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Two-dimensional Poisson Trees converge to the Brownian web

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Abstract

The *Brownian web* can be roughly described as a family of coalescing one-dimensional Brownian motions starting at all times in $\mathbb{R}$ and at all points of $\mathbb{R}$. The two-dimensional *Poisson tree* is a family of continuous time one-dimensional random walks with uniform jumps in a bounded interval. The walks start at the space–time points of a homogeneous Poisson process in $\mathbb{R}^2$ and are in fact constructed as a function of the point process. This tree was introduced by Ferrari, Landim and Thorisson. By verifying criteria derived by Fontes, Isopi, Newman and Ravishankar, we show that, when properly rescaled, and under the topology introduced by those authors, Poisson trees converge weakly to the Brownian web.

Résumé

La « toile brownienne » peut approximativement être décrite comme une famille coalescente de mouvements browniens unidimensionnels commençant, en tout temps de la droite réelle, à partir de tout point de la droite réelle. On montre qu'elle peut être approchée en un sens faible par une famille d'arbres poissonniens bidimensionnels.

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1. Introduction and results

Let S be a two-dimensional homogeneous Poisson process of parameter λ. S is a random subset of $\mathbb{R}^2$, $s \in S$ has coordinates s_1, s_2.

For $x = (x_1, x_2) \in \mathbb{R}^2$, $t \geqslant x_2$ and $r > 0$, let $M(x, t, r)$ be the following rectangle

$$M(x,t,r) := \{(x_1', x_2'):\ |x_1' - x_1| \leqslant r,\ x_2 \leqslant x_2' \leqslant t\}. \tag{1.1}$$

As t grows, the rectangle gets longer. The first time t that $M(x,t,r)$ hits some (or *another*, when $x \in S$) point of S is called $\tau(x, S, r)$; this is defined by

$$\tau(x,S,r) := \inf\{t > x_2:\ M(x,t,r) \cap (S \setminus \{x\}) \neq \emptyset\}. \tag{1.2}$$

The hitting point is the point $\alpha(x) \in S$ defined by

$$\alpha(x) := M(x, \tau(x,S,r), r) \cap (S \setminus \{x\}), \tag{1.3}$$

which consists of a unique point almost surely. If $x =$ some $s \in S$, we say that $\alpha(x) = \alpha(s)$ is the *mother* of s and that s is a *daughter* of $\alpha(s)$. Let $\alpha^0(x) = x$ and iteratively, for $n \geqslant 1$, $\alpha^n(x) = \alpha(\alpha^{n-1}(x))$. For the case of $x =$ some $s \in S$, $\alpha^n(x) = \alpha^n(s)$ is the nth grand mother of s.

Now let $G = (V, E)$ be the random directed graph with vertices $V = S$ and edges $E = \{(s, \alpha(s)):\ s \in S\}$. Ferrari, Landim and Thorisson [4] proved that G is a tree with a unique connected component and called it the two-dimensional *Poisson tree*. The drainage networks of Gangopadhyay, Roy and Sarkar [9] can be viewed as a discrete space, long range version of the Poisson tree.

The Poisson tree induces sets of continuous paths. For any $s = (s_1, s_2) \in S$, define the path X^s in $\mathbb{R}^2$ as the linearly interpolated line composed by all edges $\{(\alpha^{n-1}(s), \alpha^n(s)):\ n \in \mathbb{N}\}$ of G. Let

$$X := \{X^s:\ s \in S\}, \tag{1.4}$$

which we also call the *Poisson web*.

Clearly X depends on $\lambda > 0$ and $r > 0$; if necessary we denote it by $X(\lambda, r)$. Take $\lambda = \lambda_0 = \sqrt{3}/6$, $r = r_0 = \sqrt{3}$, and let

$$X_1 := X(\lambda_0, r_0); \qquad X_\delta := \{(\delta x_1, \delta^2 x_2) \in \mathbb{R}^2:\ (x_1, x_2) \in X_1\}, \quad \delta \in (0, 1]. \tag{1.5}$$

Namely, X_δ is the diffusive rescaling of X_1.

Our main result is a proof that X_δ converges in distribution to the Brownian web characterized in [7]. [7] introduces a metric space (Π, d) of continuous paths firstly, and then defines the Hausdorff metric space $(\mathcal{H}, d_{\mathcal{H}})$ of compact subsets of (Π, d), where $d_{\mathcal{H}}$ and d are the corresponding metric functions. Denote by $\mathcal{F}_{\mathcal{H}}$ the corresponding Borel σ-algebra generated by $d_{\mathcal{H}}$. The Brownian web is characterized there as a $(\mathcal{H}, \mathcal{F}_{\mathcal{H}})$-valued random variable $\overline{\mathcal{W}}$ (or its distribution $\mu_{\overline{\mathcal{W}}}$) whose "finite-dimensional distributions" (in a sense made precise in [7]) are *coalescing one-dimensional Brownian motions*.

Theorem 1.1. *The rescaled Poisson trees X_δ converge in distribution to the standard Brownian web as $\delta \to 0$.*

Given $t_0 \in \mathbb{R}$, $t > 0$, $a < b$, and a $(\mathcal{H}, \mathcal{F}_{\mathcal{H}})$-valued random variable V, let $\eta_V(t_0, t; a, b)$ be the $\{0, 1, 2, \ldots, \infty\}$-valued random variable giving the number of *distinct* points in $\mathbb{R} \times \{t_0 + t\}$ that are touched by paths in V which also touch some point in $[a, b] \times \{t_0\}$. By the weak convergence criteria given in [7], for any $(\mathcal{H}, \mathcal{F}_{\mathcal{H}})$-valued random variables $\{X_n\}_{n=1}^{\infty}$ with noncrossing paths, to prove that X_n converges to the standard Brownian web, one may verify the following: For some countable dense set $\mathcal{D}$ in $\mathbb{R}^2$,

(I_1) There exist $\theta_n^y \in X_n$ such that for any deterministic $y_1, \ldots, y_m \in \mathcal{D}$, $\theta_n^{y_1}, \ldots, \theta_n^{y_m}$ converge in distribution as $n \to \infty$ to coalescing Brownian motions (with unit diffusion constant) starting at $y_1, \ldots, y_m$;

(B$_1$) $\limsup_{n\to\infty} \sup_{(a,t_0)\in\mathbb{R}^2} \mathbb{P}(\eta_{X_n}(t_0,t;a,a+\epsilon) \geqslant 2) \to 0$ as $\epsilon \to 0+$;
(B$_2$) $\epsilon^{-1} \limsup_{n\to\infty} \sup_{(a,t_0)\in\mathbb{R}^2} \mathbb{P}(\eta_{X_n}(t_0,t;a,a+\epsilon) \geqslant 3) \to 0$ as $\epsilon \to 0+$.

To prove the main result, we show in Section 2 that the Poisson webs X_δ satisfy the three hypotheses above. The verification of I$_1$, on Subsection 2.1, relies on a comparison with independent paths and on the almost sure coalescence of the Poisson web paths with each other. See Lemma 2.3.

In Subsection 2.2, an FKG inequality enjoyed by the distribution of a single Poisson web path (Lemma 2.6) and the $O(t^{-1/2})$ decay of the coalescence time of two such paths (Lemma 2.7), combined with I$_1$, yield both B$_1$ and B$_2$. The argument is similar in spirit to the one for establishing weak convergence of coalescing random walks to the Brownian web in [7]. The details are nonetheless substantially different, more involved here, due to the dependence between the paths of the Poisson tree before coalescence. See the remark at the end of the paper. See also [3] for more details.

Working out a second example of a process in the basin of attraction of the Brownian web (the first example is ordinary one-dimensional coalescing random walks) that is natural on one side, and that requires substantial technical attention on another side, is the primary point of this paper. Its main result may have an applied interest, e.g. in the context of drainage networks. The convergence results here may lead to rigorous/alternative verification of some of the scaling theory for those networks. See [13]. Ordinary one-dimensional coalescing random walks starting from all space–time points have also been proposed as model of a drainage network [14], so the latter remark applies to them as well. Another application would be in obtaining aging results from the scaling limit results for systems that could be modeled by Poisson webs, like drainage networks. For the relation between aging and scaling limits, see e.g. [7,5,6,8], and references therein.

2. Proofs

Coalescing random walks. Let S be the Poisson process with parameter $\lambda > 0$, fix some $r > 0$. For any $x = (x_1, x_2) \in \mathbb{R}^2$, let $\tau^n(x) = [\alpha^n(x)]_2$, $n \geqslant 0$, be the second coordinate of $\alpha^n(x)$ and consider $\{\xi^x(t)\colon t \geqslant x_2\}$ as the continuous time Markov process defined by

$$\xi^x(t) = \left[\alpha^n(x)\right]_1, \text{the first coordinate of } \alpha^n(x); \quad t \in \left[\tau^n(x), \tau^{n+1}(x)\right), \quad n \geqslant 0. \tag{2.1}$$

We remark that for any fixed $\{x^i\}_{i=1}^m$, with $x^i = (x_1^i, x_2^i) \in \mathbb{R}^2$ for $i = 1, \ldots, m$, $\{(\xi^{x^i}(t)\colon t \geqslant x_2^i), i = 1, \ldots, m\}$ defines a finite system of coalescing random walks starting at the space–time points $x^1, \ldots, x^m$.

For $x = (x_1, x_2) \in \mathbb{R}^2$, let $x_\delta = (\delta^{-1}x_1, \delta^{-2}x_2)$, $\delta \in (0, 1]$. For the single random walk starting at $x = (x_1, x_2)$, $\xi^x(\cdot)$, defined in the last paragraph, the diffusive rescaling is

$$\xi_\delta^x(t) := \delta\xi^{x_\delta}(\delta^{-2}t), \quad \text{for } t \geqslant x_2;\ \delta \in (0, 1]. \tag{2.2}$$

Since the characterizing theorem and the weak convergence criteria given in [7] apply to continuous paths only, we need to replace the original processes by their linearly interpolated versions:

$$\bar\xi_\delta^x(t) = \delta\left\{\xi^{x_\delta}\left(\tau^n(x_\delta)\right) + \frac{\delta^{-2}t - \tau^n(x_\delta)}{\tau^{n+1}(x_\delta) - \tau^n(x_\delta)}\left(\xi^{x_\delta}\left(\tau^{n+1}(x_\delta)\right) - \xi^{x_\delta}\left(\tau^n(x_\delta)\right)\right)\right\}, \tag{2.3}$$

for $t \geqslant x_2$ such that $\delta^{-2}t \in [\tau^n(x_\delta), \tau^{n+1}(x_\delta))$, $n \geqslant 0$; $\delta \in (0, 1]$, $x \in \mathbb{R}^2$. Denote by $\bar\xi_\delta^x$ the corresponding continuous path in $\mathbb{R}^2$ and note that $\bar\xi_1^s$ is just X^s in (1.4) with $s \in S$. It is straightforward to see that $\bar\xi_\delta^x \in X_\delta$, the Poisson web defined by (1.5), if and only if $x_\delta \in S$.

Let

$$\theta_\delta^x := \begin{cases} \bar\xi_\delta^x & \text{if } x_\delta \in S, \\ \bar\xi_\delta^{(\delta[\alpha(x_\delta)]_1, \delta^2[\alpha(x_\delta)]_2)} & \text{otherwise.} \end{cases} \tag{2.4}$$

In this way, for all $x \in \mathbb{R}^2$ and $\delta \in (0,1]$, $\theta_\delta^x \in X_\delta$. Note that the paths defined by (2.3) and (2.4) depend on the choice of $\lambda > 0$ and $r > 0$. In case of necessity, we denote them by $\bar{\xi}_\delta^x(\lambda, r)$ and $\theta_\delta^x(\lambda, r)$.

The following is an application of the classical Donsker's theorem [2] to our case.

Lemma 2.1. *If $\lambda = \lambda_0 = \sqrt{3}/6$, $r = r_0 = \sqrt{3}$, then $\bar{\xi}_\delta^x$ converges in distribution as $\delta \to 0$ to B^x, the Brownian path with unit diffusion coefficient starting from space–time point $x = (x_1, x_2) \in \mathbb{R}^2$.*

For any $x^1, \ldots, x^m \in \mathbb{R}^2$, $m \in \mathbb{N}$, regard $(\bar{\xi}_\delta^{x^1}, \ldots, \bar{\xi}_\delta^{x^m})$ and $(\theta_\delta^{x^1}, \ldots, \theta_\delta^{x^m})$ as random variables in the product metric space (Π^m, d^{*m}), where d^{*m} is a metric on Π^m such that the topology generated by it coincides with the corresponding product topology. Here we choose and define

$$d^{*m}\left[(\xi^1, \ldots, \xi^m), (\zeta^1, \ldots, \zeta^m)\right] = \max_{1 \leqslant i \leqslant m} d(\xi^i, \zeta^i), \tag{2.5}$$

for all $(\xi^1, \ldots, \xi^m), (\zeta^1, \ldots, \zeta^m) \in \Pi^m$, where d was defined in [7]. The next result follows immediately from the definition.

Lemma 2.2.

$$\mathbb{P}_\lambda\left\{d^{*m}\left[(\bar{\xi}_\delta^{x^1}, \ldots, \bar{\xi}_\delta^{x^m}), (\theta_\delta^{x^1}, \ldots, \theta_\delta^{x^m})\right] \geqslant \epsilon\right\} \to 0, \quad \text{as } \delta \to 0 \tag{2.6}$$

for all $\epsilon > 0$, $\lambda > 0$, $r > 0$, and $x^1, \ldots, x^m \in \mathbb{R}^2$, $m \in \mathbb{N}$, where $\mathbb{P}_\lambda$ is the probability distribution of S, the Poisson process with parameter λ.

2.1. Convergence in finite-dimensional cases: verification of condition I_1

Let $\mathcal{D}$ be a countable dense set of points in $\mathbb{R}^2$, to verify condition I_1, by Lemma 2.2, we only need to prove the following.

Lemma 2.3. $(\bar{\xi}_\delta^{y^1}, \ldots, \bar{\xi}_\delta^{y^m})$ *converges in distribution as $\delta \to 0$ to coalescing Brownian motions (with unit diffusion constant) starting at $y^1, \ldots, y^m$ ($\in \mathcal{D}$).*

For the finite system of coalescing random walks defined in the last subsection, Ferrari, Landim and Thorisson [4] proved that, for any $x, y \in \mathbb{R}^2$, the random walks $\xi^x(t)$ and $\xi^y(t)$, $t \geqslant x_2 \vee y_2$ will meet and then coalesce almost surely. This also follows from Lemma 2.7 below. The following is a corollary of this result.

Lemma 2.4. *For any $\lambda > 0$, $r > 0$, $\mathbb{P}_\lambda\{\sup_{t \geqslant x_2} |\bar{\xi}_1^x(t) - \bar{\xi}_1^y(t)| \geqslant \sigma\} \to 0$, as $\sigma \to \infty$ for all $x, y \in \mathbb{R}^2$ with $x_2 = y_2$.*

Now, for any m distinct points $y^1, \ldots, y^m \in \mathcal{D}$ and $\delta \in (0,1]$. Let $\bar{\xi}_\delta^{y^1}, \ldots, \bar{\xi}_\delta^{y^m}$ be the m rescaled continuous random paths defined in (2.3) from the same Poisson process with $\lambda = \lambda_0 = \sqrt{3}/6$ and $r = r_0 = \sqrt{3}$. Having $(\bar{\xi}_\delta^{y^1}, \ldots, \bar{\xi}_\delta^{y^m})$ as a random element in Π^m, we want to define a function f_δ from it to Π^m. This is our main idea for the verification of condition I_1: we define what we call "δ-coalescence" of the random paths $\bar{\xi}_\delta^{y^1}, \ldots, \bar{\xi}_\delta^{y^m}$ in such a way that, in the system $f_\delta(\bar{\xi}_\delta^{y^1}, \ldots, \bar{\xi}_\delta^{y^m})$, before any δ-coalescence, the paths involved are independent.

We define f_δ by renewing the whole system step by step as follows. Consider $(\xi_\delta^{y^1}, \ldots, \xi_\delta^{y^m})$, the rescaled finite system of coalescing random walks starting at the space–time points $y^1, \ldots, y^m$. Let $\gamma_{\delta,0} = \min(y_2^1, \ldots, y_2^m)$, and we assume that $\delta > 0$ is close enough to 0, so that, in particular, the following stopping times we need are well defined.

For $(\xi_\delta^{y^1}, \dots, \xi_\delta^{y^m})$, as t $(> \gamma_{\delta,0})$ grows, let $\gamma_{\delta,k}$, $1 \leqslant k \leqslant m-1$, be the time when the kth δ-coalescence occurs. We say a δ-coalescence occurs at time t_0, if t_0 is the time when two particles get within a distance smaller than $2\sqrt{3}\delta$. Once a δ-coalescence occurs, we renew the system by coalescing the two particles to be the *left one* and then wait for the next δ-coalescence.

Denote the linearly interpolated versions of the resulting object after renewing $m-1$ times by $f_\delta(\bar\xi_\delta^{y^1}, \dots, \bar\xi_\delta^{y^m})$ and, with that, finish the definition of the function f_δ. Clearly, for $\delta \in (0,1]$ small enough, we have

$$-\infty < \gamma_{\delta,0} < \gamma_{\delta,1} < \cdots < \gamma_{\delta,m-1} < \infty \tag{2.7}$$

and the function f_δ is well defined almost surely.

Now, suppose that $\tilde\xi_\delta^{y^i}$ has the same distribution as $\bar\xi^{y^i}$, $1 \leqslant i \leqslant m$, and, as a random element in Π^m, $(\tilde\xi_\delta^{y^1}, \dots, \tilde\xi_\delta^{y^m})$ has independent components. It is easy to see that the function f_δ is also well defined for the random paths $(\tilde\xi_\delta^{y^1}, \dots, \tilde\xi_\delta^{y^m})$. Let $C_\delta \subset \Pi^m$ be such that

$$\mathbb{P}\big\{(\tilde\xi_\delta^{y^1}, \dots, \tilde\xi_\delta^{y^m}) \in C_\delta\big\} = 1 \tag{2.8}$$

and, on C_δ, f_δ is well defined.

For any $y^1, \dots, y^m \in \mathcal{D}$, let $B^{y^1}, \dots, B^{y^m}$ be m independent Brownian paths starting at space–time points $y^1, \dots, y^m$, respectively. As Arratia did in [1], we construct a set $\{\tilde B^{y^1}, \dots, \tilde B^{y^m}\}$ of m one-dimensional coalescing Brownian motions starting at $y^1, \dots, y^m$ by defining an almost surely continuous function f from Π^m to Π^m as follows. The first Brownian path of the set, $\tilde B^{y^1}$, is B^{y^1} itself. Once we have $\{\tilde B^{y^1}, \dots, \tilde B^{y^{k-1}}\}$, we define $\tilde B^{y^k}$ to be equal to B^{y^k} till that path first hits any of $\tilde B^{y^1}, \dots, \tilde B^{y^{k-1}}$, say $\tilde B^{\hat y}$; thence on it coincides with $\tilde B^{\hat y}$. This procedure is a.s. well defined, and the system resulting after $m-1$ steps is the so-called one-dimensional coalescing Brownian motions starting at space–time points $y^1, \dots, y^m$, which we denote by $f(B^{y^1}, \dots, B^{y^m})$, as a function of the m independent Brownian motions $B^{y^1}, \dots, B^{y^m}$.

Lemma 2.5. *Let $(\bar\xi_\delta^{y^1}, \dots, \bar\xi_\delta^{y^m})$ be the m rescaled continuous random paths defined in* (2.3) *from the same Poisson process with $\lambda = \lambda_0 = \sqrt{3}/6$ and $r = r_0 = \sqrt{3}$, and $(\tilde\xi_\delta^{y^1}, \dots, \tilde\xi_\delta^{y^m})$ have independent components and $\tilde\xi_\delta^{y^i}$ have the same distribution as $\bar\xi^{y^i}$ for all $1 \leqslant i \leqslant m$. Then,*

(a) $f_\delta(\bar\xi_\delta^{y^1}, \dots, \bar\xi_\delta^{y^m})$ *has the same distribution as* $f_\delta(\tilde\xi_\delta^{y^1}, \dots, \tilde\xi_\delta^{y^m})$*;*
(b) $f_\delta(\tilde\xi_\delta^{y^1}, \dots, \tilde\xi_\delta^{y^m})$ *converges in distribution to* $f(B^{y^1}, \dots, B^{y^m})$ *as* $\delta \to 0$*;*
(c) *for any* $\epsilon > 0$, $\mathbb{P}\{d^{*m}[f_\delta(\bar\xi_\delta^{y^1}, \dots, \bar\xi_\delta^{y^m}), (\bar\xi_\delta^{y^1}, \dots, \bar\xi_\delta^{y^m})] \geqslant \epsilon\} \to 0$, *as* $\delta \to 0$. d^{*m} *was defined in* (2.5).

Proof. (a), (c) Immediate from the definition of f_δ and Lemma 2.4. For (b), it is straightforward to check that, for any $c = (c^1, \dots, c^m) \in C$, $c_\delta = (c_\delta^1, \dots, c_\delta^m) \in C_\delta$, $d^{*m}(c_\delta, c) \to 0$ implies $d^{*m}(f_\delta(c_\delta), f(c)) \to 0$ as $\delta \to 0$. Thus, an extended continuous mapping theorem of Mann and Wald [11], Prohorov [12] (see also Theorem 3.27 of [10]) gives (b). □

Lemma 2.3 is an immediate consequence of Lemma 2.5. Thus, condition I_1 for the Poisson web X_δ, $\delta \in (0,1]$ follows from Lemma 2.2.

2.2. Verification of conditions B_1 and B_2

Consider the Poisson process S with parameter $\lambda > 0$ and the corresponding Poisson tree $X := X(\lambda, r)$ defined in (1.4) with respect to some fixed $r > 0$. Given $t_0 \in \mathbb{R}$, $t > 0$, $a, b \in \mathbb{R}$ with $a < b$, let $\eta_X(t_0, t; a, b)$ be the

$\{0,1,2,\ldots,\infty\}$-valued random variable defined right after the statement of Theorem 1.1. Let $\bar{\eta}_X(t_0,t;a,b)$ be another $\{0,1,2,\ldots,\infty\}$-valued random variable defined as the number of distinct points $y=(y_1,y_2)\in\mathbb{R}\times\{t_0+t\}$ such that there exists $s\in S$ with $s_2\leqslant t_0$, $\xi^s(t_0)\in[a,b]$ and $\xi^s(t_0+t)=\xi^s(y_2)=y_1$, where ξ^s is the Markov process defined in (2.1). It is straightforward to see that, for any fixed $n\in\mathbb{N}$,

$$\bar{\eta}_X(t_0,t;a,b)\geqslant n\Rightarrow\eta_X(t_0,t;a-2r,b+2r)\geqslant n\Rightarrow\bar{\eta}_X(t_0,t;a-4r,b+4r)\geqslant n. \tag{2.9}$$

This implies that, to verify conditions B_1, B_2 for the Poisson trees X_δ, we only need to verify the following B'_1 and B'_2.

(B'_1) $\limsup_{n\to\infty}\mathbb{P}(\bar{\eta}_{\delta_n}(0,t;0,\epsilon)\geqslant 2)\to 0$ as $\epsilon\to 0+$;
(B'_2) $\epsilon^{-1}\limsup_{n\to\infty}\mathbb{P}(\bar{\eta}_{\delta_n}(0,t;0,\epsilon)\geqslant 3)\to 0$ as $\epsilon\to 0+$

for any sequence of positive numbers (δ_n) such that $\lim_{n\to\infty}\delta_n=0$, where $\bar{\eta}_\delta=\bar{\eta}_{X_\delta}$, and we have used the space homogeneity of the Poisson point process to eliminate the $\sup_{(a,t_0)\in\mathbb{R}^2}$ and put $a=t_0=0$.

Here, we firstly introduce an FKG inequality for probability measures on the path space, which will play an important role in our proofs. Let $\xi=\xi^{(0,0)}$ be the random path starting at the origin defined in (2.1); denote by $\overline{\Pi}$ the space of paths where ξ takes value. We define a partial order "$\preceq$" on $\overline{\Pi}$ as follows. Given $\pi_1,\pi_2\in\overline{\Pi}$,

$$\pi_1\preceq\pi_2\quad\text{if and only if}\quad\pi_1(t)-\pi_1(s)\leqslant\pi_2(t)-\pi_2(s)\quad\text{for all } t\geqslant s\geqslant 0. \tag{2.10}$$

Define increasing events in $\overline{\Pi}$ as usual. Denote by μ_ξ the distribution of ξ on $\overline{\Pi}$.

Lemma 2.6 (FKG Inequality). *μ_ξ satisfies the FKG inequality, namely, for any increasing events $A,B\subseteq\overline{\Pi}$, $\mu_\xi(AB)\geqslant\mu_\xi(A)\mu_\xi(B)$.*

Proof. Follows by discretizing ξ as a discrete time random walk, and then using FKG for its i.i.d. increments. Details can be found in [3]. □

Let $\xi^{(0,0)}$, $\xi^{(\gamma,0)}$, $\gamma\geqslant r$, be two of the random walks defined in (2.1). Denote by Δ_γ the difference between them. Then, Δ_γ is a (space inhomogeneous) jump process in $[0,\infty)$ with absorbing state 0: In $x\in[r,\infty)$, it has rates $(2r+x\wedge(2r))\lambda$ and jump laws

$$\nu_x:=\frac{2r-x\wedge(2r)}{2r+x\wedge(2r)}\delta_{\{-x\}}+\frac{2(x\wedge(2r))}{2r+x\wedge(2r)}U\big[r-x\wedge(2r),r\big], \tag{2.11}$$

where $\delta_{\{-x\}}$ is the usual Dirac measure and $U[r-x,r]$ is the uniform distribution on $[r-x,r]$. Let $\mathcal{T}=\inf\{t>0\colon \Delta_{2r}(t)=\xi^{(0,0)}(t)-\xi^{(2r,0)}(t)=0\}$.

Lemma 2.7. *There exists a constant $c>0$ such that $\mathbb{P}(\mathcal{T}>t)\leqslant c/\sqrt{t}$ for any $t>0$, where c depends on r and λ only.*

Proof. Follows by a coupling of Δ_{2r} and a space homogeneous process which has the same transition distribution as Δ_{2r} outside a neighborhood of the origin. Inside that neighborhood they are not too different, so that the result for Δ_{2r} follows from that the analogous one for the homogeneous process, which is a random walk, and for which that result follows from standard arguments. Details can be found in [3]. □

Now, we begin to verify conditions B'_1 and B'_2. By Lemma 2.3, it is straightforward to get that,

$$\limsup_{n\to\infty}\mathbb{P}\big(\eta_{\delta_n}(0,t;0,\epsilon)\geqslant 2\big)=2\phi\big(\epsilon/\sqrt{2t}\big)-1, \tag{2.12}$$

where δ_n is any sequence of positive numbers converging to 0 as $n \to \infty$, $\eta_\delta = \eta_{X_\delta}$, and $\phi(x)$ is the standard normal distribution function. By (2.9), (2.12) with $\bar\eta_{X_\delta}$ replacing η_{X_δ} also holds. This gives B_1'.

Verifying B_2' for the Poisson web X_δ is equivalent to checking that for any $t > 0$

$$\epsilon^{-1} \limsup_{N\to\infty} \mathbb{P}\big(\bar\eta_{X_1}\big(0, tN; 0, \epsilon\sqrt{N}\,\big) \geqslant 3\big) \to 0 \quad \text{as } \epsilon \to 0+, \tag{2.13}$$

where X_1 is defined in (1.5).

For that, fix $t > 0$. On the Poisson field S with parameter $\lambda = \lambda_0 = \sqrt{3}/6$, choose $r = r_0 = \sqrt{3}$, and then define X_1 as in (1.5). We first condition the probability in (2.13) on the set of points of intersection, in increasing order, of the paths ξ^s, $s \in S$, with $[0, \epsilon\sqrt{N}\,]$, denoted $\{K_1, \dots, K_J\}$, where $J, K_1, \dots, K_J$ are random variables, with J an integer which can equal 0 (in which case set of intersection points is empty by convention). We note that by the definition of ξ^s, $s \in S$, no two distinct K_i's can be at distance smaller than r_0. For $\{x_1, \dots, x_n\} \subset [0, \epsilon\sqrt{N}]$, let $\xi_j := \xi^{(x_j,0)}$, $1 \leqslant j \leqslant n$, as in (2.1). Let $\eta' = \eta'(x_1, \dots, x_n) = |\{\xi_j(tN)\colon 1 \leqslant j \leqslant n\}|$ (conventioned to be 0 if $\{x_1, \dots, x_n\} = \emptyset$). Clearly, $J, K_1, \dots, K_J$ depend only on the points of S below time 0. Thus, since η' depends only on the points of S above and at time 0 for all $\{x_1, \dots, x_n\} \subset [0, \epsilon\sqrt{N}\,]$, given $J = n$, $K_1 = x_1, \dots, K_J = x_n$, the probability in (2.13) equals $\mathbb{P}(\eta' \geqslant 3)$.

We derive next an upper bound for the latter probability which is independent of $\{x_1, \dots, x_n\}$. First, we enlarge, if necessary, the set $\{x_1, \dots, x_n\}$ to make sure that $x_1 = 0$, $x_n = \epsilon\sqrt{N}$, and $r_0 \leqslant x_j - x_{j-1} \leqslant 2r_0$. This also ensures that $n \leqslant \epsilon\sqrt{N}/r_0 + 1$, and the enlargement can only increase the probability to be estimated. If $\eta' \geqslant 3$, then there should be some $1 \leqslant j \leqslant n-1$ such that $\xi_{j-1}(tN) < \xi_j(tN) < \xi_n(tN)$. Hence,

$$\begin{aligned}
\mathbb{P}(\eta' \geqslant 3) &\leqslant \sum_{j=2}^{n-1} \mathbb{P}\big(\xi_{j-1}(tN) < \xi_j(tN) < \xi_n(tN)\big) \\
&= \sum_{j=2}^{n-1} \int_{\overline{\Pi}_j} \mathbb{P}\big(\xi_{j-1}(tN) < \xi_j(tN) < \xi_n(tN) | \xi_j = \pi\big) \mu_{\xi_j}(\mathrm{d}\pi) \\
&= \sum_{j=2}^{n-1} \int_{\overline{\Pi}_j} \mathbb{P}\big(\xi_{j-1}(tN) < \xi_j(tN) | \xi_j = \pi\big) \mathbb{P}\big(\xi_j(tN) < \xi_n(tN) | \xi_j = \pi\big) \mu_{\xi_j}(\mathrm{d}\pi),
\end{aligned} \tag{2.14}$$

where $\overline{\Pi}_j$ is the state space of ξ_j, and μ_{ξ_j} its distribution. In the latter equality, we used the independence of $\xi_{j-1}(tN) < \xi_j(tN)$ and $\xi_j(tN) < \xi_n(tN)$ conditioned on $\xi_j = \pi$.

We claim now that $\mathbb{P}(\xi_{j-1}(tN) < \xi_j(tN) | \xi_j = \pi)$ decreases in π and $\mathbb{P}(\xi_j(tN) < \xi_n(tN) | \xi_j = \pi)$ increases in π. (The reader should check it.) This and the FKG Inequality for μ_{ξ_j} (Lemma 2.6) imply that the right-hand side of (2.14) is bounded above by

$$\sum_{j=2}^{n-1} \mathbb{P}\big(\xi_{j-1}(tN) < \xi_j(tN)\big) \mathbb{P}\big(\xi_j(tN) < \xi_n(tN)\big) \leqslant \mathbb{P}\big(\xi_0(tN) < \xi_n(tN)\big) \sum_{j=2}^{n-1} \mathbb{P}\big(\xi_{j-1}(tN) < \xi_j(tN)\big)$$

since $\mathbb{P}(\xi_j(tN) < \xi_n(tN))$ is clearly nonincreasing in j. Now the probabilities inside the sum are all bounded above by $\mathbb{P}(\xi^{(0,0)}(tN) < \xi^{(2r_0,0)}(tN))$, and we get

$$\begin{aligned}
\mathbb{P}(\eta' \geqslant 3) &\leqslant n \mathbb{P}\big(\xi^{(0,0)}(tN) < \xi^{(2r_0,0)}(tN)\big) \mathbb{P}\big(\xi^{(0,0)}(tN) < \xi^{(\epsilon\sqrt{N},0)}(tN)\big) \\
&\leqslant \frac{\epsilon\sqrt{N}}{r_0} \mathbb{P}(\mathcal{T} > tN) \mathbb{P}(\mathcal{T}_{\epsilon,N} > tN),
\end{aligned}$$

where $\mathcal{T}$ is the time when $\xi^{(0,0)}$ and $\xi^{(2r_0,0)}$ meet and coalesce, and $\mathcal{T}_{\epsilon,N}$ is the analogue time for $\xi^{(0,0)}$ and $\xi^{(\epsilon\sqrt{N},0)}$.

Now, let us consider the items at the right-hand side of the latter equation. By Lemma 2.3,

$$\limsup_{N\to\infty} \mathbb{P}(\mathcal{T}_{\epsilon,N} > tN) = \mathbb{P}(\mathcal{T}_{\epsilon,B} > t),$$

where $\mathcal{T}_{\epsilon,B}$ is the time when two i.i.d. Brownian motions starting at the same time at distance ϵ apart meet and coalesce. Thus the latter probability is an $\mathrm{O}(\epsilon)$ for every $t > 0$ fixed. By Lemma 2.7, $\mathbb{P}(\mathcal{T} > tN) \leqslant c/\sqrt{tN}$. These estimates imply that

$$\limsup_{N\to\infty} \mathbb{P}\big(\bar{\eta}_{X_1}\big(0, tN; 0, \epsilon\sqrt{N}\big) \geqslant 3\big) \leqslant \limsup_{N\to\infty} \frac{\epsilon\sqrt{N}}{r_0} \mathbb{P}(\mathcal{T} > tN)\mathbb{P}(\mathcal{T}_{\epsilon,N} > tN) = \mathrm{O}\big(\epsilon^2\big),$$

and we get B_2' for X_δ.

Remark. The argument for B_2 in [7] for establishing weak convergence of coalescing random walks to the Brownian web also relies on an FKG property of the path distributions. But in that case, it is a stronger FKG property than in the present case. It allows bounding $\mathbb{P}(\eta \geqslant 3)$ above by $[\mathbb{P}(\eta \geqslant 2)]^2$, and then the use of B_1. In particular, it is not necessary in that case to have an estimate of a microscopic quantity like $\mathbb{P}(\mathcal{T} > tN)$, on which we had to rely in here.

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