n we denote the n-th over-normalized Hermite polynomial, that is,

$$H_0(x) := 1, \quad \text{and} \quad H_n(x) := \frac{(-1)^n}{n!}\mathrm{e}^{x^2/2}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\big(\mathrm{e}^{-x^2/2}\big),\ n \geqslant 1.$$

Furthermore, we set $H_{-1}(x) := 0$. It can be shown as in Theorem 1.1.1 of Nualart [22] that for all $p \geqslant 1$,

$$\operatorname{span}\big\{H_n\big(B^H(\varphi)\big)\colon\ n \in \mathbb{N},\ \varphi \in \Lambda_H^*,\ \|\varphi\|_{\Lambda_H} = 1\big\}$$

is dense in $L^p(\Omega)$.

Definition 3.3. Let $u = \{u_t,\ t \in \mathbb{R}\}$ be a measurable process. We say that $u \in \operatorname{Dom}^* \delta$ if and only if there exists a $\delta(u) \in \bigcup_{p>1} L^p(\Omega)$ such that for all $n \in \mathbb{N}$ and $\varphi \in \Lambda_H^*$ with $\|\varphi\|_{\Lambda_H} = 1$, the following conditions are satisfied:

(i) for almost all $t \in \mathbb{R}$: $u_t H_{n-1}(B^H(\varphi)) \in L^1(\Omega)$,
(ii) $\mathbb{E}[u_\cdot H_{n-1}(B^H(\varphi))]\mathbf{D}_+^\alpha \mathbf{D}_-^\alpha \varphi(\cdot) \in L^1(\mathbb{R})$, and
(iii) $c_H^2 \int_{\mathbb{R}} \mathbb{E}[u_t H_{n-1}(B^H(\varphi))]\mathbf{D}_+^\alpha \mathbf{D}_-^\alpha \varphi(t)\,\mathrm{d}t = \mathbb{E}[\delta(u) H_n(B^H(\varphi))]$.

Note that if $u \in \operatorname{Dom}^*\delta$, then $\delta(u)$ is uniquely defined, and the mapping $\delta : \operatorname{Dom}^*\delta \to \bigcup_{p>1} L^p(\Omega)$ is linear.

Remark 3.4.

1. Let $n \in \mathbb{N}$ and $\varphi \in \Lambda_H^*$. By (3.7), the process

   $$H_{n-1}\big(B^H(\varphi)\big)\mathbf{D}_+^\alpha \mathbf{D}_-^\alpha \varphi(t)$$

   is in $L^p(\Omega, L^q(\mathbb{R}))$ for all

   $$p \in [1, \infty) \quad \text{and} \quad q \in \left(\frac{1}{3/2 - H},\ \frac{1}{1/2 - H}\right).$$

   By twice applying Hölder’s inequality, it follows that if

   $$u \in L^p\big(\Omega, L^q(\mathbb{R})\big)$$

   for some

   $$p \in (1, \infty] \quad \text{and} \quad q \in \left(\frac{1}{1/2 + H},\ \infty\right],$$

   then

   $$u_t H_{n-1}\big(B^H(\varphi)\big)\mathbf{D}_+^\alpha \mathbf{D}_-^\alpha \varphi(t) \in L^1\big(\Omega, L^1(\mathbb{R})\big) = L^1(\Omega \times \mathbb{R}), \tag{3.9}$$

   which implies that u satisfies conditions (i) and (ii) of Definition 3.3.
2. The extended divergence operator δ is closed in the following sense:

Let

$$p \in (1, \infty] \quad \text{and} \quad q \in \left(\frac{1}{1/2 + H},\ \infty\right].$$

Let $\{u^k\}_{k=1}^{\infty}$ be a sequence in $\mathrm{Dom}^* \delta \cap L^p(\Omega, L^q(\mathbb{R}))$ and $u \in L^p(\Omega, L^q(\mathbb{R}))$ such that

$$\lim_{k\to\infty} u^k = u \quad \text{in } L^p\big(\Omega, L^q(\mathbb{R})\big).$$

It follows that for all $n \in \mathbb{N}$ and $\varphi \in \Lambda_H^*$,

$$\lim_{k\to\infty} u_t^k H_{n-1}\big(B^H(\varphi)\big)\mathbf{D}_+^\alpha \mathbf{D}_-^\alpha \varphi(t) = u_t H_{n-1}\big(B^H(\varphi)\big)\mathbf{D}_+^\alpha \mathbf{D}_-^\alpha \varphi(t)$$

in $L^1(\Omega \times \mathbb{R})$. If there exists a $\hat{p} \in (1, \infty]$ and an $X \in L^{\hat{p}}(\Omega)$ such that

$$\lim_{k\to\infty} \delta(u^k) = X \quad \text{in } L^{\hat{p}}(\Omega),$$

then $u \in \mathrm{Dom}^* \delta$, and $\delta(u) = X$.

Proposition 3.5.

$$\mathrm{Dom}^* \delta \cap \bigcup_{p>1} L^p(\Omega, \Lambda_H) = \mathrm{Dom}\, \delta,$$

and the extended divergence operator δ *restricted to* $\mathrm{Dom}\,\delta$ *coincides with the standard divergence operator defined by* (3.3).

Proof. Let $u \in \mathrm{Dom}\,\delta = \bigcup_{p>1} \mathrm{Dom}\,\delta_p$. Then, there exists a $p > 1$ such that $u \in \mathrm{Dom}\,\delta_p$, and $\delta_p(u) \in L^p(\Omega)$. In particular, $u \in L^p(\Omega, \Lambda_H)$. Hence, it follows from Theorem 5.3 of [32] that $u \in L^p(\Omega, L^{1/H}(\mathbb{R}))$. Therefore, by Remark 3.4.1, u satisfies conditions (i) and (ii) of Definition 3.3.

Now, let $n \in \mathbb{N}$ and $\varphi \in \Lambda_H^*$ with $\|\varphi\|_{\Lambda_H} = 1$. The duality relation (3.2), the expression

$$DH_n\big(B^H(\varphi)\big) = H_{n-1}\big(B^H(\varphi)\big)\varphi$$

and the fractional integration by parts formula (3.8), yield

$$\begin{aligned}\mathbb{E}\big[\delta_p(u)H_n\big(B^H(\varphi)\big)\big] &= \mathbb{E}\big\langle u, DH_n\big(B^H(\varphi)\big)\big\rangle_{\Lambda_H} = \mathbb{E}\big[H_{n-1}\big(B^H(\varphi)\big)\langle u, \varphi\rangle_{\Lambda_H}\big] \\ &= c_H^2 \mathbb{E}\big[H_{n-1}\big(B^H(\varphi)\big)\langle \mathbf{D}_-^\alpha u, \mathbf{D}_-^\alpha \varphi\rangle_{L^2(\mathbb{R})}\big] \\ &= c_H^2 \mathbb{E}\Big[H_{n-1}\big(B^H(\varphi)\big) \int_{\mathbb{R}} u_t \mathbf{D}_+^\alpha \mathbf{D}_-^\alpha \varphi(t)\,\mathrm{d}t\Big]. \end{aligned} \tag{3.10}$$

Since (3.9) is valid, Fubini's theorem implies that (3.10) is equal to

$$c_H^2 \int_{\mathbb{R}} \mathbb{E}\big[u_t H_{n-1}\big(B^H(\varphi)\big)\big]\mathbf{D}_+^\alpha \mathbf{D}_-^\alpha \varphi(t)\,\mathrm{d}t,$$

which shows that u also fulfills condition (iii) of Definition 3.3. Hence,

$$\mathrm{Dom}\,\delta \subset \mathrm{Dom}^* \delta \cap \bigcup_{p>1} L^p(\Omega, \Lambda_H).$$

and the operator δ from Definition 3.3 is an extension of the one defined by (3.3).

If $u \in \mathrm{Dom}^* \delta \cap \bigcup_{p>1} L^p(\Omega, \Lambda_H)$, then there exists a $p > 1$ such that $u \in L^p(\Omega, \Lambda_H)$ and $\delta(u) \in L^p(\Omega)$. Let $n \in \mathbb{N}$ and $\varphi \in \Lambda_H^*$. Theorem 5.3 of [32] implies that $u \in L^p(\Omega, L^{1/H}(\mathbb{R}))$, and it follows that (3.9) holds. Therefore, Fubini's theorem applies, and we get

$$\begin{aligned}\mathbb{E}\big\langle u, DH_n\big(B^H(\varphi)\big)\big\rangle_{\Lambda_H} &= \mathbb{E}\bigg[c_H^2 \int_{\mathbb{R}} \mathbf{D}_-^\alpha u_t H_{n-1}\big(B^H(\varphi)\big)\mathbf{D}_-^\alpha \varphi(t)\,\mathrm{d}t\bigg] \\ &= \mathbb{E}\bigg[c_H^2 \int_{\mathbb{R}} u_t \mathbf{D}_+^\alpha \mathbf{D}_-^\alpha \varphi(t)\,\mathrm{d}t\, H_{n-1}\big(B^H(\varphi)\big)\bigg] \\ &= c_H^2 \int_{\mathbb{R}} \mathbb{E}\big[u_t H_{n-1}\big(B^H(\varphi)\big)\big]\mathbf{D}_+^\alpha \mathbf{D}_-^\alpha \varphi(t)\,\mathrm{d}t \\ &= \mathbb{E}\big[\delta(u) H_n\big(B^H(\varphi)\big)\big].\end{aligned}$$

It can be deduced from this by an approximation argument that

$$\mathbb{E}\langle u, DF\rangle_{\Lambda_H} = \mathbb{E}\big[\delta(u)F\big]$$

for all $F \in \mathcal{S}$, which shows that $u \in \operatorname{Dom}\delta$, and therefore,

$$\operatorname{Dom}^*\delta \cap \bigcup_{p>1} L^p(\Omega, \Lambda_H) \subset \operatorname{Dom}\delta. \qquad \square$$

Proposition 3.6. *Let* $u \in \operatorname{Dom}^*\delta$ *such that* $\mathbb{E}[u_.] \in L^2(\mathbb{R})$. *Then* $\mathbb{E}[u_.] \in \Lambda_H$.

Proof. By Definition 3.3, $\delta(u) \in L^p(\Omega)$ for some $p > 1$. Let $\varphi \in \Lambda_H^*$ with $\|\varphi\|_{\Lambda_H} = 1$. For $n = 1$, condition (iii) of Definition 3.3 yields

$$\bigg|c_H^2 \int_{\mathbb{R}} \mathbb{E}[u_t]\mathbf{D}_+^\alpha \mathbf{D}_-^\alpha \varphi(t)\,\mathrm{d}t\bigg| = \big|\mathbb{E}\big[\delta(u)B^H(\varphi)\big]\big| \leqslant \big\|\delta(u)\big\|_{L^p(\Omega)}\big\|B^H(\varphi)\big\|_{L^{\tilde{p}}(\Omega)},$$

where $\tilde{p} = \frac{p}{p-1}$. Since there exists a constant $\gamma_{\tilde{p}}$ such that for all $\varphi \in \Lambda_H$,

$$\big\|B^H(\varphi)\big\|_{L^{\tilde{p}}(\Omega)} = \gamma_{\tilde{p}}\big\|B^H(\varphi)\big\|_{L^2(\Omega)} = \gamma_{\tilde{p}}\|\varphi\|_{\Lambda_H},$$

the mapping

$$\varphi \mapsto c_H^2 \int_{\mathbb{R}} \mathbb{E}[u_t]\mathbf{D}_+^\alpha \mathbf{D}_-^\alpha \varphi(t)\,\mathrm{d}t$$

is a continuous linear functional on $\Lambda_H^* = I_-^\alpha(\mathcal{E}_H) \subset \Lambda_H$, which can be extended to a continuous linear functional on Λ_H. Therefore, there exists a $\psi \in \Lambda_H$ such that for all $\varphi \in \Lambda_H^*$,

$$c_H^2 \int_{\mathbb{R}} \mathbb{E}[u_t]\mathbf{D}_+^\alpha \mathbf{D}_-^\alpha \varphi(t)\,\mathrm{d}t = \langle \psi, \varphi\rangle_{\Lambda_H} = c_H^2 \int_{\mathbb{R}} \mathbf{D}_-^\alpha \psi(t)\mathbf{D}_-^\alpha \varphi(t)\,\mathrm{d}t. \tag{3.11}$$

It follows from the integration by parts formula (3.8) that (3.11) is equal to

$$c_H^2 \int_{\mathbb{R}} \psi(t)\mathbf{D}_+^\alpha \mathbf{D}_-^\alpha \varphi(t)\,\mathrm{d}t.$$

Hence,

$$\bigg|\int_{\mathbb{R}} \psi(t)\mathbf{D}_+^\alpha \mathbf{D}_-^\alpha \varphi(t)\,\mathrm{d}t\bigg| \leqslant \big\|\mathbb{E}[u_.]\big\|_{L^2(\mathbb{R})}\|\mathbf{D}_+^\alpha \mathbf{D}_-^\alpha \varphi\|_{L^2(\mathbb{R})}$$

for all $\varphi \in \Lambda_H^*$. In the proof of Lemma 5.9 in Pipiras and Taqqu [24] it is shown that $\mathbf{D}_-^\alpha(\mathcal{E}_H)$ is dense in $L^2(\mathbb{R})$. Analogously, it can be shown that $\mathbf{D}_+^\alpha \mathbf{D}_-^\alpha(\Lambda_H^*) = \mathbf{D}_+^\alpha(\mathcal{E}_H)$ is dense in $L^2(\mathbb{R})$. This implies that $\psi \in L^2(\mathbb{R})$, and it follows that $\mathbb{E}[u_.] = \psi$. □

Theorem 3.7 (Fubini theorem). *Let $(Y, \mathcal{Y}, \mu)$ be a measure space and $u_t^y(\omega) \in L^0(\Omega \times \mathbb{R} \times Y)$ such that*

(i) *for almost all $y \in Y$: $u^y \in \mathrm{Dom}^*\delta$;*
(ii) *for almost all $(\omega, t) \in \Omega \times \mathbb{R}$: $u_t(\omega) \in L^1(Y)$ and $\int_Y |u^y| \,\mathrm{d}\mu(y) \in L^2(\Omega \times \mathbb{R})$;*
(iii) *for almost all $\omega \in \Omega$: $\delta(u)(\omega) \in L^1(Y)$ and $\int_Y |\delta(u^y)| \,\mathrm{d}\mu(y) \in L^2(\Omega)$.*

Then

$$\int_Y u^y \,\mathrm{d}\mu(y) \in \mathrm{Dom}^*\delta \quad \textit{and} \quad \delta\left[\int_Y u^y \,\mathrm{d}\mu(y)\right] = \int_Y \delta(u^y) \,\mathrm{d}\mu(y).$$

Proof. It follows from assumption (ii) that $\int_Y u^y \,\mathrm{d}\mu(y) \in L^2(\Omega \times \mathbb{R})$. By Remark 3.4.1, $\int_Y u^y \,\mathrm{d}\mu(y)$ satisfies conditions (i) and (ii) of Definition 3.3. Assumption (iii) implies that $\int_Y \delta(u^y) \,\mathrm{d}\mu(y) \in L^2(\Omega)$. Now, let $n \in \mathbb{N}$ and $\varphi \in \Lambda_H^*$ such that $\|\varphi\|_{\Lambda_H} = 1$. Then

$$\begin{aligned} c_H^2 \int_{\mathbb{R}} \mathbb{E}\left[\int_Y u_t^y \,\mathrm{d}\mu(y) H_{n-1}\big(B^H(\varphi)\big)\right] \mathbf{D}_+^\alpha \mathbf{D}_-^\alpha \varphi(t) \,\mathrm{d}t &= c_H^2 \int_Y \int_{\mathbb{R}} \mathbb{E}\big[u_t^y H_{n-1}\big(B^H(\varphi)\big)\big] \mathbf{D}_+^\alpha \mathbf{D}_-^\alpha \varphi(t) \,\mathrm{d}t \,\mathrm{d}\mu(y) \\ &= \int_Y \mathbb{E}\big[\delta(u^y) H_n\big(B^H(\varphi)\big)\big] \,\mathrm{d}\mu(y) \\ &= \mathbb{E}\left[\int_Y \delta(u^y) \,\mathrm{d}\mu(y) H_n\big(B^H(\varphi)\big)\right], \end{aligned}$$

where the first and the third equality follow from the standard version of Fubini's theorem. It can be applied because

$$\int_{\Omega \times \mathbb{R} \times Y} |u_t^y| \big|H_{n-1}\big(B^H(\varphi)\big)\big| \big|\mathbf{D}_+^\alpha \mathbf{D}_-^\alpha \varphi(t)\big| \,\mathrm{d}P \,\mathrm{d}t \,\mathrm{d}\mu(y) \leqslant \left\| \int_Y |u^y| \,\mathrm{d}\mu(y) \right\|_{L^2(\Omega \times \mathbb{R})} \|\mathbf{D}_+^\alpha \mathbf{D}_-^\alpha \varphi\|_{L^2(\mathbb{R})} < \infty,$$

and

$$\int_{\Omega \times Y} \big|\delta(u^y)\big| \big|H_n\big(B^H(\varphi)\big)\big| \,\mathrm{d}P \,\mathrm{d}\mu(y) \leqslant \left\| \int_Y \big|\delta(u^y)\big| \,\mathrm{d}\mu(y) \right\|_{L^2(\Omega)} < \infty.$$

Hence, $\int_Y u^y \,\mathrm{d}\mu(y)$ also satisfies condition (iii) of Definition 3.3, and the proposition is proved. □

4. Itô and Tanaka formula

In this section we establish versions of the formulas of Itô and Tanaka for any value of the Hurst parameter $H \in (0, \frac{1}{2})$. The basic ingredient is the next lemma, which contains Itô's formula for smooth functions f such that f and all its derivatives do not grow too fast. In the whole section, a and b are two are two real numbers such that $-\infty < a < b < \infty$.

Definition 4.1. We say that a function $f:\mathbb{R}\to\mathbb{R}$ satisfies the growth condition (GC) if there exist positive constants c and λ such that

$$\lambda < \frac{1}{4}\big[\big(|a|\vee|b|\big)\big]^{-2H} \quad \text{and} \quad \big|f(x)\big| \leqslant c\,\mathrm{e}^{\lambda x^2} \quad \text{for all } x\in\mathbb{R}.$$

Remark 4.2.

1. If f satisfies the growth condition (GC) for positive constants c and λ, then there exists a positive constant ξ such that
$$\lambda = \frac{1}{4}\big[\big(|a|\vee|b|+\xi\big)\big]^{-2H} \quad \text{and} \quad \big|f(x)\big| \leqslant c\,\mathrm{e}^{\lambda x^2} \quad \text{for all } x\in\mathbb{R}. \tag{4.1}$$
Hence, $f(B_t^H)\in L^2(\Omega)$ for all $t\in(a-\xi,b+\xi)$.
2. If $f\in C^n(\mathbb{R})$ for some $n\geqslant 1$ and $f^{(n)}$ satisfies (GC) for some positive constants c and λ, then it is easy to see that there exists a positive constant c' such that for all $j=0,\dots,n$, $f^{(j)}$ satisfies (GC) for the constants c' and λ.

Lemma 4.3 (Itô formula). *Let $f\in C^\infty(\mathbb{R})$ such that for all $n\geqslant 0$, $f^{(n)}$ satisfies the growth condition* (GC) *of Definition* 4.1. *Then*

$$f'(B_t^H)1_{(a,b]}(t)\in \operatorname{Dom}^*\delta,$$

and

$$\delta\big[f'(B_t^H)1_{(a,b]}(t)\big] = f(B_b^H)-f(B_a^H)-\frac{1}{2}\int_a^b f''(B_t^H)\,\mathrm{d}|t|^{2H}.$$

Proof. It follows from the growth condition (GC) that

$$f'(B_t^H)1_{(a,b]}(t)\in L^2(\Omega\times\mathbb{R}) \tag{4.2}$$

and

$$f(B_b^H)-f(B_a^H)-\frac{1}{2}\int_a^b f''(B_t^H)\,\mathrm{d}|t|^{2H}\in L^2(\Omega).$$

By Remark 3.4.1, (4.2) implies that the conditions (i) and (ii) of Definition 3.3 are satisfied. It remains to verify condition (iii) of Definition 3.3, which reads as follows:

$$\begin{aligned} &c_H^2\int_a^b \mathbb{E}\big[f'(B_t^H)H_{n-1}\big(B^H(\varphi)\big)\big](\mathbf{D}_+^\alpha\mathbf{D}_-^\alpha\varphi)(t)\,\mathrm{d}t\\ &\quad = \mathbb{E}\bigg[\bigg\{f(B_b^H)-f(B_a^H)-\frac{1}{2}\int_a^b f''(B_t^H)\,\mathrm{d}|t|^{2H}\bigg\}H_n\big(B^H(\varphi)\big)\bigg], \end{aligned} \tag{4.3}$$

for all $n\in\mathbb{N}$ and $\varphi\in\Lambda_H^*$ with $\|\varphi\|_{\Lambda_H}=1$. Let us first assume that $a=0<b<\infty$. Then, (4.3) simplifies to

$$c_H^2\int_0^b \mathbb{E}\big[f'(B_t^H)H_{n-1}\big(B^H(\varphi)\big)\big](\mathbf{D}_+^\alpha\mathbf{D}_-^\alpha\varphi)(t)\,\mathrm{d}t$$

$$= \mathbb{E}\left[\left\{f(B_b^H) - f(0) - H\int_0^b f''(B_t^H)t^{2H-1}\,\mathrm{d}t\right\}H_n\big(B^H(\varphi)\big)\right]. \tag{4.4}$$

Denote

$$p(\sigma, y) := (2\pi\sigma)^{-1/2}\exp\left(-\frac{1}{2}\frac{y^2}{\sigma}\right),\ \ \sigma > 0,\ \ y \in \mathbb{R}, \tag{4.5}$$

and notice that $\frac{\partial p}{\partial \sigma} = \frac{1}{2}\frac{\partial^2 p}{\partial y^2}$. Hence, for all $n \in \mathbb{N}$ and $t \in (0, b]$,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}\big[f^{(n)}(B_t^H)\big] &= \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}} p(t^{2H}, y)f^{(n)}(y)\,\mathrm{d}y = \int_{\mathbb{R}} \frac{\partial}{\partial\sigma}p(t^{2H}, y)2Ht^{2H-1}f^{(n)}(y)\,\mathrm{d}y \\
&= Ht^{2H-1}\int_{\mathbb{R}} \frac{\partial^2}{\partial y^2}p(t^{2H}, y)f^{(n)}(y)\,\mathrm{d}y = Ht^{2H-1}\int_{\mathbb{R}} p(t^{2H}, y)f^{(n+2)}(y)\,\mathrm{d}y \\
&= Ht^{2H-1}\mathbb{E}\big[f^{(n+2)}(B_t^H)\big].
\end{aligned} \tag{4.6}$$

For $n = 0$, the left-hand side of (4.4) is zero. On the other hand, it follows from (4.6) that

$$\mathbb{E}\big[f(B_b^H)\big] - f(0) - H\int_0^b \mathbb{E}\big[f''(B_t^H)\big]t^{2H-1}\,\mathrm{d}t = 0,$$

which shows that (4.4) is fulfilled for $n = 0$.

Now, let $n \geqslant 1$. It follows from the integration by parts formula (3.8) that for all $t \in (0, b]$,

$$\langle 1_{(0,t]}, \varphi\rangle_{\Lambda_H} = c_H^2\int_0^t (\mathbf{D}_+^\alpha\mathbf{D}_-^\alpha\varphi)(s)\,\mathrm{d}s.$$

This and (4.6) imply that for all $t \in (0, b]$,

$$\begin{aligned}
&\frac{d}{\mathrm{d}t}\big(\mathbb{E}\big[f^{(n)}(B_t^H)\big]\langle 1_{(0,t]}, \varphi\rangle_{\Lambda_H}^n\big) \\
&\quad = Ht^{2H-1}\mathbb{E}\big[f^{(n+2)}(B_t^H)\big]\langle 1_{(0,t]}, \varphi\rangle_{\Lambda_H}^n + c_H^2 n\mathbb{E}\big[f^{(n)}(B_t^H)\big]\langle 1_{(0,t]}, \varphi\rangle_{\Lambda_H}^{n-1}(\mathbf{D}_+^\alpha\mathbf{D}_-^\alpha\varphi)(t).
\end{aligned}$$

Hence,

$$\begin{aligned}
\mathbb{E}\big[f^{(n)}(B_b^H)\big]\langle 1_{(0,b]}, \varphi\rangle_{\Lambda_H}^n &= H\int_0^b \mathbb{E}\big[f^{(n+2)}(B_t^H)\big]\langle 1_{(0,t]}, \varphi\rangle_{\Lambda_H}^n t^{2H-1}\,\mathrm{d}t \\
&\quad + c_H^2 n\int_0^b \mathbb{E}\big[f^{(n)}(B_t^H)\big]\langle 1_{(0,t]}, \varphi\rangle_{\Lambda_H}^{n-1}(\mathbf{D}_+^\alpha\mathbf{D}_-^\alpha\varphi)(t)\,\mathrm{d}t.
\end{aligned} \tag{4.7}$$

It follows from Theorem 1.1.2 and Proposition 1.3.1 in Nualart [22] that for all $k \geqslant 1$ and $\varphi \in \Lambda_H$ with $\|\varphi\|_{\Lambda_H} = 1$,

$$H_{k-1}\big(B^H(\varphi)\big)\varphi(t) \in \operatorname{Dom}\delta$$

and

$$\delta\big[H_{k-1}\big(B^H(\varphi)\big)\varphi(t)\big] = kH_k\big(B^H(\varphi)\big).$$

Hence, by iteratively applying the duality relation (3.2), we obtain

$$\mathbb{E}\big[f^{(n)}(B_b^H)\big]\langle 1_{(0,b]}, \varphi\rangle_{\Lambda_H}^n = n!\,\mathbb{E}\big[f(B_b^H)H_n\big(B^H(\varphi)\big)\big],$$
$$\mathbb{E}\big[f^{(n)}(B_t^H)\big]\langle 1_{(0,t]}, \varphi\rangle_{\Lambda_H}^{n-1} = (n-1)!\,\mathbb{E}\big[f'(B_t^H)H_{(n-1)}\big(B^H(\varphi)\big)\big] \quad \text{and}$$
$$\mathbb{E}\big[f^{(n+2)}(B_t^H)\big]\langle 1_{(0,t]}, \varphi\rangle_{\Lambda_H}^n = n!\,\mathbb{E}\big[f''(B_t^H)H_n\big(B^H(\varphi)\big)\big].$$

This together with (4.7) implies (4.4) for $n \geqslant 1$. Analogously, it can be shown that (4.3) is true in the case $-\infty < a < b = 0$. Now, it follows by additivity that (4.3) also holds in the general case $-\infty < a < b < \infty$. □

It can be derived from Theorem 8.1 in Berman [5] that the process $\{B_t^H,\ a \leqslant t \leqslant b\}$ has a continuous local time, that is, there exists a two-parameter process

$$\{\ell_{(a,t]}^y,\ a < t \leqslant b,\ y \in \mathbb{R}\}$$

which is continuous in t and y such that for every continuous function $g : \mathbb{R} \to \mathbb{R}$,

$$\int_a^t g(B_s^H)\,\mathrm{d}s = \int_{\mathbb{R}} g(y)\ell_{(a,t]}^y\,\mathrm{d}y, \quad a < t \leqslant b.$$

We define the weighted local time

$$\{L_{(a,t]}^y,\ a < t \leqslant b,\ y \in \mathbb{R}\}$$

as follows:

$$L_{(a,t]}^y := 2H\int_a^t \operatorname{sign}(s)|s|^{2H-1}\ell_{(a,\cdot]}^y(\mathrm{d}s).$$

It is also continuous in t and y, and for all continuous functions $g : \mathbb{R} \to \mathbb{R}$,

$$\int_a^t g(B_s^H)\,\mathrm{d}|s|^{2H} = \int_{\mathbb{R}} g(y)L_{(a,t]}^y\,\mathrm{d}y. \tag{4.8}$$

Theorem 4.4 (Tanaka formula). *Let* $y \in \mathbb{R}$. *Then*

$$1_{(y,\infty)}(B_t^H)1_{(a,b]}(t) \in \operatorname{Dom}^* \delta,$$

and

$$\delta\big[1_{(y,\infty)}(B_t^H)1_{(a,b]}(t)\big] = (B_b^H - y)^+ - (B_a^H - y)^+ - \frac{1}{2}L_{(a,b]}^y.$$

Proof. We set for all $k \geqslant 1$,

$$f_k(x) := \int_{-\infty}^x \int_{-\infty}^v p\left(\frac{1}{k}, z - y\right)\mathrm{d}z\,\mathrm{d}v,\ x \in \mathbb{R},$$

where the function p is given by (4.5). Then, for all $x \in \mathbb{R}$,

$$f_k(x) \to (x-y)^+ \quad \text{and} \quad f_k'(x) = \int_{-\infty}^x p\left(\frac{1}{k}, z - y\right)\mathrm{d}z \to \frac{1}{2}1_{\{y\}}(x) + 1_{(y,\infty)}(x), \quad \text{as } k \to \infty.$$

The functions f_k satisfy the conditions of Lemma 4.3. Therefore, for all $k \geqslant 1$, $f_k'(B_t^H)1_{(a,b]}(t) \in \mathrm{Dom}^* \delta$, and

$$\delta\big[f_k'(B_t^H)1_{(a,b]}(t)\big] = f_k(B_b^H) - f_k(B_a^H) - \frac{1}{2}\int_a^b f_k''(B_t^H)\,\mathrm{d}|t|^{2H}.$$

By Remark 3.4.2, the theorem follows from the following three facts:

(A) $f_k'(B_t^H)1_{(a,b]}(t) \to 1_{(y,\infty)}(B_t^H)1_{(a,b]}(t)$ in $L^2(\Omega \times \mathbb{R})$,
(B) $f_k(B_b^H) - f_k(B_a^H) \to (B_b^H - y)^+ - (B_a^H - y)^+$ in $L^2(\Omega)$, and
(C) $\int_a^b f_k''(B_t^H)\,\mathrm{d}|t|^{2H} \to L^y_{(a,b]}$ in $L^2(\Omega)$.

(A) and (B) are obvious. For the special case $a = 0 < b < \infty$, (C) is part of the statement of Proposition 2 in Coutin, Nualart and Tudor [7]. By using the fact that time-reversed fBm is again a fBm it can easily be checked that (C) is also true if $-\infty < a < b = 0$. Now, it follows by additivity that (C) is true for general $-\infty < a < b < \infty$. □

Remark 4.5. It immediately follows from Theorem 4.4 that for all $y \in \mathbb{R}$, the processes $1_{(-\infty,y]}(B_t^H)1_{(a,b]}(t)$ and $\mathrm{sign}(B_t^H - y)1_{(a,b]}(t)$ belong to $\mathrm{Dom}^* \delta$ and

$$\delta\big[-1_{(-\infty,y]}(B_t^H)1_{(a,b]}(t)\big] = (B_b^H - y)^- - (B_a^H - y)^- - \frac{1}{2}L^y_{(a,b]}$$

and

$$\delta\big[\mathrm{sign}(B_t^H - y)1_{(a,b]}(t)\big] = |B_b^H - y| - |B_a^H - y| - L^y_{(a,b]}.$$

If f is a convex function, we denote by f_-' its left-derivative and by f'' the measure given by $f''([y,z)) = f_-'(z) - f_-'(y)$, $-\infty < y < z < \infty$.

Theorem 4.6 (Itô–Tanaka formula). *Let f be a convex function such that*

(i) $f(B_a^H), f(B_b^H) \in L^2(\Omega)$,
(ii) $f_-'(B_t^H)1_{(a,b]}(t) \in L^2(\Omega \times \mathbb{R})$, *and*
(iii) $\int_{\mathbb{R}} |L|^y_{(a,b]} f''(\mathrm{d}y) \in L^2(\Omega)$,

where

$$|L|^y_{(a,t]} := 2H \int_a^t |s|^{2H-1} \ell^y_{(a,\cdot]}(\mathrm{d}s).$$

Then

$$f_-'(B_t^H)1_{(a,b]}(t) \in \mathrm{Dom}^* \delta \tag{4.9}$$

and

$$\delta\big[f_-'(B_t^H)1_{(a,b]}(t)\big] = f(B_b^H) - f(B_a^H) - \frac{1}{2}\int_{\mathbb{R}} L^y_{(a,b]} f''(\mathrm{d}y). \tag{4.10}$$

Proof. f'_- can be written as $f'_-(x) = f'_-(0) + g + h$, where

$$g(x) = \int_{[0,\infty)} 1_{(y,\infty)}(x) f''(\mathrm{d}y)$$

and

$$h(x) = -\int_{(-\infty,0)} 1_{(-\infty,y)}(x) f''(\mathrm{d}y).$$

Note that

$$f(x) = F(x) + G(x) + H(x),$$

where

$$F(x) = f(0) + f'_-(0)x,$$

$$G(x) = 1_{(0,\infty)}(x) \int_0^x g(z)\,\mathrm{d}z = \int_{[0,\infty)} (x-y)^+ f''(\mathrm{d}y),$$

and

$$H(x) = -1_{(-\infty,0)}(x) \int_x^0 h(z)\,\mathrm{d}z = \int_{(-\infty,0)} (x-y)^- f''(\mathrm{d}y).$$

It is enough to prove (4.9) and (4.10) for the cases

(A) $f(x) = F(x)$,
(B) $f(x) = G(x)$,
(C) $f(x) = H(x)$

separately. In case (A), (4.9) and (4.10) follow from (2.1). To prove (4.9) and (4.10) in the cases (B) and (C), let us first assume that $a = 0 < b < \infty$. Then, it can easily be checked that the conditions (i)–(iii) still hold if f is replaced by G or H. Therefore, in the cases (B) and (C), (4.9) and (4.10) follow from Theorems 4.4 and 3.7. If $-\infty < b < 0 = a$, the cases (B) and (C) can be dealt with analogously. For the general case $-\infty < a < b < \infty$ the result follows by additivity. □

Remark 4.7.

1. If in Theorem 4.6, $0 \leqslant a$, then condition (iii) reduces to $\int_{\mathbb{R}} L^y_{(a,b]} f''(\mathrm{d}y) \in L^2(\Omega)$.
2. If in Theorem 4.6, $f \in C^2(\mathbb{R})$, then by (4.8), formula (4.10) can be written as

$$\delta\big[f'(B^H_t)1_{(a,b]}(t)\big] = f(B^H_b) - f(B^H_a) - \frac{1}{2}\int_a^b f''(B^H_t)\,\mathrm{d}|t|^{2H}.$$

Corollary 4.8. *Let $f \in C^2(\mathbb{R})$ such that f'' satisfies the growth condition* (GC). *Then*

$$f'(B^H_t)1_{(a,b]}(t) \in \mathrm{Dom}^*\,\delta \tag{4.11}$$

and

$$\delta\big[f'(B_t^H)1_{(a,b]}(t)\big] = f(B_b^H) - f(B_a^H) - \frac{1}{2}\int_a^b f''(B_t^H)\,\mathrm{d}|t|^{2H}. \tag{4.12}$$

Proof.

$$f(x) = F(x) + G(x) + H(x),$$

where

$$F(x) = f(0) + f'(0)x,$$

$$G(x) = \int_0^t g(y)\,\mathrm{d}y, \quad g(y) = \int_0^y f_+''(z)\,\mathrm{d}z,$$

and

$$H(x) = \int_0^t h(y)\,\mathrm{d}y, \quad h(y) = \int_0^y f_-''(z)\,\mathrm{d}z.$$

Clearly, (4.11) and (4.12) hold for F. To see that (4.11) and (4.12) hold for G, note that f_+'' satisfies the growth condition (GC). It follows that also g and G fulfill (GC). Therefore G satisfies conditions (i) and (ii) of Theorem 4.6. The fact that

$$\int_{\mathbb{R}} |L|_{(a,b]}^y f_+''(\mathrm{d}y) = 2H\int_a^b f_+''(B_s^H)|s|^{2H-1}\,\mathrm{d}s,$$

shows that condition (iii) of Theorem 4.6 is also satisfied. Hence, Theorem 4.6 and Remark 13.2 imply that (4.11) and (4.12) hold for G. Analogously, it can be shown that (4.11) and (4.12) hold for H, and the corollary is proved. □

Corollary 4.9. *Let $f:\mathbb{R}\to\mathbb{R}$ be a function that satisfies one of the following two conditions*:

(i) $f \in C^2(\mathbb{R})$ *and* f'' *satisfies the growth condition* (GC).
(ii) f *is convex and satisfies the assumptions of Theorem* 4.6.

Moreover, let $A \in \mathcal{F}$ such that $f_-'(B_t^H)(\omega) = 0$ $P \times \mathrm{d}t$-almost everywhere on the product space $A \times (a,b]$. Then $\delta[f_-'(B_t^H)1_{(a,b]}(t)] = 0$ almost everywhere on A.

Proof. There exists a measurable subset $\widetilde{\Omega} \subset \Omega$ with $P[\widetilde{\Omega}] = 1$ such that for all $\omega \in \widetilde{\Omega}$, the function

$$B_t^H(\omega), \quad t \in (a,b],$$

is continuous and has no interval of constancy. For $\omega \in \widetilde{\Omega}$, we set

$$m(\omega) = \min_{a\leqslant t\leqslant b} B_t^H(\omega) \quad \text{and} \quad M(\omega) = \max_{a\leqslant t\leqslant b} B_t^H(\omega).$$

It follows from the assumptions and Corollary 4.8 or Theorem 4.6, respectively, that for almost all $\omega \in A \cap \widetilde{\Omega}$:

$$f_-' = 0 \quad \text{on the interval } \big(m(\omega), M(\omega)\big]$$

and

$$\delta\big[f'(B_t^H)1_{(a,b]}(t)\big](\omega) = f\big(B_b^H(\omega)\big) - f\big(B_a^H(\omega)\big) - \frac{1}{2}\int_{\mathbb{R}} L^y_{(a,b]}(\omega) f''(\mathrm{d}y) = 0.$$

This proves the corollary. □

Let $f \in C^2(\mathbb{R})$. Then we can define $\delta[f'(B_t^H)1_{(a,b]}(t)]$ by localization in the following way:
For all $k \geqslant 1$, we set

$$f_k''(x) := \begin{cases} f''(-k) & \text{if } x < -k, \\ f''(x) & \text{if } -k \leqslant x \leqslant k, \\ f''(k) & \text{if } x > k, \end{cases} \tag{4.13}$$

$$f_k'(x) := f'(0) + \int_0^x f_k''(y)\,\mathrm{d}y, \quad x \in \mathbb{R}, \tag{4.14}$$

and

$$f_k(x) = f(0) + \int_0^x f_k'(y)\,\mathrm{d}y, \quad x \in \mathbb{R}. \tag{4.15}$$

The sequence of sets

$$\Omega_k = \Big\{\sup_{a<t\leqslant b} |B_t^H| < k\Big\}, \quad k \geqslant 1,$$

is increasing, and

$$\bigcup_{k=1}^{\infty} \Omega_k = \Omega, \tag{4.16}$$

almost surely. It can easily be seen that

$$\text{all } f_k \text{ satisfy the conditions of Corollary 4.8} \tag{4.17}$$

and for all $k \geqslant 1$,

$$f_k'(B_t^H)1_{(a,b]}(t) = f'(B_t^H)1_{(a,b]}(t) \ \text{ on } \Omega_k \times \mathbb{R}. \tag{4.18}$$

If we define

$$\delta\big[f'(B_t^H)1_{(a,b]}(t)\big] := \delta\big[f_k'(B_t^H)1_{(a,b]}(t)\big] \quad \text{on } \Omega_k \tag{4.19}$$

for all $k \geqslant 1$, we get from Corollary 4.8 that

$$\delta\big[f'(B_t^H)1_{(a,b]}(t)\big] = f(B_b^H) - f(B_a^H) - \frac{1}{2}\int_a^b f''(B_t^H)\,\mathrm{d}|t|^{2H}. \tag{4.20}$$

It follows from Corollary 4.9 that we obtain the same formula (4.20) if in (4.19) we replace the pair of sequences $\{\Omega_k\}_{k=1}^\infty$ and $\{f_k\}_{k=1}^\infty$ by another pair $\{\widetilde{\Omega}_k\}_{k=1}^\infty$, $\{\tilde{f}_k\}_{k=1}^\infty$ that satisfies (4.16), (4.17) and (4.18).
For a convex function $f:\mathbb{R} \to \mathbb{R}$, we can modify (4.13), (4.14) and (4.15) to

$$f_k''(E) := f''\big([-k,k) \cap E\big), \quad E \text{ a Borel set},$$

$$f_{-,k}'(x) := \begin{cases} f_-'(0) + f_k''([0,x)) & \text{if } x \geqslant 0, \\ f_-'(0) - f_k''([x,0)) & \text{if } x < 0 \end{cases}$$

and

$$f_k(x) = f(0) + \int_0^x f'_{-,k}(y)\,\mathrm{d}y, \qquad x \in \mathbb{R}.$$

Then the functions f_k satisfy the conditions of Theorem 4.6, and the same localization procedure as above yields

$$\delta\big[f'_-(B^H_t)1_{(a,b]}(t)\big] = f(B^H_b) - f(B^H_a) - \frac{1}{2}\int_{\mathbb{R}} L^y_{(a,b]} f''(\mathrm{d}y). \tag{4.21}$$

As above, it follows from Corollary 4.9 that a different localization yields the same formula (4.21).

5. Symmetric integration with respect to fractional Brownian motion

In this section we show that the Russo–Vallois symmetric integral of a general smooth function of B^H with respect to B^H exists if and only if $H > \frac{1}{6}$. Throughout the section, a and b are two real numbers such that $-\infty < a < b < \infty$, and $g : \mathbb{R} \to \mathbb{R}$ is a continuous function.

Definition 5.1. If the limit in probability

$$P - \lim_{\varepsilon \searrow 0} \int_a^b g(B^H_t) \frac{B^H_{t+\varepsilon} - B^H_{t-\varepsilon}}{2\varepsilon}\,\mathrm{d}t \tag{5.1}$$

exists, we call it symmetric integral and denote it by

$$\int_a^b g(B^H_t)\,\mathrm{d}^0 B^H_t.$$

Remark 5.2. The symmetric integral was introduced by Russo and Vallois [28]. According to their definition, the symmetric integral of a stochastic process $(X_t)_{t\in[a,b]}$ with respect to another stochastic process $(Y_t)_{t\in[a,b]}$ is given by

$$P - \lim_{\varepsilon \searrow 0} \int_a^b X_t \frac{Y_{(t+\varepsilon)\wedge b} - Y_{(t-\varepsilon)\vee a}}{2\varepsilon}\,\mathrm{d}t.$$

Our Definition 5.1 looks slightly different. However, since g is continuous, we have for $\varepsilon \in (0, \frac{b-a}{2})$,

$$\begin{aligned}
&\int_a^b g(B^H_t) \frac{B^H_{t+\varepsilon} - B^H_{t-\varepsilon}}{2\varepsilon}\,\mathrm{d}t - \int_a^b g(B^H_t) \frac{B^H_{(t+\varepsilon)\wedge b} - B^H_{(t-\varepsilon)\vee a}}{2\varepsilon}\,\mathrm{d}t \\
&\quad = \frac{1}{2\varepsilon}\int_{b-\varepsilon}^b g(B^H_t)(B^H_{t+\varepsilon} - B^H_b)\,\mathrm{d}t - \frac{1}{2\varepsilon}\int_a^{a+\varepsilon} g(B^H_t)(B^H_{t-\varepsilon} - B^H_a)\,\mathrm{d}t \\
&\quad \overset{(\varepsilon\searrow 0)}{\longrightarrow} 0 \quad \text{almost surely.}
\end{aligned}$$

Hence, the limit (5.1) exists if and only if

$$P - \lim_{\varepsilon \searrow 0} \int_a^b g(B_t^H) \frac{B^H_{(t+\varepsilon)\wedge b} - B^H_{(t-\varepsilon)\vee a}}{2\varepsilon} \,\mathrm{d}t \tag{5.2}$$

exists, and if they exist, the limits (5.1) and (5.2) are the same.

Note that if $h:\mathbb{R} \to \mathbb{R}$ is a continuous function, then

$$\begin{aligned}\lim_{\varepsilon \searrow 0} \int_a^b h(t) \frac{h(t+\varepsilon) - h(t-\varepsilon)}{2\varepsilon} \,\mathrm{d}t &= \lim_{\varepsilon \searrow 0} \frac{1}{2\varepsilon} \left(\int_a^b h(t)h(t+\varepsilon)\,\mathrm{d}t - \int_{a-\varepsilon}^{b-\varepsilon} h(t+\varepsilon)h(t)\,\mathrm{d}t \right) \\ &= \lim_{\varepsilon \searrow 0} \frac{1}{2\varepsilon} \left(\int_{b-\varepsilon}^{b} h(t)h(t+\varepsilon)\,\mathrm{d}t - \int_{a-\varepsilon}^{a} h(t)h(t+\varepsilon)\,\mathrm{d}t \right) \\ &= \frac{1}{2} h^2(b) - \frac{1}{2} h^2(a).\end{aligned}$$

It follows that for all $H \in (0,1)$,

$$\int_a^b B_t^H \frac{B^H_{t+\varepsilon} - B^H_{t-\varepsilon}}{2\varepsilon} \,\mathrm{d}t \xrightarrow{(\varepsilon \searrow 0)} \frac{1}{2}(B_b^H)^2 - \frac{1}{2}(B_a^H)^2 \quad \text{almost surely},$$

which implies that

$$\int_a^b B_t^H \,\mathrm{d}^0 B_t^H = \frac{1}{2}(B_b^H)^2 - \frac{1}{2}(B_a^H)^2$$

for all $H \in (0,1)$. Since for $H \in [\frac{1}{2}, 1)$, B^H has finite quadratic variation, it follows from Theorem 2.1 of Russo and Vallois [29] that for all $H \in [\frac{1}{2}, 1)$ and $g \in C^1(\mathbb{R})$,

$$\int_a^b g(B_t^H) \,\mathrm{d}^0 B_t^H = G(B_b^H) - G(B_a^H), \tag{5.3}$$

where G is given by

$$G(x) := \int_0^x g(y)\,\mathrm{d}y, \quad x \in \mathbb{R}.$$

In Theorem 4.1 of Russo and Vallois [30] it is proved that for $H = \frac{1}{2}$, formula (5.3) even holds if $g \in L^2_{\mathrm{loc}}(\mathbb{R})$.

For $H \in (0, \frac{1}{2})$ the paths of B^H are rougher than the paths of Brownian motion (recall that B^H has infinite quadratic variation if $H \in (0, \frac{1}{2})$). However, it follows from what is shown in Section 4 of Alòs, León and Nualart [1] that if $H \in (\frac{1}{4}, \frac{1}{2})$, then (5.3) is still true for all $g \in C^1(\mathbb{R})$. In Theorem 4.1 of Gradinaru, Russo and Vallois [12], formula (5.3) is proved for $H = \frac{1}{4}$ and $g \in C^3(\mathbb{R})$.

In this section we show that for general $g \in C^3(\mathbb{R})$, $H = \frac{1}{6}$ is the critical value for the existence of the symmetric integral in (5.3). The main result of this section is the following

Theorem 5.3.

(a) *Let* $g \in C^3(\mathbb{R})$. *Then for every* $H \in (\frac{1}{6}, \frac{1}{2})$,

$$\int_a^b g(B_t^H)\,\mathrm{d}^0 B_t^H = G(B_b^H) - G(B_a^H),$$

where G *is given by* $G(x) := \int_0^x g(y)\,\mathrm{d}y$, $x \in \mathbb{R}$.

(b) *If* $H \in (0, \frac{1}{6}]$, *then*

$$\int_a^b (B_t^H)^2\,\mathrm{d}^0 B_t^H$$

does not exist.

For the proof of Theorem 5.3 we need the following three lemmas and the subsequent proposition.

Lemma 5.4. *Let* $H \in (0, \frac{1}{2})$ *and* $g \in C^2(\mathbb{R})$ *such that* g'' *satisfies the growth condition* (GC) *of Definition* 4.1. *Then there exists a* $\xi > 0$ *such that for all* $r, s, t \in (a - \xi, b + \xi)$,

$$\mathbb{E}\big[g(B_t^H)\big(g(B_s^H) - g(B_r^H)\big)\big] = \int_r^s \left\{\mathbb{E}\big[g'(B_t^H)g'(B_v^H)\big]\frac{\partial}{\partial v}R_H(t,v) + \frac{1}{2}\mathbb{E}\big[g(B_t^H)g''(B_v^H)\big]\frac{\partial |v|^{2H}}{\partial v}\right\}\mathrm{d}v.$$

Proof. By Remarks 4.2, there exist positive constants c, λ, ξ such that g, g' and g'' satisfy (4.1) for c, λ and ξ. Hence, it follows from Corollary 4.8 that for all $r, s \in (a - \xi, b + \xi)$,

$$g(B_s^H) - g(B_r^H) = \delta\big[g'(B_v^H)1_{(r,s]}(v)\big] + \frac{1}{2}\int_r^s g''(B_v^H)\frac{\partial |v|^{2H}}{\partial v}\,\mathrm{d}v,$$

where $1_{(r,s]}(v) := -1_{(s,r]}(v)$ if $s < r$. By (iii) of Definition 3.3, this means that for all $n \in \mathbb{N}$ and $\varphi \in \Lambda_H^*$ with $\|\varphi\|_{\Lambda_H} = 1$, the following equation holds:

$$\begin{aligned} &c_H^2 \int_r^s \mathbb{E}\big[g'(B_v^H)H_{n-1}\big(B^H(\varphi)\big)\big]\mathbf{D}_+^\alpha \mathbf{D}_-^\alpha \varphi(v)\,\mathrm{d}v \\ &\quad = \mathbb{E}\big[\big(g(B_s^H) - g(B_r^H)\big)H_n\big(B^H(\varphi)\big)\big] - \frac{1}{2}\int_r^s \mathbb{E}\big[g''(B_v^H)H_n\big(B^H(\varphi)\big)\big]\frac{\partial |v|^{2H}}{\partial v}\,\mathrm{d}v. \end{aligned} \tag{5.4}$$

An inspection of the arguments that lead to Corollary 4.8 shows that (5.4) still holds if φ is replaced by the function $1_{(0,t]}$. Then, it follows by approximation that the following version of (5.4) is also true:

$$\begin{aligned} &c_H^2 \int_r^s \mathbb{E}\big[g'(B_v^H)g'(B_t^H)\big]\mathbf{D}_+^\alpha \mathbf{D}_-^\alpha 1_{(0,t]}(v)\,\mathrm{d}v \\ &\quad = \mathbb{E}\big[\big(g(B_s^H) - g(B_r^H)\big)g(B_t^H)\big] - \frac{1}{2}\int_r^s \mathbb{E}\big[g''(B_v^H)g(B_t^H)\big]\frac{\partial |v|^{2H}}{\partial v}\,\mathrm{d}v. \end{aligned}$$

It remains to be shown that

$$c_H^2 \mathbf{D}_+^{1/2-H} \mathbf{D}_-^{1/2-H} 1_{(0,t]}(v) = \frac{\partial}{\partial v} R(t,v).$$

But this follows immediately from

$$R_H(t,v) = \langle 1_{(0,t]}, 1_{(0,v]} \rangle_{\Lambda_H} = c_H^2 \langle \mathbf{D}_-^{1/2-H} 1_{(0,t]}, \mathbf{D}_-^{1/2-H} 1_{(0,v]} \rangle_{L^2(\mathbb{R})} = c_H^2 \int_0^v (\mathbf{D}_+^{1/2-H} \mathbf{D}_-^{1/2-H} 1_{(0,t]})(x)\, \mathrm{d}x,$$

which is a consequence of the fractional integration by parts formula (see Corollary 2 to Theorem 6.2 in [32]). □

Lemma 5.5. *For every* $H \in (0, \frac{1}{2})$ *there exists a constant* $\alpha_H > 0$ *such that for all* $h > 0$ *and* $x \in \mathbb{R} \setminus \{0\}$,

$$\frac{1}{2h} \big| |x+h|^{2H} - |x-h|^{2H} \big| \leqslant \alpha_H |x|^{2H-1}. \tag{5.5}$$

Proof. Fix $h > 0$. For all $x \in [h, \infty)$,

$$\frac{(x+h)^{2H} - (x-h)^{2H}}{2h} \frac{1}{x^{2H-1}} = \frac{(1+y)^{2H} - (1-y)^{2H}}{2y},$$

where $y := \frac{h}{x} \in (0, 1]$. The function

$$\frac{(1+y)^{2H} - (1-y)^{2H}}{2y}$$

is continuous on $(0, 1]$, and

$$\lim_{y \searrow 0} \frac{(1+y)^{2H} - (1-y)^{2H}}{2y} = 2H.$$

Therefore,

$$\alpha_H := \sup_{y \in (0,1]} \frac{(1+y)^{2H} - (1-y)^{2H}}{2y} < \infty,$$

which shows that

$$\frac{(x+h)^{2H} - (x-h)^{2H}}{2h} \leqslant \alpha_H x^{2H-1} \tag{5.6}$$

for all $h > 0$ and $x \in [h, \infty)$. For $h > 0$ and $x \in (0, h)$, we have

$$\frac{1}{2h} \big| |x+h|^{2H} - |x-h|^{2H} \big| \leqslant \frac{1}{2h} (2h)^{2H} \leqslant \alpha_H h^{2H-1} \leqslant \alpha_H x^{2H-1}. \tag{5.7}$$

Since both sides of (5.5) are even in x, the lemma follows from (5.6) and (5.7). □

Lemma 5.6. *For all* $H > 0$ *there exists a constant* $\beta_H > 0$ *such that for every* $\eta \in (0, 1)$ *and all* $t, s \in \mathbb{R}$ *such that* $t \neq s$,

$$\frac{1}{2\eta} \int_{s-\eta}^{s+\eta} |v-s|^{2H} |v-t|^{2H-1} \, \mathrm{d}v \leqslant \begin{cases} \beta_H, & \text{if } H > \frac{1}{4}, \\ \beta_H |t-s|^{4H-1}, & \text{if } H \in (0, \frac{1}{4}]. \end{cases} \tag{5.8}$$

Proof. Since both sides of (5.8) are even in $t-s$, it is enough to prove the lemma for the case $s-t>0$. It can easily be checked that there exists a constant $\beta_H>0$ such that for all $z>0$,

$$\frac{1}{2z}\int_{-z}^{z}|y|^{2H}|y+1|^{2H-1}\,\mathrm{d}y\leqslant\begin{cases}\beta_H z^{4H-1}, & \text{if } H>\frac14,\\ \beta_H, & \text{if } H\in(0,\frac14].\end{cases}$$

It follows that for every $\eta\in(0,1)$ and all $t,s\in\mathbb{R}$ such that $s-t>0$,

$$\begin{aligned}\frac{1}{2\eta}\int_{s-\eta}^{s+\eta}|v-s|^{2H}|v-t|^{2H-1}\,\mathrm{d}v&=\frac{1}{2\eta}\int_{-\eta}^{\eta}|x|^{2H}|x+s-t|^{2H-1}\,\mathrm{d}x\\
&=(s-t)^{4H-1}\frac{s-t}{2\eta}\int_{-\eta/(s-t)}^{\eta/(s-t)}|y|^{2H}|y+1|^{2H-1}\,\mathrm{d}y\\
&\leqslant\begin{cases}(s-t)^{4H-1}\beta_H(\frac{\eta}{s-t})^{4H-1}, & \text{if } H>\frac14,\\ (s-t)^{4H-1}\beta_H, & \text{if } H\in(0,\frac14]\end{cases}\\
&\leqslant\begin{cases}\beta_H, & \text{if } H>\frac14,\\ (s-t)^{4H-1}\beta_H, & \text{if } H\in(0,\frac14]\end{cases}\end{aligned}$$

which proves the lemma. □

Proposition 5.7. *Let $H\in(\frac16,\frac12)$ and $g\in C^3(\mathbb{R})$ such that g''' satisfies the growth condition* (GC) *of Definition* 4.1. *Then*

$$L^2-\lim_{\varepsilon\searrow 0}\int_a^b g(B_t^H)\frac{B_{t+\varepsilon}^H-B_{t-\varepsilon}^H}{2\varepsilon}\,\mathrm{d}t=G(B_b^H)-G(B_a^H),$$

where G is given by $G(x):=\int_0^x g(y)\,\mathrm{d}y$, $x\in\mathbb{R}$.

Proof. By Remarks 4.2, there exist positive constants c,λ,ξ such that g,g',g'' and g''' satisfy (GC) for c,λ,ξ. It follows from Lemma 1.2.2 of Nualart [22] that for all $t\in(a,b]$ and $\varepsilon>0$,

$$g(B_t^H)(B_{t+\varepsilon}^H-B_{t-\varepsilon}^H)=\delta\big[g(B_t^H)1_{(t-\varepsilon,t+\varepsilon]}\big]+g'(B_t^H)\langle 1_{(0,t]},1_{(t-\varepsilon,t+\varepsilon]}\rangle_{\Lambda_H}.$$

Hence,

$$\int_a^b g(B_t^H)\frac{B_{t+\varepsilon}^H-B_{t-\varepsilon}^H}{2\varepsilon}\,\mathrm{d}t=\frac{1}{2\varepsilon}(A_\varepsilon+B_\varepsilon),$$

where

$$A_\varepsilon:=\int_a^b\delta\big[g(B_t^H)1_{(t-\varepsilon,t+\varepsilon]}\big]\,\mathrm{d}t$$

and

$$B_\varepsilon:=\int_a^b g'(B_t^H)\langle 1_{(0,t]},1_{(t-\varepsilon,t+\varepsilon]}\rangle_{\Lambda_H}\,\mathrm{d}t.$$

It can easily be checked that

$$\frac{1}{2\varepsilon}B_\varepsilon \overset{(\varepsilon\searrow 0)}{\longrightarrow} \frac{1}{2}\int_a^b g'(B_t^H)\,\mathrm{d}|t|^{2H} \quad \text{in } L^2(\Omega).$$

It remains to prove that

$$L^2-\lim_{\varepsilon\searrow 0}\frac{1}{2\varepsilon}A_\varepsilon = G(B_b^H)-G(B_a^H)-\frac{1}{2}\int_a^b g'(B_t^H)\,\mathrm{d}|t|^{2H}. \tag{5.9}$$

Assume that

$$\lim_{\varepsilon,\eta\searrow 0}\frac{1}{4\varepsilon\eta}\mathbb{E}[A_\varepsilon A_\eta] \text{ exists.} \tag{5.10}$$

This implies that $(\frac{1}{2\varepsilon}A_\varepsilon)_{\varepsilon>0}$ is Cauchy as $\varepsilon\searrow 0$, and therefore converges in $L^2(\Omega)$. It follows from Theorem 3.7 that

$$A_\varepsilon = \delta\left[\int_a^b g(B_t^H)1_{(t-\varepsilon,t+\varepsilon]}(v)\,\mathrm{d}t\right] = \delta\left[1_{(a-\varepsilon,b+\varepsilon]}(v)\int_{(v-\varepsilon)\vee a}^{(v+\varepsilon)\wedge b} g(B_t^H)\,\mathrm{d}t\right],$$

and it can easily be checked that

$$\frac{1}{2\varepsilon}1_{(a-\varepsilon,b+\varepsilon]}(v)\int_{(v-\varepsilon)\vee a}^{(v+\varepsilon)\wedge b} g(B_t^H)\,\mathrm{d}t \overset{(\varepsilon\searrow 0)}{\longrightarrow} g(B_v^H)1_{(a,b]}(v) \quad \text{in } L^2(\Omega\times\mathbb{R}).$$

Hence, it follows from Remark 3.4.2 that

$$L^2-\lim_{\varepsilon\searrow 0}\frac{1}{2\varepsilon}A_\varepsilon = \delta\big[g(B_t^H)1_{(a,b]}(t)\big]. \tag{5.11}$$

Since by Corollary 4.8,

$$\delta\big[g(B_t^H)1_{(a,b]}(t)\big] = G(B_b^H)-G(B_a^H)-\frac{1}{2}\int_a^b g'(B_t^H)\,\mathrm{d}|t|^{2H},$$

(5.11) implies (5.9).

To complete the proof we have to show (5.10). Clearly, for all $t\in(a,b]$ and $\varepsilon\in(0,\xi)$,

$$g(B_t^H)1_{(t-\varepsilon,t+\varepsilon]}(v)\in\mathbb{L}^{1,2}$$

(check Definition 1.3.2 in Nualart [22]). Hence, it follows by property (3) on page 39 of Nualart [22] that for all $t,s\in(a,b]$ and $\varepsilon,\eta\in(0,\xi)$,

$$\begin{aligned}
&\mathbb{E}\big[\delta\big[g(B_t^H)1_{(t-\varepsilon,t+\varepsilon]}\big]\delta\big[g(B_s^H)1_{(s-\eta,s+\eta]}\big]\big]\\
&\quad= \mathbb{E}\big[g(B_t^H)g(B_s^H)\big]\langle 1_{(t-\varepsilon,t+\varepsilon]},1_{(s-\eta,s+\eta]}\rangle_{\Lambda_H}\\
&\qquad+\mathbb{E}\big[g'(B_t^H)g'(B_s^H)\big]\langle 1_{(0,t]},1_{(s-\eta,s+\eta]}\rangle_{\Lambda_H}\langle 1_{(t-\varepsilon,t+\varepsilon]},1_{(0,s]}\rangle_{\Lambda_H}\\
&\quad= \mathbb{E}\big[g(B_t^H)g(B_s^H)\big]\big[R_H(t+\varepsilon,s+\eta)-R_H(t+\varepsilon,s-\eta)-R_H(t-\varepsilon,s+\eta)+R_H(t-\varepsilon,s-\eta)\big]\\
&\qquad+\mathbb{E}\big[g'(B_t^H)g'(B_s^H)\big]\big[R_H(t,s+\eta)-R_H(t,s-\eta)\big]\big[R_H(t+\varepsilon,s)-R_H(t-\varepsilon,s)\big].
\end{aligned}$$

Therefore,

$$\frac{1}{4\varepsilon\eta}\mathbb{E}[A_\varepsilon A_\eta] = \frac{1}{4\varepsilon\eta}(C_{\varepsilon,\eta} + D_{\varepsilon,\eta}),$$

where

$$C_{\varepsilon,\eta} := \int_a^b \int_a^b \mathbb{E}\big[g(B_t^H)g(B_s^H)\big] \times \big[R_H(t+\varepsilon, s+\eta) - R_H(t+\varepsilon, s-\eta) - R_H(t-\varepsilon, s+\eta) + R_H(t-\varepsilon, s-\eta)\big]\,\mathrm{d}s\,\mathrm{d}t$$

and

$$D_{\varepsilon,\eta} := \int_a^b \int_a^b \mathbb{E}\big[g'(B_t^H)g'(B_s^H)\big]\big[R(t, s+\eta) - R(t, s-\eta)\big]\big[R(t+\varepsilon, s) - R(t-\varepsilon, s)\big]\,\mathrm{d}s\,\mathrm{d}t.$$

Furthermore,

$$C_{\varepsilon,\eta} = C_{\varepsilon,\eta}^1 + C_{\varepsilon,\eta}^2 + C_{\varepsilon,\eta}^2,$$

where

$$C_{\varepsilon,\eta}^1 := \int_a^b \int_a^b \mathbb{E}\big[g(B_t^H)\big(g(B_{s-\eta}^H) - g(B_{s+\eta}^H)\big)\big]\big[R_H(t+\varepsilon, s) - R_H(t-\varepsilon, s)\big]\,\mathrm{d}s\,\mathrm{d}t,$$

$$C_{\varepsilon,\eta}^2 := \int_a^b \left(\int_b^{b+\eta} - \int_a^{a+\eta}\right)\mathbb{E}\big[g(B_t^H)g(B_{s-\eta}^H)\big]\big[R_H(t+\varepsilon, s) - R_H(t-\varepsilon, s)\big]\,\mathrm{d}s\,\mathrm{d}t$$

and

$$C_{\varepsilon,\eta}^3 := \int_a^b \left(\int_{b-\eta}^{b} - \int_{a-\eta}^{a}\right)\mathbb{E}\big[g(B_t^H)g(B_{s+\eta}^H)\big[R_H(t+\varepsilon, s) - R_H(t-\varepsilon, s)\big]\big]\,\mathrm{d}s\,\mathrm{d}t.$$

It can easily be checked that

$$\lim_{\varepsilon,\eta\searrow 0} \frac{1}{4\varepsilon\eta}(C_{\varepsilon,\eta}^2 + C_{\varepsilon,\eta}^3) = \int_a^b \left(\mathbb{E}\big[g(B_t^H)g(B_b^H)\big]\frac{\partial R_H}{\partial t}(t,b) - \mathbb{E}\big[g(B_t^H)g(B_a^H)\big]\frac{\partial R_H}{\partial t}(t,a)\right)\mathrm{d}t.$$

It follows from Lemma 5.4 that

$$-C_{\varepsilon,\eta}^1 = E_{\varepsilon,\eta} + F_{\varepsilon,\eta},$$

where

$$E_{\varepsilon,\eta} := \int_a^b \int_a^b \int_{s-\eta}^{s+\eta} \mathbb{E}\big[g'(B_t^H)g'(B_v^H)\big]\frac{\partial R_H}{\partial v}(t,v)\,\mathrm{d}v\big[R_H(t+\varepsilon, s) - R_H(t-\varepsilon, s)\big]\,\mathrm{d}s\,\mathrm{d}t$$

and

$$F_{\varepsilon,\eta} := \frac{1}{2}\int_a^b \int_a^b \int_{s-\eta}^{s+\eta} \mathbb{E}\big[g(B_t^H)g''(B_v^H)\big]\frac{\partial |v|^{2H}}{\partial v}\,\mathrm{d}v\big[R_H(t+\varepsilon, s) - R_H(t-\varepsilon, s)\big]\,\mathrm{d}s\,\mathrm{d}t.$$

Obviously,

$$\lim_{\varepsilon,\eta\searrow 0}\frac{1}{4\varepsilon\eta}F_{\varepsilon,\eta}=H\int_a^b\int_a^b \mathbb{E}\big[g(B_t^H)g''(B_s^H)\big]\operatorname{sign}(s)|s|^{2H-1}\frac{\partial R_H}{\partial t}(t,s)\,\mathrm{d}s\,\mathrm{d}t.$$

Therefore, it is enough to prove that

$$\lim_{\varepsilon,\eta\searrow 0}\frac{1}{4\varepsilon\eta}(E_{\varepsilon,\eta}-D_{\varepsilon,\eta})\text{ exists.}$$

We have

$$\begin{aligned}&E_{\varepsilon,\eta}-D_{\varepsilon,\eta}\\&\quad=\int_a^b\int_a^b\int_{s-\eta}^{s+\eta}\mathbb{E}\big[g'(B_t^H)\big(g'(B_v^H)-g'(B_s^H)\big)\big]\frac{\partial R_H}{\partial v}(t,v)\,\mathrm{d}v\big[R_H(t+\varepsilon,s)-R_H(t-\varepsilon,s)\big]\,\mathrm{d}s\,\mathrm{d}t.\end{aligned}\tag{5.12}$$

By Lemma 5.4 and the estimate

$$\left|\frac{\partial R_H}{\partial w}(t,w)\right|\leqslant H\big(|w|^{2H-1}+|w-t|^{2H-1}\big),\tag{5.13}$$

we obtain

$$\begin{aligned}&\big|\mathbb{E}\big[g'(B_t^H)\big(g'(B_v^H)-g'(B_s^H)\big)\big]\big|\\&\quad\leqslant H\int_s^v\big|\mathbb{E}\big[g''(B_t^H)g''(B_w^H)\big]\big|\big(|w|^{2H-1}+|w-t|^{2H-1}\big)\,\mathrm{d}w+H\int_s^v\big|\mathbb{E}\big[g'(B_t^H)g'''(B_w^H)\big]\big||w|^{2H-1}\,\mathrm{d}w\\&\quad\leqslant\gamma_H|v-s|^{2H},\end{aligned}\tag{5.14}$$

where

$$\gamma_H:=\max_{t\in[a,b],\ w\in[a-\xi,b+\xi]}\big|\mathbb{E}\big[g''(B_t^H)g''(B_w^H)\big]\big|+\frac{1}{2}\max_{t\in[a,b],\ w\in[a-\xi,b+\xi]}\big|\mathbb{E}\big[g'(B_t^H)g'''(B_w^H)\big]\big|.$$

It follows from Lemma 5.5 that

$$\frac{1}{\varepsilon}\big[R_H(t+\varepsilon,s)-R_H(t-\varepsilon,s)\big]\leqslant\alpha_H\big(|t|^{2H-1}+|t-s|^{2H-1}\big).\tag{5.15}$$

By plugging (5.14), (5.13) and (5.15) into (5.12) we obtain

$$\frac{1}{4\varepsilon\eta}|E_{\varepsilon,\eta}-D_{\varepsilon,\eta}|\leqslant\frac{\gamma_H H\alpha_H}{4\eta}\int_a^b\int_a^b\int_{s-\eta}^{s+\eta}|v-s|^{2H}\big(|v|^{2H-1}+|v-t|^{2H-1}\big)\big(|t|^{2H-1}+|t-s|^{2H-1}\big)\,\mathrm{d}v\,\mathrm{d}s\,\mathrm{d}t.$$

Obviously, for all $t,s\in\mathbb{R}$, such that $t\neq 0$ and $t\neq s$,

$$\frac{1}{2\eta}\int_{s-\eta}^{s+\eta}|v-s|^{2H}\big(|v|^{2H-1}+|v-t|^{2H-1}\big)\big(|t|^{2H-1}+|t-s|^{2H-1}\big)\,\mathrm{d}v\to 0$$

as $\eta\searrow 0$. On the other hand, it follows from Lemma 5.6 that for all $\eta\in(0,\xi\wedge 1)$,

$$\frac{1}{2\eta}\int_{s-\eta}^{s+\eta}|v-s|^{2H}\big(|v|^{2H-1}+|v-t|^{2H-1}\big)\big(|t|^{2H-1}+|t-s|^{2H-1}\big)\,\mathrm{d}v$$
$$\leqslant\begin{cases}2\beta_H(|t|^{2H-1}+|t-s|^{2H-1}), & \text{if } H\in(\frac{1}{4},\frac{1}{2}),\\ \beta_H(|s|^{4H-1}+|t-s|^{4H-1})(|t|^{2H-1}+|t-s|^{2H-1}), & \text{if } H\in(\frac{1}{6},\frac{1}{4}].\end{cases}$$

Hence, it follows from Lebesgue's dominated convergence theorem that

$$\lim_{\varepsilon,\eta\searrow 0}\frac{1}{4\varepsilon\eta}(E_{\varepsilon,\eta}-D_{\varepsilon,\eta})=0,$$

and the proposition is proved. □

Proof of Theorem 5.3(a). Let $H\in(\frac{1}{6},\frac{1}{2})$ and $g\in C^3(\mathbb{R})$. For all $k\geqslant 1$, we set

$$g_k'''(x):=\begin{cases}g'''(-k), & \text{if } x<-k,\\ g'''(x), & \text{if } -k\leqslant x\leqslant k,\\ g'''(k), & \text{if } x>k,\end{cases}$$

$$g_k''(x):=g''(0)+\int_0^x g_k'''(y)\,\mathrm{d}y,\quad x\in\mathbb{R},$$

$$g_k'(x):=g'(0)+\int_0^x g_k''(y)\,\mathrm{d}y,\quad x\in\mathbb{R},$$

$$g_k(x):=g(0)+\int_0^x g_k'(y)\,\mathrm{d}y,\quad x\in\mathbb{R},$$

and

$$G_k(x):=\int_0^x g_k(y)\,\mathrm{d}y,\quad x\in\mathbb{R}.$$

The sequence of sets

$$\Omega_k:=\Big\{\max_{a\leqslant t\leqslant b}|B_t^H|\leqslant k\Big\},\quad k\geqslant 1,$$

is increasing, and

$$\bigcup_{k=1}^{\infty}\Omega_k=\Omega\quad\text{almost surely.}\tag{5.16}$$

On each Ω_k we have

$$\int_a^b g(B_t^H)\frac{B_{t+\varepsilon}^H-B_{t-\varepsilon}^H}{2\varepsilon}\,\mathrm{d}t=\int_a^b g_k(B_t^H)\frac{B_{t+\varepsilon}^H-B_{t-\varepsilon}^H}{2\varepsilon}\,\mathrm{d}t,\quad\text{for all }\varepsilon>0,\tag{5.17}$$

and

$$G(B_b^H)-G(B_a^H)=G_k(B_b^H)-G_k(B_a^H).\tag{5.18}$$

Since for all $k \geqslant 1$, g_k satisfies the assumptions of Proposition 5.7, it follows that

$$L^2 - \lim_{\varepsilon \searrow 0} \int_a^b g_k(B_t^H) \frac{B_{t+\varepsilon}^H - B_{t-\varepsilon}^H}{2\varepsilon} \, \mathrm{d}t = G_k(B_b^H) - G_k(B_a^H).$$

This together with (5.16), (5.17) and (5.18) proves part (a) of Theorem 5.3. □

Proof of Theorem 5.3(b). It follows from what we have shown in the proof of Proposition 5.7 that

$$L^2 - \lim_{\varepsilon \searrow 0} \int_a^b (B_t^H)^2 \frac{B_{t+\varepsilon}^H - B_{t-\varepsilon}^H}{2\varepsilon} \, \mathrm{d}t \tag{5.19}$$

exists if and only if

$$\lim_{\varepsilon, \eta \searrow 0} \frac{G_{\varepsilon,\eta}}{\varepsilon \eta} \tag{5.20}$$

exists, where

$$\begin{aligned} G_{\varepsilon,\eta} &:= \int_a^b \int_a^b \int_{s-\eta}^{s+\eta} \mathbb{E}\big[B_t^H (B_v^H - B_s^H)\big] \frac{\partial R_H}{\partial v}(t,v) \, \mathrm{d}v \big[R_H(t+\varepsilon, s) - R_H(t-\varepsilon, s)\big] \, \mathrm{d}s \, \mathrm{d}t \\ &= \int_a^b \int_a^b \int_{s-\eta}^{s+\eta} \big[R_H(t,v) - R_H(t,s)\big] \frac{\partial R_H}{\partial v}(t,v) \, \mathrm{d}v \big[R_H(t+\varepsilon, s) - R_H(t-\varepsilon, s)\big] \, \mathrm{d}s \, \mathrm{d}t \\ &= \frac{1}{2} \int_a^b \int_a^b \big[R_H(t, s+\eta) - R_H(t, s-\eta)\big]\big[R_H(t, s+\eta) + R(t, s-\eta) - 2R(t,s)\big] \\ &\quad \times \big[R_H(t+\varepsilon, s) - R_H(t-\varepsilon, s)\big] \, \mathrm{d}s \, \mathrm{d}t. \end{aligned}$$

Note that

$$G_{\varepsilon,\eta} = G_{\varepsilon,\eta}^1 + G_{\varepsilon,\eta}^2 + G_{\varepsilon,\eta}^3 + G_{\varepsilon,\eta}^4,$$

where

$$\begin{aligned} G_{\varepsilon,\eta}^1 &= \frac{1}{4} \int_a^b \int_a^b \big[|s+\eta|^{2H} - |s-\eta|^{2H}\big]\big[R_H(t, s+\eta) + R(t, s-\eta) - 2R(t,s)\big] \\ &\quad \times \big[R_H(t+\varepsilon, s) - R_H(t-\varepsilon, s)\big] \, \mathrm{d}s \, \mathrm{d}t, \end{aligned}$$

$$\begin{aligned} G_{\varepsilon,\eta}^2 &= \frac{1}{8} \int_a^b \int_a^b \big[|t-s+\eta|^{2H} - |t-s-\eta|^{2H}\big]\big[R_H(t, s+\eta) + R(t, s-\eta) - 2R(t,s)\big] \\ &\quad \times \big[|t+\varepsilon|^{2H} - |t-\varepsilon|^{2H}\big] \, \mathrm{d}s \, \mathrm{d}t, \end{aligned}$$

$$\begin{aligned} G_{\varepsilon,\eta}^3 &= \frac{1}{16} \int_a^b \int_a^b \big[|t-s+\eta|^{2H} - |t-s-\eta|^{2H}\big]\big[|s+\eta|^{2H} + |s-\eta|^{2H} - 2|s|^{2H}\big] \\ &\quad \times \big[|t-s-\varepsilon|^{2H} - |t-s+\varepsilon|^{2H}\big] \, \mathrm{d}s \, \mathrm{d}t \end{aligned}$$

and

$$G^4_{\varepsilon,\eta} = \frac{1}{16}\int_a^b\int_a^b \bigl[|t-s+\eta|^{2H} - |t-s-\eta|^{2H}\bigr]\bigl[|t-s+\eta|^{2H} + |t-s-\eta|^{2H} - 2|t-s|^{2H}\bigr]$$
$$\times \bigl[|t-s+\varepsilon|^{2H} - |t-s-\varepsilon|^{2H}\bigr]\,\mathrm{d}s\,\mathrm{d}t. \tag{5.21}$$

It can easily be checked that for all $H > 0$,

$$\lim_{\varepsilon\eta\searrow 0}\frac{1}{\varepsilon\eta}G^1_{\varepsilon,\eta} = \lim_{\varepsilon\eta\searrow 0}\frac{1}{\varepsilon\eta}G^2_{\varepsilon,\eta} = \lim_{\varepsilon\eta\searrow 0}\frac{1}{\varepsilon\eta}G^3_{\varepsilon,\eta} = 0.$$

Hence, if we can show that for $H \in (0, \frac{1}{6}]$, $\frac{1}{\varepsilon\eta}G^4_{\varepsilon,\eta}$ does not converge as $\varepsilon, \eta \searrow 0$, then the limit (5.20) cannot exist. By transforming the double integral in (5.21) a couple of times we obtain

$$\frac{8}{\varepsilon\eta}G^4_{\varepsilon,\eta} = \frac{1}{\varepsilon\eta}\int_0^{b-a}\int_0^t \bigl[|t-s+\eta|^{2H} - |t-s-\eta|^{2H}\bigr]\bigl[|t-s+\eta|^{2H} + |t-s-\eta|^{2H} - 2|t-s|^{2H}\bigr]$$
$$\times \bigl[|t-s+\varepsilon|^{2H} - |t-s-\varepsilon|^{2H}\bigr]\,\mathrm{d}s\,\mathrm{d}t$$
$$= \frac{1}{\varepsilon\eta}\int_0^{b-a}\int_0^v \bigl[|w+\eta|^{2H} - |w-\eta|^{2H}\bigr]\bigl[|w+\eta|^{2H} + |w-\eta|^{2H} - 2|w|^{2H}\bigr]$$
$$\times \bigl[|w+\varepsilon|^{2H} - |w-\varepsilon|^{2H}\bigr]\,\mathrm{d}w\,\mathrm{d}v$$
$$= \frac{1}{\varepsilon\eta}\int_0^{b-a} (b-a-w)\bigl[|w+\eta|^{2H} - |w-\eta|^{2H}\bigr]\bigl[|w+\eta|^{2H} + |w-\eta|^{2H} - 2|w|^{2H}\bigr]$$
$$\times \bigl[|w+\varepsilon|^{2H} - |w-\varepsilon|^{2H}\bigr]\,\mathrm{d}w.$$

It can easily be checked that for all $H > 0$,

$$\frac{1}{\varepsilon\eta}\int_0^{b-a} w\bigl[|w+\eta|^{2H} - |w-\eta|^{2H}\bigr]\bigl[|w+\eta|^{2H} + |w-\eta|^{2H} - 2|w|^{2H}\bigr]$$
$$\times \bigl[|w+\varepsilon|^{2H} - |w-\varepsilon|^{2H}\bigr]\,\mathrm{d}w \to 0, \quad \text{as } \varepsilon, \eta \searrow 0.$$

To show that

$$\frac{1}{\varepsilon\eta}\int_0^{b-a} \bigl[|w+\eta|^{2H} - |w-\eta|^{2H}\bigr]\bigl[|w+\eta|^{2H} + |w-\eta|^{2H} - 2|w|^{2H}\bigr]\bigl[|w+\varepsilon|^{2H} - |w-\varepsilon|^{2H}\bigr]\,\mathrm{d}w \tag{5.22}$$

does not converge as $\varepsilon, \eta \searrow 0$, we set $c := b - a$ and let $\varepsilon = d\eta$ for some $d > 0$. Then (5.22) becomes

$$\frac{1}{d\eta^2}\int_0^c \bigl[|w+\eta|^{2H} - |w-\eta|^{2H}\bigr]\bigl[|w+\eta|^{2H} + |w-\eta|^{2H} - 2|w|^{2H}\bigr]\bigl[|w+d\eta|^{2H} - |w-d\eta|^{2H}\bigr]\,\mathrm{d}w$$
$$= \frac{\eta^{6H-1}}{d}\int_0^{c/\eta} \bigl[|x+1|^{2H} - |x-1|^{2H}\bigr]\bigl[|x+1|^{2H} + |x-1|^{2H} - 2|x|^{2H}\bigr]$$
$$\times \bigl[|x+d|^{2H} - |x-d|^{2H}\bigr]\,\mathrm{d}x. \tag{5.23}$$

For $H \in (0, \frac{1}{6})$, the constant d can be chosen such that

$$\int_0^\infty \big[|x+1|^{2H} - |x-1|^{2H}\big]\big[|x+1|^{2H} + |x-1|^{2H} - 2|x|^{2H}\big]\big[|x+d|^{2H} - |x-d|^{2H}\big]\,\mathrm{d}x \in \mathbb{R} \setminus \{0\}.$$

Then, (5.23) explodes for $\eta \searrow 0$. For $H = \frac{1}{6}$, the expression

$$\frac{1}{d}\int_0^\infty \big[|x+1|^{2H} - |x-1|^{2H}\big]\big[|x+1|^{2H} + |x-1|^{2H} - 2|x|^{2H}\big]\big[|x+d|^{2H} - |x-d|^{2H}\big]\,\mathrm{d}x$$

obviously is not constant in d. Hence, we have shown that for $H \in (0, \frac{1}{6}]$, the limit (5.20) does not exist, which implies that the limit (5.19) does not exist either. Since in any finite Wiener chaos convergence in L^2 is equivalent to convergence in probability, it follows that

$$P - \lim_{\varepsilon \searrow 0} \int_a^b (B_t^H)^2 \frac{B_{t+\varepsilon}^H - B_{t-\varepsilon}^H}{2\varepsilon}\,\mathrm{d}t$$

does not exist. □

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