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Uniqueness for reflecting Brownian motion in lip domains ☆

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Abstract

A lip domain is a Lipschitz domain where the Lipschitz constant is strictly less than one. We prove strong existence and pathwise uniqueness for the solution $X = \{X_t,\ t \geqslant 0\}$ to the Skorokhod equation

$$dX_t = dW_t + \mathbf{n}(X_t)\,dL_t,$$

in planar lip domains, where $W = \{W_t,\ t \geqslant 0\}$ is a Brownian motion, $\mathbf{n}$ is the inward pointing unit normal vector, and $L = \{L_t,\ t \geqslant 0\}$ is a local time on the boundary which satisfies some additional regularity conditions. Counterexamples are given for some Lipschitz (but not lip) three dimensional domains.

Résumé

Un domaine lip est un domaine lipschitzien où la constante lipschitzienne est inférieure à 1. Nous démontrons l'existence forte et l'unicité trajectorielle pour la solution $X = \{X_t,\ t \geqslant 0\}$ de l'équation de Skorokhod

$$dX_t = dW_t + \mathbf{n}(X_t)\,dL_t$$

dans les domaines lip du plan, où $W = \{W_t,\ t \geqslant 0\}$ est un mouvement brownien, $\mathbf{n}$ est le vecteur normal et $L = \{L_t,\ t \geqslant 0\}$ est un temps local sur la frontière qui satisfait certaines conditions de régularité. Quelques contre-exemples sont donnés pour des domaines lipschitziens (mais pas lip) en trois dimensions.

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1. Introduction

1.1. Main results

We start with an informal presentation of our main results. The rigorous statement is postponed until the next section because it requires a number of technical definitions.

Suppose that $D \subset \mathbb{R}^d$, $d \geqslant 2$, is a Lipschitz domain and $x_0 \in \overline{D}$. Let $\mathbf{n}(x)$ denote the inward-pointing unit normal vector at those points $x \in \partial D$ for which such a vector can be uniquely defined (such x form a subset of ∂D of full surface measure), and let $W = \{W_t,\ t \geqslant 0\}$ be a d-dimensional Brownian motion. Consider the following equation for reflecting Brownian motion (RBM) in $\overline{D}$, known as the (stochastic) Skorokhod equation,

$$X_t = x_0 + W_t + \int_0^t \mathbf{n}(X_s)\, dL_s \quad \text{for } t \geqslant 0. \tag{1.1}$$

Here $L = \{L_t,\ t \geqslant 0\}$ is the local time of $X = \{X_t,\ t \geqslant 0\}$ on ∂D, that is, a continuous nondecreasing process that increases only when X is on the boundary ∂D. See Definition 2.1 for a precise statement of what it means to be a solution to (1.1). Our main results, informally stated, are the following. See Theorems 2.3, 2.4, and 2.5 below for a precise statement.

Theorem 1.1. (i) *If D is a bounded Lipschitz domain, then weak uniqueness holds for* (1.1).

(ii) *If D is a bounded planar Lipschitz domain whose Lipschitz constant is strictly less than* 1, *then strong existence and pathwise uniqueness hold for* (1.1).

We do not prove that Theorem 1.1(ii) is sharp, but we have the following counterexample indicating that difficulties can arise for Lipschitz domains for which the Lipschitz constant is greater than 1.

Theorem 1.2. *There exists a Lipschitz domain $D \subset \mathbb{R}^3$ whose Lipschitz constant is strictly greater than* 1 *where weak uniqueness for* (1.1) *fails.*

The counterexample of Theorem 1.2 will be based on a slightly different definition of the local time than that in Theorem 1.1 (see Section 2 for details). Note also that the domain in Theorem 1.2 is unbounded, while Theorem 1.1 involves bounded domains. Although we do not carry it out in this paper, Theorem 1.1 can be modified to handle certain unbounded domains and the example in Theorem 1.2 can be modified to be a bounded domain. Our proofs take an even more complicated route. We first construct a strong solution in any "special" unbounded Lipschitz domain (i.e., lying above the graph of a Lipschitz function that has Lipschitz constant strictly less than one) and then we prove the analogous result for bounded lip domains through a localization argument.

1.2. A new method

We develop a new method for proving pathwise uniqueness for stochastic differential equations. Common methods used to prove pathwise uniqueness include (i) Picard iteration, (ii) solving the corresponding deterministic Skorokhod equation, or (iii) using Itô's formula in a clever way. The method we use to prove pathwise uniqueness for (1.1) is quite different from the usual ones. We believe that our method has other applications, for example, to reflecting Brownian motion with oblique angle of reflection. Some of its elements have appeared in [7] and [12], but each of these papers contains an error; see Remark 5.8.

The first step in our method is to prove weak uniqueness for the joint distribution of the driving Brownian motion W and the solution X of the stochastic differential equation (1.1). The second step is to prove strong existence under the assumptions of Theorem 1.1(ii). Given a Brownian motion W in $\mathbb{R}^d$, we construct a strong

solution (X, L) to (1.1) where **n** is replaced by an oblique vector field. We then take a sequence of oblique vector fields converging to **n** and show that the corresponding solutions converge a.s. to a strong solution of (1.1). Weak uniqueness and strong existence together imply pathwise uniqueness; this idea is classical (see [6, Theorem 4.2], for example), but as far as we know, it has not been successfully implemented in the past. A proof of what we need for the present context is given in Section 6.

1.3. Lip domains

One reason for the intense interest in Lipschitz domains in analysis and probability is that they are often a critical case: many theorems can be proved for Lipschitz domains, while their analogues for less smooth domains are not true. Consequently the proofs needed are often quite delicate.

Lipschitz domains whose Lipschitz constant is strictly less than one are called *lip domains*; the term was coined in [15]. These domains have appeared in a natural way in several recent articles involving reflecting Brownian motion [1–3,15,17], and implicitly in two other papers [8,19]. The crucial property of a lip domain, exploited in each paper listed above, is that one can define a partial order and construct a pair of ("coupled") reflecting Brownian motions in the domain with the property that the two reflecting Brownian particles remain in the same order forever. We point out that a version of this "monotonicity" property proved in Theorem 5.3 below is different from that used in the papers listed above in that here we consider two reflecting Brownian motions corresponding to two distinct reflection direction vector fields. The fact that difficulties can arise in 3-dimensional Lipschitz domains when the Lipschitz constant is greater than 1, as is established in this paper, makes lip domains a natural class to consider in the present context.

1.4. Correction

We correct an error in the proof of weak uniqueness for the stochastic Skorokhod equation (1.1) in [5]; see Remark 4.1. To complete the program started in [5], we impose in Section 2 the additional but natural conditions (2.2) and (2.3) on the local time. These additional assumptions allow us to remove one of the hypotheses in [5]; see Theorem 2.3 for a precise statement. Note that the extra assumptions do not weaken the part of Theorem 1.1 dealing with strong existence.

1.5. Literature review

The construction of reflecting Brownian motion as a strong Markov process in domains that are Lipschitz or even less smooth can be found in [30,31], [10] and [20–22]. The question of when the Skorokhod equation holds (in a variety of contexts) is considered in [20,22,23,11,32,26,44]. For results on weak uniqueness, see [41] and [43], for example, for results on RBM with oblique reflection and [5] for results on RBM with normal reflection. Lions and Sznitman [36] proved pathwise uniqueness for RBM in C^2 domains. Dupuis and Ishii [24] considered pathwise uniqueness for RBM with oblique reflection. Their domains could be non-smooth, but the angle of reflection must be nearly C^2; in the case of RBM with normal reflection, this means the domain must be nearly C^2. The paper [12] is concerned with pathwise uniqueness for RBM with normal reflection in $C^{1+\alpha}$ domains, but contains a gap (see Remark 5.9). It is at present an open problem as to whether pathwise uniqueness holds for the Skorokhod equation in $C^{1+\alpha}$ domains in dimensions three and higher.

1.6. Organization of the paper

Section 2 introduces some definitions and gives the precise statements of our main results. Section 3 recalls a number of results about RBM. Section 4 proves weak uniqueness for RBM, while Section 5 presents the strong existence argument. The proof of pathwise uniqueness is given in Section 6, while the counterexamples are given in Section 7.

2. Main results

If $x \in \mathbb{R}^d$, we will often write $x = (\tilde{x}, \hat{x})$, where $\tilde{x} = (x_1, \ldots, x_{d-1}) \in \mathbb{R}^{d-1}$ and $\hat{x} = x_d \in \mathbb{R}$. We will use $|\cdot|$ for the usual Euclidean norm in $\mathbb{R}^{d-1}$ or $\mathbb{R}^d$. The open ball of radius r about x will be denoted $B(x, r)$. We will use the letter c with subscripts to denote finite strictly positive constants whose exact value is unimportant and which may vary from place to place. The Euclidean boundary and closure of a domain D in $\mathbb{R}^d$ will be denoted by ∂D and $\overline{D}$, respectively.

For a process X and a Borel set $A \subset \mathbb{R}^d$, let

$$T_A = T(A) = \inf\{t > 0\colon\ X_t \in A\}, \qquad \tau_A = \tau(A) = \inf\{t > 0\colon\ X_t \notin A\},$$

i.e., T_A and τ_A are the first hitting time of A and the first exit time from A, respectively. Unless specified otherwise, these random times will be defined relative to the reflecting Brownian motion X.

We say that $\Phi : \mathbb{R}^{d-1} \to \mathbb{R}$ is a Lipschitz function with Lipschitz constant κ if

$$\left|\Phi(\tilde{x}) - \Phi(\tilde{y})\right| \leqslant \kappa|\tilde{x} - \tilde{y}|$$

for all $\tilde{x}, \tilde{y} \in \mathbb{R}^{d-1}$. A *Lipschitz domain* is an open connected set D, either bounded or unbounded, such that for some $\kappa < \infty$ and every point $x \in \partial D$ there exist a neighborhood U^x of x and a Lipschitz function Φ^x whose Lipschitz constant is no larger than κ such that $D \cap U^x = \{(\tilde{y}, \hat{y}) \in U^x\colon\ \hat{y} > \Phi^x(\tilde{y})\}$ in some orthonormal coordinate system which may depend on x. The infimum of the set of κ for which the above holds is called the Lipschitz constant of the domain D. (In what follows, the issue of whether the infimum is attained never arises.) Strictly speaking, in the case of unbounded domains the definition we gave above is for a uniformly Lipschitz domain. Since the only unbounded domains we will consider are uniformly Lipschitz ones, we will refer to them simply as Lipschitz domains as well. If a Lipschitz domain D has Lipschitz constant strictly less than 1, then we will call D a *lip domain*. See [34] for further information on Lipschitz domains.

Consider a Lipschitz domain D. Let $\mathcal{N}_0$ denote the set of points $x = (\tilde{x}, \hat{x}) \in \partial D$ such that if Φ^x is the function in the definition of a Lipschitz domain, then $\Phi^x(\tilde{y})$ is differentiable at $\tilde{y} = \tilde{x}$. Let the inward pointing unit normal vector at $x \in \mathcal{N}_0$ be denoted by $\mathbf{n}(x)$. Such a set $\mathcal{N}_0$ and the vector field $\mathbf{n}(x)$ are typically only Lebesgue measurable. However, there is then a Borel subset $\mathcal{N}$ of $\mathcal{N}_0$ such that $\mathcal{N}_0 \setminus \mathcal{N}$ is of zero Lebesgue measure and $\mathbf{n}(x)$ restricted to $\mathcal{N}$ is Borel measurable. For $x \in \partial D$ and $\varepsilon > 0$, define

$$\mathbf{N}_\varepsilon(x) = \left\{ v\colon\ |v| = 1,\ v = \sum_{i=1}^{m} a_i \mathbf{n}(x_i) \text{ for some } m \geqslant 1,\ a_i \geqslant 0,\ x_i \in \mathcal{N} \cap B(x, \varepsilon) \right\}.$$

We let $\mathbf{N}_0(x) = \{\mathbf{n}(x)\}$ for $x \in \mathcal{N}$. Since a Lipschitz function is differentiable almost everywhere (see Exercise 3.37 on p. 103 of [29]), we see that $\partial D \setminus \mathcal{N}$ has zero surface measure. For $x \notin \mathcal{N}$, we let $\mathbf{N}_0(x) = \bigcap_{\varepsilon>0} \mathbf{N}_\varepsilon(x)$ unless this set is empty. In the latter case we set $\mathbf{N}_0(x) = \{(0, 0, \ldots, 0, 1)\}$. Our definition of the family of "constraint directions" $\mathbf{N}_0(x)$ for $x \notin \mathcal{N}$ is consistent with the assumptions commonly used in the literature, see, e.g., Section 2.2 in [25].

We would like to point out that for $x \in \mathcal{N}$, we do not necessarily have $\bigcap_{\varepsilon>0} \mathbf{N}_\varepsilon(x) = \{\mathbf{n}(x)\}$ and so one could use $\bigcap_{\varepsilon>0} \mathbf{N}_\varepsilon(x)$ as an alternative definition of $\mathbf{N}_0(x)$. An example in Section 7 shows that there need not be pathwise uniqueness for the Skorokhod equation in some Lipschitz domains if one were to adopt this alternative definition of $\mathbf{N}_0(x)$.

Throughout this paper, for a Lipschitz domain D in $\mathbb{R}^d$, we let ν denote the surface measure on ∂D.

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geqslant 0}, \mathbf{P})$ be a complete filtered probability space satisfying the usual conditions; that is, the filtration $\{\mathcal{F}_t\}_{t\geqslant 0}$ is right-continuous and $\mathcal{F}_0$ contains all sets of zero $\mathbf{P}$-measure. We say that a d-dimensional process W is a Brownian motion with respect to the filtration $\{\mathcal{F}_t, t \geqslant 0\}$ if (i) $t \mapsto W_t$ is continuous and $W_0 = 0$ a.s.; (ii) for every $t \geqslant 0$, W_t is $\mathcal{F}_t$-measurable; and (iii) for every $t > s \geqslant 0$, $W_t - W_s$ is independent of $\mathcal{F}_s$ and $W_t - W_s$ has a normal distribution with mean zero and covariance matrix $(t - s)I$, where I is the $d \times d$ identity matrix.

Let $x_0 \in \overline{D}$. In Definition 2.1 we will give a precise meaning to what we mean by existence and uniqueness of solutions to the following stochastic differential equation:

$$X_t = x_0 + W_t + \int_0^t \mathbf{n}(X_s)\, dL_s \quad \text{and} \quad X_t \in \overline{D} \quad \text{for all } t \geqslant 0. \tag{2.1}$$

Remark 2.2 following Definition 2.1 discusses some subtle points and should be regarded as a complement to the definition.

We will always assume that our filtrations $\{\mathcal{F}_t\}_{t\geqslant 0}$ are right-continuous and complete with respect to whichever probability measure is being discussed.

Definition 2.1. Let D be a Lipschitz domain in $\mathbb{R}^d$.

(1) A weak solution to (2.1) is a triplet of continuous processes (X, W, L) on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geqslant 0}, \mathbf{P})$ such that
   (a) X is adapted to $\{\mathcal{F}_t\}_{t\geqslant 0}$,
   (b) L is a nondecreasing $\{\mathcal{F}_t\}_{t\geqslant 0}$-adapted process that increases only when $X_t \in \partial D$, i.e., $\int_0^\infty \mathbf{1}_D(X_s)\, dL_s = 0$,
   (c) if $A \subset \partial D$ and $\nu(A) = 0$ then

$$\int_0^\infty \mathbf{1}_A(X_s)\, dL_s = 0, \quad \text{a.s.}, \tag{2.2}$$

   (d) whenever f is a nonnegative function in $L^1(\partial D, \nu)$ then for all $0 < t < u < \infty$,

$$\int_t^u f(X_s)\, dL_s < \infty, \quad \text{a.s.}, \tag{2.3}$$

   (e) W is a d-dimensional Brownian motion with respect to the filtration $\{\mathcal{F}_t\}_{t\geqslant 0}$,
   (f) (X, W, L) satisfies (2.1) for some Borel measurable map $x \mapsto \mathbf{n}(x)$ on ∂D such that

$$\mathbf{n}(x) \in \mathbf{N}_0(x) \quad \text{when } x \in \partial D, \tag{2.4}$$

   (g)

$$\int_0^\infty \mathbf{1}_{\partial D}(X_s)\, ds = 0. \tag{2.5}$$

(2) We say that weak uniqueness holds for (2.1) if whenever (X, W, L) and $(\widetilde{X}, \widetilde{W}, \widetilde{L})$ are weak solutions to (2.1), then the process (X, L) has the same law as the process $(\widetilde{X}, \widetilde{L})$.

(3) Pathwise uniqueness is said to hold for (2.1) if whenever

$$\big(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geqslant 0}, \mathbf{P}, (X, W, L)\big)$$

and

$$\big(\Omega, \mathcal{F}, \{\mathcal{G}_t\}_{t\geqslant 0}, \mathbf{P}, (\widetilde{X}, W, \widetilde{L})\big)$$

are two weak solutions to (2.1) with a common Brownian motion W and probability space $(\Omega, \mathcal{F}, \mathbf{P})$ but possibly different filtrations $\{\mathcal{F}_t\}_{t\geqslant 0}$ and $\{\mathcal{G}_t\}_{t\geqslant 0}$, then

$$\mathbf{P}\big((X_t, L_t) = (\widetilde{X}_t, \widetilde{L}_t) \text{ for all } t \geqslant 0\big) = 1.$$

(4) Consider a Brownian motion W on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ and let $\{\mathcal{F}_t^W\}_{t\geqslant 0}$ be the augmented filtration generated by W under $\mathbf{P}$. A strong solution to (2.1), relative to $(\Omega, \mathcal{F}, \mathbf{P})$ and W, is a pair of continuous processes (X, L) such that

$$\big(\Omega, \mathcal{F}, \{\mathcal{F}_t^W\}_{t\geqslant 0}, \mathbf{P}, (X, W, L)\big)$$

is a weak solution to (2.1). In particular, X and L are both adapted to $\{\mathcal{F}_t^W\}_{t\geqslant 0}$.

(5) We say that strong uniqueness holds for (2.1) if for every pair of strong solutions (X, L) and $(\widetilde{X}, \widetilde{L})$ to (2.1), relative to the same probability space $(\Omega, \mathcal{F}, \mathbf{P})$ and Brownian motion W, we have

$$\mathbf{P}\big((X_t, L_t) = (\widetilde{X}_t, \widetilde{L}_t) \text{ for all } t \geqslant 0\big) = 1.$$

Clearly pathwise uniqueness implies strong uniqueness. It is known (cf. Yamada and Watanabe [45]) that pathwise uniqueness implies weak uniqueness.

Remark 2.2. (i) Recall that $\mathcal{N}$ denotes the Borel set of points in ∂D where the normal vector is well defined in the classical sense. Since $\nu(\partial D \setminus \mathcal{N}) = 0$, condition (2.2) implies that the integral $\int_0^t \mathbf{n}(X_s)\, dL_s$ has the same value for any Borel measurable choice of $\mathbf{n}(x)$ when $x \in \partial D \setminus \mathcal{N}$; in other words, condition (2.4) is irrelevant as long as (2.2) is satisfied. Note, however, that this is not the case in Theorem 2.5 below.

(ii) Suppose that X is a (component of a) weak solution to (2.1). We will argue that L in (2.1) is uniquely determined by X. Since X is adapted to the filtration $\{\mathcal{F}_t\}_{t\geqslant 0}$ and $\mathbf{n}$ is Borel measurable, then $\mathbf{n}(X)$ is adapted. Definition 2.1(1) implies that X is a continuous $\mathbb{R}^d$-valued semimartingale. Therefore X has a unique Doob–Meyer decomposition:

$$X_t = x_0 + B_t + A_t \quad \text{for all } t \geqslant 0,$$

where, with probability one, B is a continuous $\mathbb{R}^d$-valued local martingale with $B_0 = 0$ and A is a continuous $\mathbb{R}^d$-valued process locally of finite variation with $A_0 = 0$, both adapted to the augmented filtration generated by X. The amount of time the process X spends in ∂D has zero Lebesgue measure, so it follows from (2.1) and Definition 2.1(1)(b) that $W_t = \int_0^t 1_D(X_s)\, dX_s$ is adapted to the augmented filtration generated by X, and so is the process $t \mapsto \int_0^t \mathbf{n}(X_s)\, dL_s$. Hence by the uniqueness of the Doob–Meyer decomposition for X, $A_t = \int_0^t \mathbf{n}(X_s)\, dL_s$, which by (2.2), equals $\int_0^t \mathbf{n}(X_s)\mathbf{1}_{\{X_s\in\mathcal{N}\}}\, dL_s$. Since $|\mathbf{n}(x)| = 1$ and $\mathbf{n}(x)$ is uniquely defined for $x \in \partial D \cap \mathcal{N}$, then by (2.2) again,

$$L_t = \int_0^t \mathbf{n}(X_s)\mathbf{1}_{\{X_s\in\mathcal{N}\}} \cdot dA_s \quad \text{for all } t \geqslant 0,$$

and we conclude that L is uniquely determined by A and X, and hence by X alone. This shows that we could have removed L from the statements of parts (2), (3) and (5) of Definition 2.1 without changing the meaning of weak uniqueness, pathwise uniqueness, and strong uniqueness, respectively.

(iii) Even when D is a half space, it is possible that $\int_0^u f(X_s)\, dL_s$ is infinite with probability one for each $u > 0$ if f is only required to be in $L^1(\partial D)$. Therefore in a condition such as (2.3) it is essential that the interval over which we integrate be separated from the point 0.

(iv) Our definition of strong solution seems to be weaker than that used by other authors, cf. [33] and [38]. However these two notions are equivalent under the assumption of weak uniqueness and existence of weak solutions with random starting distributions; see Corollary 3 in [45] as well as the first part of the proof for Theorem 5.9 below. We will prove all assertions related to strong solutions that are used in this paper, so the difference plays no role.

The first of our main theorems, to be proved in Section 4, is the following improved and corrected result from [5] concerning weak uniqueness.

Theorem 2.3. *Weak uniqueness holds for* (2.1) *in bounded Lipschitz domains* $D \subset \mathbb{R}^d$, $d \geqslant 2$.

The following is our main new result, to be proved in Section 6.

Theorem 2.4. *If* $D \subset \mathbb{R}^2$ *is a bounded lip domain, then for every* $x_0 \in \overline{D}$ *we have a strong solution and pathwise uniqueness for* (2.1).

The following counterexample will be proved in Section 7. Note that in this theorem, conditions (2.2) and (2.3) are not required to hold.

Theorem 2.5. *For every* $\kappa > 1$ *there exists a Lipschitz function* $\Phi : \mathbb{R}^2 \to \mathbb{R}$ *with Lipschitz constant* κ *with the following property.*

Let D *be the region in* $\mathbb{R}^3$ *above the graph of* Φ. *Then there exist a Brownian motion* W, *and two pairs of processes* $(X^{(1)}, L^{(1)})$ *and* $(X^{(2)}, L^{(2)})$ *such that for* $i = 1, 2$, *the pair* $(X^{(i)}, L^{(i)})$ *satisfies all the conditions in Definition* 2.1(1) *and* 2.1(4) *to be a strong solution to* (2.1) *relative to* W *except conditions* (2.2)–(2.3), *but the processes* $\{X_t^{(1)}, t \geqslant 0\}$ *and* $\{X_t^{(2)}, t \geqslant 0\}$ *have different distributions.*

Remark 2.6. The above result also shows that the deterministic version of the Skorokhod equation in a Lipschitz domain in $\mathbb{R}^3$ might not have a unique solution, for otherwise we would have pathwise uniqueness for (2.1).

Remark 2.7. Theorem 2.5 above leaves open the following questions connected with pathwise uniqueness for solutions to (2.1) in Lipschitz domains. Is it the case that only one of (2.2) or (2.3) is necessary? Our counterexample is for $d = 3$; is it the case that (2.2) and (2.3) are not needed if the domain lies in the plane? Our example requires that the Lipschitz constant of D be larger than one; is it the case that for lip domains (2.2) and (2.3) are unnecessary?

3. Preliminaries

Most of this section will be devoted to a review of known results for a family of solutions to (2.1).

We start with a general remark concerning our notational conventions for probability measures in this and the next section. The symbol $\mathbb{P}$ will refer to the distributions of a specific family of solutions to (2.1), namely, the family constructed in [10]. We will use $\mathbf{P}$ to denote the law of an arbitrary weak solution to (2.1), and $\mathcal{P}$ will stand for a collection of $\mathbf{P}$'s. The details are given later in this section.

Let $D \subset \mathbb{R}^d$ be a Lipschitz domain that is not necessarily bounded. We will denote by $\{\mathbb{P}^x\}_{x \in \overline{D}}$ the laws of RBM constructed in [10] via Dirichlet form theory. We will make this statement more precise in Properties 3.1 and Remarks 3.2 and 3.3 below, but we point out here that as a consequence of [10] and [11], there exist a d-dimensional Brownian motion W with respect to the filtration of X and a continuous increasing process L adapted to the filtration of X such that (2.1) holds. Remark 2.2(ii) shows that we may restrict our attention to X and consider $\mathbb{P}^x$ to be the law of X when $x_0 = x$ in (2.1). We will refer to $(\mathbb{P}^x, x \in \overline{D}; X_t, t \geqslant 0)$ as *standard* reflecting Brownian motion in $\overline{D}$. Expectation with respect to $\mathbb{P}^x$ will be denoted by $\mathbb{E}^x$. We will sometimes talk about RBM in a domain; this should be interpreted as RBM in $\overline{D}$ when the domain referred to is D.

In [10] and [11], standard RBM was constructed only on bounded Lipschitz domains, but see Remark 3.2 and also [22] for the unbounded Lipschitz domain case.

To simplify our presentation of the results from [10], we will limit ourselves to the following special type of Lipschitz domain. Let Φ be a bounded Lipschitz function mapping $\mathbb{R}^{d-1} \to \mathbb{R}$ with Lipschitz constant κ (in this section and Section 4 we do not assume that $\kappa < 1$). Consider unbounded domains of the form $U = \{x\colon \hat{x} > \Phi(\tilde{x})\}$. Obviously, $\partial U = \{x \in \mathbb{R}^d\colon \hat{x} = \Phi(\tilde{x})\}$.

The following hold.

Properties 3.1. *Suppose $d \geqslant 3$. Let U be the special Lipschitz domain in $\mathbb{R}^d$ mentioned above and let ν be the surface measure on ∂U.*

(i) *The family $(\mathbb{P}^x, X_t, x \in \overline{U})$ is a strong Markov process associated with the Dirichlet form $(\mathcal{E}, \mathcal{D})$, where $\mathcal{E}(f,f) = \frac{1}{2}\int_U |\nabla f(x)|^2\,dx$ and $\mathcal{D}$ is the completion of the class of restrictions to $\overline{U}$ of C^∞ functions on $\mathbb{R}^d$ with compact support under the metric $\mathcal{E}(f,f)^{1/2} + \|f\|_{L^2(U)}$; in other words, $\mathcal{D}$ is the Sobolev space $W^{1,2}(U)$. This property uniquely determines the family $\{\mathbb{P}^x\}_{x\in\overline{U}}$.*

(ii) *Standard RBM has a jointly continuous transition density function $p(t,x,y)$ on $(0,\infty)\times\overline{U}\times\overline{U}$; the density $p(t,x,y)$ is symmetric in x and y, and there exist constants $k_1, k_2 \in (0,\infty)$ depending only on κ such that*

$$p(t,x,y) \leqslant k_1 t^{-d/2}\exp\bigl(-k_2|x-y|^2/t\bigr), \quad x,y\in\overline{U},\ t>0. \tag{3.1}$$

(iii) *There exist constants $c_1, c_2 \in (0,\infty)$ such that*

$$\mathbb{P}^x\Bigl(\sup_{s\leqslant t}|X_s - x| \geqslant \lambda\Bigr) \leqslant c_1 e^{-c_2\lambda^2/t}, \qquad \lambda>0,\ x\in\overline{U},\ t>0. \tag{3.2}$$

(iv) *The Green function $G(x,y)$ for X on $\overline{U}$ is defined as $\int_0^\infty p(t,x,y)\,dt$. Recall ∂U has zero Lebesgue measure. Clearly*

$$\mathbb{E}^x\left[\int_0^\infty f(X_s)\,ds\right] = \int_U G(x,y)f(y)\,dy$$

whenever $x\in\overline{U}$ and $f\geqslant 0$ on $\overline{U}$. The Green function $G(x,y)$ of X_t is jointly continuous except on the diagonal and is strictly positive everywhere in $\overline{U}\times\overline{U}$.

(v) *There exist constants $k_3, k_4 > 0$ depending only on κ such that*

$$k_3|x-y|^{2-d} \leqslant G(x,y) \leqslant k_4|x-y|^{2-d}, \quad x,y\in\overline{U}. \tag{3.3}$$

(vi) *For an open set $D\subset\mathbb{R}^d$, we say that a locally bounded function h is harmonic with respect to X in $\overline{U}\cap D$ if for every $B(x,r)$ with $\overline{B(x,r)}\subset D$, we have $h(y) = \mathbb{E}^y[h(X_{\tau(B(x,r))})]$ for $y\in\overline{U}\cap B(x,r)$. If f is a bounded function on $\overline{U}\cap\partial B(x,r)$ and $h(y) = \mathbb{E}^y[f(X_{\tau(B(x,r))})]$ for $y\in\overline{U}\cap B(x,r)$, then h is harmonic with respect to X in $\overline{U}\cap B(x,r)$.*

(vii) *The following Harnack inequality holds. There exists a constant $c_1\in(0,\infty)$ depending only on κ, such that if h is nonnegative and harmonic with respect to X in $\overline{U}\cap B(x,r)$, then*

$$h(y) \leqslant c_1 h(z), \quad y,z\in\overline{U}\cap B(x,r/2).$$

(viii) *If h is harmonic with respect to X in $\overline{U}\cap B(x,r)$, then there exist $c_1>0$ and $\alpha>0$ not depending on x or r such that*

$$\bigl|h(y)-h(z)\bigr| \leqslant c_1\Bigl(\sup_{w\in\overline{U}\cap B(x,r)}\bigl|h(w)\bigr|\Bigr)\left(\frac{|y-z|}{r}\right)^\alpha, \quad y,z\in\overline{U}\cap B(x,r/2).$$

(ix) *The local time L in the Skorokhod decomposition* (2.1) *for standard RBM X in $\overline{U}$ is a positive continuous additive functional of X with corresponding Revuz measure $\nu/2$, that is, for every $\lambda>0$ and every $y\in\overline{U}$,*

$$\mathbb{E}^y\left[\int_0^\infty e^{-\lambda t}\,dL_t\right] = \frac{1}{2}\int_{\partial U} G^\lambda(y,x)\nu(dx),$$

where $G^\lambda(y,x) = \int_0^\infty e^{-\lambda t}p(t,y,x)\,dt$ is the λ-resolvent density for standard RBM. Furthermore, $t\mapsto L_t$ increases only when X is in ∂U.

Remark 3.2. The estimate in (ii) is [10], Theorem 3.1. The symmetry of $p(t,x,y)$ is a consequence of the Dirichlet form construction. Theorem 3.4 of [10] gives a corresponding lower bound for the transition density, and then the arguments in Section 4 of [27] show that p is continuous in x and y. The estimate in (iii) is [10], Theorem 3.2. The continuity of the Green function off the diagonal follows easily from the continuity of the transition densities. The estimate in (v) is [10], Corollaries 3.3 and 3.5. (vi) is a definition, while (vii) and (viii) are [10], Theorems 3.9 and Corollaries 3.8, respectively.

(i) and (ix) were proved in the case of bounded Lipschitz domains in [10], Section 4 and [11], respectively. To extend the results to the case of a domain such as U, one can proceed as follows. As a consequence of Proposition 2.3 and Remark 1 of [22], for any Lipschitz domain D in $\mathbb{R}^d$, one can always construct RBM $X = (X_t, \mathbb{P}_x, x \in \overline{D} \setminus \mathcal{N}_0)$ on $\overline{D}$ via the Dirichlet form approach as a continuous strong Markov process starting from every point in $\overline{D}$ except a boundary subset $\mathcal{N}_0$ of zero capacity and this process is conservative. Since by (ii) X has a jointly continuous transition density function, the RBM X can be defined to start from every point in $\overline{D}$ (cf. [31]). This in particular applies to the special Lipschitz domain U here and so (i) holds. That RBM X on $\overline{U}$ has a Skorokhod decomposition and that the local time L is a positive continuous additive functional of X with Revuz measure $\nu/2$ is a consequence of Theorem 2.6 and Remark 1 in [22]. So the conclusion of (ix) follows.

Remark 3.3. By the uniqueness of the Laplace transform and standard arguments, we obtain from Properties 3.1(ix), for any non-negative Borel measurable functions f and g, any $a < b$ in $(0,\infty)$, and any $y \in \overline{U}$,

$$\mathbb{E}^y\left[\int_a^b g(s) f(X_s)\, dL_s\right] = \frac{1}{2}\int_a^b \int_{\partial U} g(s) f(x) p(s,y,x) \nu(dx)\, ds. \tag{3.4}$$

In view of (3.1) and (3.4), if $f \geqslant 0$ and $0 < t < u < \infty$, there exists a constant c_1 depending only on t, u, and the domain U such that for $y \in \overline{U}$

$$\mathbb{E}^y\left[\int_t^u f(X_s)\, dL_s\right] \leqslant \frac{1}{2}\int_t^u \int_{\partial U} f(x) p(s,y,x) \nu(dx)\, ds \leqslant c_1 \int_{\partial U} f(x) \nu(dx). \tag{3.5}$$

Taking $f = \mathbf{1}_A$ with $\nu(A) = 0$ and using the fact that t and u are arbitrary, we conclude that (2.2) holds. The above inequality also shows that (2.3) holds. Therefore $\mathbb{P}^{x_0}$ is a weak solution to (2.1) in the sense of Definition 2.1(1) with $D = U$, even though this definition is more restrictive than the typical definition for RBM on smooth domains because of the extra conditions (2.2) and (2.3). On a smooth domain $D \subset \mathbb{R}^d$ (for example, a C^2 domain), given a d-dimensional Brownian motion W and $x_0 \in \overline{D}$, RBM can be defined as the unique continuous solution (X, L) to (2.1) that is adapted to the filtration generated by W such that L is non-decreasing and increases only when X is on the boundary of D (see [36]). The existence and uniqueness for such a solution follows from the fact that the deterministic Skorokhod problem is uniquely solvable in C^2 domains. That conditions (2.2) and (2.3) are satisfied by such a solution is a consequence of the construction. But for general Lipschitz domains, our Theorem 2.5 shows that solutions to the deterministic Skorokhod problem are not unique; therefore we need conditions (2.2) and (2.3) as part of a definition for RBM to insure even weak uniqueness for solutions to (2.1).

Suppose D is not a special Lipschitz domain U but an arbitrary bounded Lipschitz domain. The analogue of Properties 3.1(ix) follows from [11]. The argument above leading to (3.4) and (3.5) then shows that the $\mathbb{P}^{x_0}$ constructed in [10] is a weak solution to (2.1) in the sense of Definition 2.1(1) as well.

We finish this section by stating two results which can serve as substitutes for the strong Markov property.

Consider the case where Ω is the canonical probability space, that is, Ω is the collection of continuous functions from $[0,\infty)$ to $\mathbb{R}^d$. We furnish Ω with the σ-field $\mathcal{F}$ generated by the cylindrical Borel sets. In this case Ω supports shift operators, that is, maps $\theta_t : \Omega \to \Omega$ such that $X_s(\theta_t\omega) = X_{s+t}(\omega)$. Let $\mathcal{P}(z)$, $z \in \overline{D}$, denote the collection of

all probability measures $\mathbf{P}$ on Ω such that the coordinate process $t \mapsto X_t(\omega) = \omega(t)$ is a weak solution to (2.1) with $x_0 = z$ under $\mathbf{P}$ with respect to the augmented natural filtration generated by the coordinate map.

We recall Proposition 2.3 of [5]. If S is a finite stopping time with respect to $\{\mathcal{F}_t\}_{t\geqslant 0}$, $\mathcal{F}_S$ is the usual σ-field of events prior to S; that is, $\mathcal{F}_S = \{A \in \mathcal{F}_\infty\colon\ A \cap \{S \leqslant t\} \in \mathcal{F}_t \text{ for every } t \geqslant 0\}$.

Proposition 3.4. *Fix $z \in \overline{D}$. Suppose $\mathbf{P} \in \mathcal{P}(z)$, S is a finite stopping time with respect to $\{\mathcal{F}_t\}_{t\geqslant 0}$, and $\mathbf{P}_S(\omega, d\omega')$ is a regular conditional probability for the law of $X_\cdot \circ \theta_S$ under $\mathbf{P}[\,\cdot \mid \mathcal{F}_S]$. Then $\mathbf{P}_S(\omega, \cdot) \in \mathcal{P}(X_S(\omega))$ for $\mathbf{P}$-almost every ω.*

For completeness, we sketch a proof.

Proof. If $A(\omega) = \{\omega'\colon\ X_0(\omega') = X_S(\omega)\}$, then

$$A(\omega) \circ \theta_S = \big\{\omega'\colon\ X_0 \circ \theta_S(\omega') = X_S(\omega)\big\} = \big\{\omega'\colon\ X_S(\omega') = X_S(\omega)\big\}.$$

Therefore

$$\mathbf{P}\big(A(\omega) \circ \theta_S \mid \mathcal{F}_S\big) = \mathbf{1}_{\{X_S(\omega)\}}(X_S) = 1, \quad \text{a.s.}$$

The proof that L is a local time on the boundary satisfying (2.2) and (2.3) for almost every ω is similar.

The law of $[X_t - X_0 - \int_0^t \mathbf{n}(X_s)\,dL_s] \circ \theta_S$ given $\mathcal{F}_S$ is the law of $[X_{t+S} - X_S - \int_S^{S+t} \mathbf{n}(X_s)\,dL_s]$ given $\mathcal{F}_S$. It is routine to check that the conditions of Lévy's theorem (see [4, Corollary I.5.10]) are satisfied, and hence this is a Brownian motion with respect to the filtration generated by $t \mapsto X_{t+S}$. □

Let Ω and $\mathcal{F}$ be as above. We note the following analogue of Proposition 3.4, where $\mathcal{F}_S$ is replaced by the σ-field generated by the random variable X_S. An almost identical proof yields

Proposition 3.5. *Suppose $\mathbf{P} \in \mathcal{P}(x_0)$, S is a finite stopping time with respect to $\{\mathcal{F}_t\}_{t\geqslant 0}$, and $\mathbf{P}_S(\omega, d\omega')$ is a regular conditional probability for the law of $X_\cdot \circ \theta_S$ under $\mathbf{P}[\,\cdot \mid X_S]$. Then $\mathbf{P}_S(\omega, \cdot) \in \mathcal{P}(X_S(\omega))$ for $\mathbf{P}$-almost every ω.*

4. Weak uniqueness

In this section we will prove Theorem 2.3.

Remark 4.1. In [5], an assertion similar to Theorem 2.3 was made. However, there is a gap in the proof of the main theorem of [5]: the third sentence of the proof of Corollary 4.6 there is incorrect. The proof of Theorem 2.3 given below will follow the argument in [5] for the most part. The extra assumptions (2.2) and (2.3), absent from [5], allow us to carry that argument to completion. On the other hand, in [5] an assumption was required that L could be approximated by certain increasing processes. That assumption is not needed here.

We suppose in most of this section that $d \geqslant 3$; we will remove this restriction when we give the proof of Theorem 2.3. We first consider the following set-up.

Recall we write $x = (\tilde{x}, \hat{x})$ for $x \in \mathbb{R}^d$, where $\tilde{x} \in \mathbb{R}^{d-1}$ and $\hat{x} \in \mathbb{R}$. Let $\Phi : \mathbb{R}^{d-1} \to \mathbb{R}$ be a bounded Lipschitz function with Lipschitz constant $\kappa \in (0, \infty)$, let $k_0 > \|\Phi\|_\infty + 1$, $D = \{y\colon\ \Phi(\tilde{y}) < \hat{y} < k_0\}$, and $K = \{x\colon\ \hat{x} \geqslant k_0\}$. Consider the subprocess of the standard RBM X in the special Lipschitz domain $U = \{y \in \mathbb{R}^d\colon\ \hat{y} > \Phi(\tilde{y})\}$ (defined in Section 3), killed upon hitting K. We call such a subprocess of X standard RBM on $\overline{D}$ with absorption on K. It has $\overline{D} \setminus K$ as its state space. It then follows that if G^D is the Green function for RBM in $\overline{D}$ with absorption in K

and G is the Green function for standard RBM in $\overline{U}$, then $G^D(x,y) \leqslant G(x,y)$ for all x and y in $\overline{D} \setminus K$. In fact, by the strong Markov property of X, we have

$$G^D(x,y) = G(x,y) - \mathbb{E}^x\big[G(X_{T_K}, y)\big] \quad \text{for } x, y \in \overline{D} \setminus K \text{ with } x \neq y.$$

Since X behaves like a Brownian motion in U and ∂K is a horizontal hyperplane, we have by Proposition 3.1 that $G^D(x,y)$ is jointly continuous except along the diagonal and is positive in $(\overline{D} \setminus K) \times (\overline{D} \setminus K)$. Moreover $G^D(x,y)$ is symmetric in x and y in $\overline{D} \setminus K$ and for each $x \in \overline{D} \setminus K$, $y \mapsto G^D(x,y)$ can be extended continuously to $\overline{U} \setminus \{x\}$ with $G^D(x,y) = 0$ for $y \in K$. Let $\partial_r D = \{x \in \mathbb{R}^d\colon \hat{x} = \Phi(\tilde{x})\}$ be the reflecting part of the boundary of D in $\overline{D} \setminus K$. We will use $\mathbb{P}^x$ and $\mathbb{E}^x$ to denote the probability and expectation for this standard RBM X in $\overline{D}$ with absorption on K as well as for standard RBM in $\overline{D}$ without absorption; no confusion should result since we will always specify the possible values of the time t.

Lemma 4.2. *Suppose $d \geqslant 3$. There exist constants $c_1, c_2 \in (0,\infty)$ such that*

$$G^D(x,y) \leqslant c_1 \exp\big(-c_2|x-y|\big) \quad \textit{for } x, y \in D \cup \partial_r D \textit{ with } |x-y| \geqslant 1.$$

Proof. Fix $y \in D \cup \partial_r D$. By the strong Markov property,

$$G^D(x,y) = \mathbb{E}^x\big[G^D(X_{\tau(B(x,r))}, y)\big]$$

if $x \in D \cup \partial_r D$ with $|x-y| > r$. This shows that $G^D(\cdot, y)$ is harmonic with respect to X in $(D \cup \partial_r D) \setminus \{y\}$. Since the dimension $d \geqslant 3$ and by the definition of the domain D, there is an integer $m_0 \geqslant 1$, independent of y, such that for every $m \geqslant m_0$, $(D \cup \partial_r D) \setminus \overline{B(y,m)}$ is connected. For positive integers $m \geqslant m_0$, let $S_m = T_{B(y,m)}$. Suppose $x \in D$ with $|x-y| \geqslant m+1$ and $\hat{x} \geqslant k_0 - \frac{1}{2} \geqslant \|\Phi\|_\infty + \frac{1}{2}$. The tube $\mathcal{T}(x) = \{z \in D\colon |\tilde{z} - \tilde{x}| < \frac{1}{4}, \hat{z} > \hat{x} - \frac{1}{4}\}$ lies in $D \setminus \overline{B(y,m)}$, and so by the support theorem for standard d-dimensional Brownian motion (see [4], Theorem I.6.6), there exists $c_3 > 0$ not depending on such x so that $\mathbb{P}^x(T_K < S_m) \geqslant c_3$. Note that $z \mapsto \mathbb{P}^z(T_K < S_m)$ is harmonic with respect to X in the connected set $(D \cup \partial_r D) \setminus \overline{B(y,m)}$. For general $x \in D \cup \partial_r D$ with $|x-y| \geqslant m+1$, by the Harnack inequality for standard RBM (Properties 3.1(vii)) used repeatedly to a chain of balls in $(D \cup \partial_r D) \setminus \overline{B(y,m)}$ that connects x to some point $z \in D$ with $|z-y| \geqslant m+1$ and $\hat{z} \geqslant k_0 - \frac{1}{2} \geqslant \|\Phi\|_\infty + \frac{1}{2}$, there exists $c_4 > 0$ such that $\mathbb{P}^x(T_K < S_m) \geqslant c_4$ whenever $x \in D \cup \partial_r D$ with $|x-y| \geqslant m+1$. The number of times the Harnack inequality needs to be used depends only on κ and k_0, so c_4 does not depend on x nor on m.

We now show $T_K < \infty$, $\mathbb{P}^{x_0}$-a.s. for each $x_0 \in D \cup \partial_r D$. Since we are in a Lipschitz domain, $\hat{\mathbf{n}}(x) \geqslant 0$ for almost every $x \in \partial_r D$, so $\widehat{X}_t \geqslant \hat{x}_0 + \widehat{W}_t$ for every t. Since $\widehat{W}$ will eventually exceed $2k_0 + 2$ with probability one, then X must eventually hit K with probability one.

If $x \in D \cup \partial_r D$ and $|x-y| \geqslant m+1$, then because G^D is 0 on $\{x : \hat{x} = k_0\}$, we have

$$\begin{aligned} G^D(x,y) &= \mathbb{E}^x\big[G^D(X_{S_m}, y)\big] = \mathbb{E}^x\big[G^D(X_{S_m}, y);\ S_m < T_K\big] \\ &\leqslant \Big(\sup_{z \in \partial B(y,m) \cap \overline{D}} G^D(z,y)\Big)\mathbb{P}^x(S_m < T_K) \\ &\leqslant (1-c_4)\Big(\sup_{z \in \partial B(y,m) \cap \overline{D}} G^D(z,y)\Big). \end{aligned}$$

Therefore

$$\sup_{z \in B(y,m+1)^c \cap \overline{D}} G^D(z,y) \leqslant (1-c_4) \sup_{z \in \partial B(y,m) \cap \overline{D}} G^D(z,y).$$

Since $\sup_{z \in \partial B(y,1) \cap \overline{D}} G^D(z,y)$ is bounded by (3.3), then by induction

$$\sup_{z \in B(y,m)^c \cap \overline{D}} G^D(z,y) \leqslant c_5(1-c_4)^m,$$

and the lemma follows. □

In order to tie in with the set-up of [5], we define the following. Consider some $w_0 \in \partial_r D$ and $r_0 > 0$. It is easy to deduce from known results (see, e.g., [34]) that there exists a positive constant $c \in (0, \infty)$, depending only on κ, such that

$$D_0 = \big\{x \in D\colon\ |\tilde{x} - \tilde{w}_0| < r_0,\ |\hat{x} - \hat{w}_0| < c\, r_0\big\},$$

is star-shaped with respect to some point $z_0 \in D_0$. That is, there exists a Lipschitz function $\phi : \partial B(0,1) \to (0, \infty)$ such that in spherical coordinates (r, θ) centered at z_0, $D_0 = \{(r, \theta)\colon\ 0 \leqslant r < \phi(\theta)\}$. Fix some c and z_0 with the above properties, choose a $\rho_0 \in (0, \operatorname{dist}(z_0, \partial D_0)/4)$ and let K_0 be the closure of $B(z_0, \rho_0)$.

Let H_0 be a C^∞ function whose support is contained in D_0 and is disjoint from K_0 and let E_0 be a Lipschitz domain that is star-shaped with respect to z_0, that contains the support of H_0, that contains K_0, and whose closure is contained in D_0. Let $\partial_r D_0 = (\partial_r D) \cap \overline{D}_0$ and $A_0 = \partial D_0 \setminus \partial_r D$. For $x \in \partial_r D_0$ set

$$V_\delta(x) = \big\{y \in D_0\colon\ |\tilde{y} - \tilde{x}| < |\hat{y} - \hat{x}|/(2\kappa), \hat{x} < \hat{y} < \hat{x} + \delta\big\},$$

and choose δ small enough so that $V_\delta(x)$ does not intersect E_0 for any $x \in \partial_r D_0$. Then set

$$N(f)(x) = \sup_{y \in V_\delta(x)} \big|f(y)\big|, \quad x \in \partial_r D_0.$$

The following is Proposition 3.5 of [5]. See Remark 4.4 following Proposition 4.3 for the clarification of some subtle points.

Proposition 4.3. *There exists a nonnegative and bounded Borel measurable function u defined on $\overline{D}_0$ such that*

(i) *u is C^∞ in $D_0 \setminus K_0$,*
(ii) *$-\frac{1}{2}\Delta u = H_0$ in $D_0 \setminus K_0$,*
(iii) *$\partial u/\partial \mathbf{n}$ exists ν-a.e. on ∂D_0,*
(iv) *$u = 0$ ν-a.e. on A_0,*
(v) *u is continuous on D_0 and $u = 0$ on K_0,*
(vi) *$\partial u/\partial \mathbf{n} = 0$ ν-a.e. on $\partial D_0 \setminus A_0$, and*
(vii) *$\int_{\partial_r D_0} |N(\nabla u)(x)|^2 \nu(dx) < \infty$.*

Remark 4.4. (a) In [5] u is defined only on D_0. Since the support of H_0 is contained in D_0, (ii) tells us that u is harmonic in a neighborhood of ∂D_0. Since u is bounded, nontangential limits exist at ν-a.e. point of ∂D_0; see [4, Section III.4]. We define u on ∂D_0 to be equal to the nontangential limit when it exists and 0 otherwise. This allows us to define $\partial u/\partial \mathbf{n}$ at ν-a.e. point of ∂D_0.

(b) Proposition 4.3(v) was not stated in [5], but is immediate from the proof there.

Corollary 4.5. *Let $q_i = 1 - 1/i$ and $F_i(x) = u(z_0 + q_i(x - z_0))$, where u is the function described in Proposition* 4.3. *Then*

(i) *each F_i is C^∞ on $\overline{D}_0 \setminus \overline{B(z_0, \rho_0/q_i)}$,*
(ii) *the $\{F_i, i \geqslant 1\}$ are nonnegative and uniformly bounded on $\overline{D}_0$,*
(iii) *$F_i \to u$ in $D_0 \setminus K_0$ as $i \to \infty$,*
(iv) *$-\frac{1}{2}\Delta F_i \to H_0$ uniformly in $D_0 \setminus \overline{B(z_0, \rho_0/q_j)}$ if $i \geqslant j$ and $i \to \infty$,*
(v) *F_i is continuous on $\overline{D}_0$ and $F_i = 0$ in $B(z_0, \rho_0/q_i)$,*
(vi) *$F_i \to 0$ ν-a.e. on A_0 as $i \to \infty$,*
(vii) *$\partial F_i/\partial \mathbf{n} \to 0$ ν-a.e. on $\partial_r D_0$ as $i \to \infty$, and*
(viii) *$\int_{\partial_r D_0} \sup_i |\nabla F_i(x)|^2 \nu(dx) < \infty$.*

Proof. The formula for F_i and Proposition 4.3(ii) show that

$$-\frac{1}{2}\Delta F_i(x) = -\frac{1}{2}q_i^2\Delta u\big(z_0 + q_i(x - z_0)\big) = q_i^2 H_0\big(z_0 + q_i(x - z_0)\big)$$

if $x \in D_0 \setminus \overline{B}(z_0, \rho/q_j)$, and then (iv) follows. Since $\Delta u = -2H_0$ and the support $\operatorname{supp}(H_0)$ of H_0 is a positive distance from both ∂D_0 and K_0, then u is harmonic for x that are in $D_0 \setminus (K_0 \cup \operatorname{supp}(H_0))$. If $x \in A_0$, then $F_i(x) = u(z_0 + q_i(x - z_0)) \to 0$ ν-a.e. as $i \to \infty$ by Proposition 4.3(iv) and the Fatou theorem for harmonic functions in Lipschitz domains (see [4, Section III.4]). As each component of ∇u is a harmonic function in $D_0 \setminus (K_0 \cup \operatorname{supp}(H_0))$, by the (local) Fatou theorem and Proposition 4.3(vi)–(vii), we see that $\nabla F_i(x) = q_i \nabla u(z_0 + q_i(x - z_0))$ converges ν-a.e. to $\nabla u(x)$ as $i \to \infty$ for $x \in \partial_r D_0$. Hence (vii) holds. Part (viii) follows from the bound $|\nabla F_i| \leqslant N(\nabla u)$ on $\partial_r D_0$ for large i. □

For questions of weak uniqueness we may assume without loss of generality that Ω is the canonical probability space (see the paragraphs preceding Proposition 3.4) and therefore supports shift operators θ_t; see [42, Chapter 6].

For the rest of this section, $\mathbf{P}^{x_0}$ will denote the law of a weak solution to (2.1) in $\overline{D}$, killed upon hitting K, with $X_0 = x_0 \in \overline{D}$. The corresponding expectation will be denoted $\mathbf{E}^{x_0}$.

Lemma 4.6. *There exists a positive constant $c_1 < \infty$ such that for all $x_0 \in \overline{D}_0$,*

$$\mathbf{E}^{x_0}[T_{K_0 \cup A_0}] \leqslant c_1, \quad \mathbf{E}^{x_0}[L_{T_{K_0 \cup A_0}}] \leqslant c_1.$$

Proof. Recall that we write $x = (\tilde{x}, \hat{x})$ for $x \in \mathbb{R}^d$, where $\tilde{x} \in \mathbb{R}^{d-1}$ and $\hat{x} \in \mathbb{R}$. We will use similar notation for X, W and $\mathbf{n}$, i.e., $X_t = (\widetilde{X}_t, \widehat{X}_t)$, $W_t = (\widetilde{W}_t, \widehat{W}_t)$ and $\mathbf{n} = (\tilde{\mathbf{n}}, \hat{\mathbf{n}})$.

Note that $x_0 \in \overline{D}_0$ implies $\hat{x}_0 \geqslant -k_0 + 1$. Since $\hat{\mathbf{n}}(x) \geqslant 0$ for all $x \in \partial_r D_0$, we have $\widehat{X}_t \geqslant x_0 + \widehat{W}_t$ for all t. Let $c_2 > 0$ be such that $\widehat{W}_1 \geqslant 2k_0 + 2$ with probability greater than c_2. If $\widehat{X}_t \geqslant k_0 + 1$, then $T_{K_0 \cup A_0} \leqslant t$ since $K \cap \partial D_0 \subset A_0$, so for $x \in \overline{D}_0$,

$$\mathbf{P}^x(T_{K_0 \cup A_0} \leqslant 1) \geqslant c_2.$$

By Proposition 3.4, the law of $X_\cdot \circ \theta_j$ under a regular conditional probability for $\mathbf{E}^{x_0}[\,\cdot \mid \mathcal{F}_j]$ is a weak solution to (2.1) starting at X_j, so

$$\begin{aligned}
\mathbf{P}^{x_0}(T_{K_0 \cup A_0} > j + 1) &\leqslant \mathbf{P}^{x_0}(T_{K_0 \cup A_0} \circ \theta_j > 1, T_{K_0 \cup A_0} > j) \\
&= \mathbf{E}^{x_0}\big[cPx(T_{K_0 \cup A_0} \circ \theta_j > 1 \mid \mathcal{F}_j); T_{K_0 \cup A_0} > j\big] \\
&\leqslant (1 - c_2)\mathbf{P}^{x_0}(T_{K_0 \cup A_0} > j).
\end{aligned}$$

By induction, $\mathbf{P}^{x_0}(T_{K_0 \cup A_0} > j) \leqslant (1 - c_2)^j$, and the first desired inequality is now immediate.

There exists $c_3 > 0$ such that $\hat{\mathbf{n}}(x) \geqslant c_3$ for all $x \in \partial_r D$. By the support theorem for standard Brownian motion, the probability of the union of the two events $\{\widehat{W}_1 \geqslant 2k_0 + 2\}$ and $\{\inf_{t \leqslant 1} \widehat{W}_t \geqslant -1\}$ is $c_4 > 0$. On this event, as we observed above, $T_{K_0 \cup A_0} \leqslant 1$, while

$$k_0 + 1 \geqslant \widehat{X}_{1 \wedge T_{K_0 \cup A_0}} \geqslant \hat{x}_0 + c_3 L_{1 \wedge T_{K_0 \cup A_0}} + \inf_{t \leqslant 1 \wedge T_{K_0 \cup A_0}} \widehat{W}_t$$

and $\hat{x}_0 > -k_0 + 1$ since $x_0 \in \overline{D}_0$, so

$$L_{T_{K_0 \cup A_0}} = L_{1 \wedge T_{K_0 \cup A_0}} \leqslant (2k_0 + 2)/c_3.$$

It follows that with probability at least $c_4 > 0$ we have $L_{T_{K_0\cup A_0}} \leqslant c_5$ for a constant $c_5 < \infty$. Observing that

$$\mathbf{P}^{x_0}\big(L_{T_{K_0\cup A_0}} > c_5(j+1)\big) \leqslant \mathbf{P}^{x_0}\big(L_{T_{K_0\cup A_0}} \circ \theta_{U_j} > c_5,\ L_{T_{K_0\cup A_0}} > c_5 j\big)$$

where $U_j = \inf\{t \geqslant 0\colon\ L_t \geqslant c_5 j\}$, we argue similarly to the first paragraph of this proof to conclude that $\mathbf{P}^{x_0}(L_{T_{K_0\cup A_0}} > c_5 j) \leqslant (1-c_4)^j$, and the second desired inequality follows. □

Recall that $\mathbb{P}^{x_0}$ and $\mathbb{E}^{x_0}$ denote the probability and expectation for standard RBM in $\overline{D}$ started at x_0 and killed upon hitting K.

Theorem 4.7. *If H_0 is a C^∞-function whose support is contained in D_0 and is disjoint from K_0, then for $x \in D_0$,*

$$\mathbf{E}^{x_0}\left[\int_0^{T_{K_0\cup A_0}} H_0(X_s)\,ds\right] = \mathbb{E}^{x_0}\left[\int_0^{T_{K_0\cup A_0}} H_0(X_s)\,ds\right].$$

Proof. If $x_0 \in K_0$, both sides are 0, so we assume $x_0 \notin K_0$. Let functions u and F_i be as described in Corollary 4.5 and let $f(x) = \sup_i |\nabla F_i(x)|^2$. Corollary 4.5(viii) implies that $\int_{\partial_r D_0} f(x)\nu(dx) < \infty$, so by (2.3) we have $\int_\varepsilon^{t\wedge T_{K_0\cup A_0}} f(X_s)\,dL_s < \infty$, $\mathbf{P}^{x_0}$-a.s. for each $t < \infty$ and $\varepsilon > 0$. Since $x_0 \in D_0$, it takes a positive amount of time for X_t to reach ∂D, and L does not increase during that time. So for each ω there exists ε_0 depending on ω such that $\int_0^{\varepsilon_0} f(X_s)\,dL_s = 0$. We conclude that $\int_0^{t\wedge T_{K_0\cup A_0}} f(X_s)\,dL_s < \infty$, $\mathbf{P}^{x_0}$-a.s. for each $t < \infty$. The function $t \to \int_0^{t\wedge T_{K_0\cup A_0}} f(X_s)\,dL_s$ is continuous from the right by dominated convergence and continuous from the left by monotone convergence. Let

$$S_M = \inf\left\{t \geqslant 0\colon\ t + L_t + \int_0^t f(X_s)\,dL_s \geqslant M\right\} \wedge T_{K_0\cup A_0}. \tag{4.1}$$

By the continuity and finiteness of $t \to \int_0^{t\wedge T_{K_0\cup A_0}} f(X_s)\,dL_s$ and Lemma 4.6, we see that for $\mathbf{P}^{x_0}$-almost every ω there exists M_0 depending on ω such that $S_M = T_{K_0\cup A_0}$ if $M \geqslant M_0$.

Let $U_j = \inf\{t > 0\colon\ |X_t - z_0| < \rho_0(1 + j^{-1})\}$ for $j \geqslant 1$. Applying Itô's formula with F_i for i large enough so that $q_i^{-1} < 1 + j^{-1}$, we obtain

$$\begin{aligned}
&\mathbf{E}^{x_0}\big[F_i(X_{S_M\wedge U_j}) - F_i(x_0)\big]\\
&\quad= \mathbf{E}^{x_0}\left[\int_0^{S_M\wedge U_j} \nabla F_i(X_s)\cdot dX_s\right] + \frac{1}{2}\mathbf{E}^{x_0}\left[\int_0^{S_M\wedge U_j} \Delta F_i(X_s)\,ds\right]\\
&\quad= \mathbf{E}^{x_0}\left[\int_0^{S_M\wedge U_j} \nabla F_i(X_s)\cdot dW_s\right] + \mathbf{E}^{x_0}\left[\int_0^{S_M\wedge U_j} \nabla F_i(X_s)\cdot \mathbf{n}(X_s)\,dL_s\right] + \frac{1}{2}\mathbf{E}^{x_0}\left[\int_0^{S_M\wedge U_j} \Delta F_i(X_s)\,ds\right].
\end{aligned}$$

Note that the expectation of the stochastic integral term is 0 because $\nabla F_i(X_s)$ is bounded in absolute value for $s \leqslant S_M \wedge U_j$ and W is a Brownian motion. We therefore have

$$\begin{aligned}
&\mathbf{E}^{x_0}\big[F_i(X_{S_M\wedge U_j});\, T_{K_0\cup A_0} = S_M \leqslant U_j\big] + \mathbf{E}^{x_0}\big[F_i(X_{S_M\wedge U_j});\, U_j < S_M = T_{K_0\cup A_0}\big]\\
&\quad + \mathbf{E}^{x_0}\big[F_i(X_{S_M\wedge U_j});\, S_M < T_{K_0\cup A_0}\big] - F_i(x_0)
\end{aligned}$$

$$= \mathbf{E}^{x_0}\left[\int_0^{S_M\wedge U_j} \nabla F_i(X_s)\cdot \mathbf{n}(X_s)\,dL_s\right] + \frac{1}{2}\mathbf{E}^{x_0}\left[\int_0^{S_M\wedge U_j} \Delta F_i(X_s)\,ds\right]. \tag{4.2}$$

We will examine what happens to the six terms in (4.2) as $i\to\infty$, starting with the terms on the right hand side. Let $C_1 = \{x\in\partial_r D_0\colon\ (\partial F_i/\partial\mathbf{n})(x)\not\to 0\}$. By Corollary 4.5(vii), $\nu(C_1)=0$, and so by (2.2) we conclude that $\int_0^{T_{K_0\cup A_0}} \mathbf{1}_{C_1}(X_s)\,dL_s = 0$ a.s. The fact that $\nu(C_1)=0$ implies also that

$$\mathbf{E}^{x_0}\left[\int_0^{S_M\wedge U_j} \nabla F_i(X_s)\cdot \mathbf{n}(X_s)\,dL_s\right] = \mathbf{E}^{x_0}\left[\int_0^{S_M\wedge U_j} \frac{\partial F_i}{\partial\mathbf{n}}(X_s)\mathbf{1}_{C_1^c}(X_s)\,dL_s\right]. \tag{4.3}$$

The definition of f gives $|(\partial F_i/\partial\mathbf{n})(X_s)|\leqslant (f(X_s))^{1/2}$. This and (4.1) imply that

$$\int_0^{S_M\wedge U_j} \frac{\partial F_i}{\partial\mathbf{n}}(X_s)\mathbf{1}_{C_1^c}(X_s)\,dL_s \leqslant \int_0^{S_M\wedge U_j} \big(f(X_s)\big)^{1/2}\mathbf{1}_{C_1^c}(X_s)\,dL_s \leqslant \left(\int_0^{S_M}\big|f(X_s)\big|\,dL_s\right)^{1/2}(L_{S_M})^{1/2}\leqslant M.$$

By the dominated convergence theorem and Corollary 4.5(vii), the right hand side of (4.3) tends to 0 as $i\to\infty$. We have shown that the first term on the right hand side of (4.2) tends to 0 as $i\to\infty$. The limit of the second term on the right hand side is

$$-\mathbf{E}^{x_0}\left[\int_0^{S_M\wedge U_j} H_0(X_s)\,ds\right],$$

by Corollary 4.5(iv).

Now we examine the terms in the left hand side of (4.2) as $i\to\infty$. Let $C_2 = \{x\in A_0\colon\ F_i(x)\not\to 0\}$ and recall from Corollary 4.5(vi) that C_2 has null surface measure.

We claim that

$$\mathbf{P}^{x_0}(X_{T_{K_0\cup A_0}}\in C_2) = 0. \tag{4.4}$$

Let $C_2^*(\varepsilon) = \{z\in C_2\colon \operatorname{dist}(z,\partial_r D)>\varepsilon\}$ and $D_0^*(\varepsilon) = \{z\in\overline{D}_0\colon \operatorname{dist}(z,\partial_r D)\leqslant\varepsilon\}$. If (4.4) does not hold, there exists $\varepsilon>0$ such that $\mathbf{P}^{x_0}(X_{T_{K_0\cup A_0}}\in C_2^*(3\varepsilon))>0$. Let $\alpha_1 = \inf\{t\geqslant 0\colon\ X_t\notin D_0^*(2\varepsilon)\}$, $\beta_i = \inf\{t\geqslant\alpha_i\colon\ X_t\in D_0^*(\varepsilon)\}$ and $\alpha_{i+1} = \inf\{t\geqslant\beta_i\colon\ X_t\notin D_0^*(2\varepsilon)\}$ for $i=1,2,\ldots$. Note that since $t\mapsto X_t$ is continuous, $\alpha_i\to\infty$ $\mathbf{P}^{x_0}$-a.s. as $i\to\infty$. So there must exist $i\geqslant 1$ such that $\mathbf{P}^{x_0}(X_{T_{K_0\cup A_0}}\in C_2^*(3\varepsilon), \alpha_i < T_{K_0\cup A_0}\leqslant\beta_i)>0$. Away from $\partial_r D$ the process X_t behaves just like Brownian motion in $\mathbb{R}^d$. Hence $X_{t+\alpha_i}$ is a Brownian motion started at X_{α_i} for $t\leqslant\beta_i-\alpha_i$ and therefore

$$\mathbb{P}^{X_{\alpha_i}}\big(X_{T_{K_0\cup A_0}\wedge T_{D_0^*(\varepsilon)}}\in C_2^*(3\varepsilon)\big)>0. \tag{4.5}$$

But harmonic measure for Brownian motion and surface measure are mutually absolutely continuous in Lipschitz domains (see [4, Section III.5]), which contradicts (4.5). Therefore (4.4) holds.

By (4.4) and the bounded convergence theorem,

$$\begin{aligned}&\lim_{i\to\infty}\mathbf{E}^{x_0}\big[F_i(X_{S_M\wedge U_j});\, T_{K_0\cup A_0}=S_M\leqslant U_j\big]\\ &\quad=\lim_{i\to\infty}\mathbf{E}^{x_0}\big[F_i(X_{T_{K_0\cup A_0}})\mathbf{1}_{C_2^c}(X_{S_M});\ T_{K_0\cup A_0}=S_M\leqslant U_j\big]=0.\end{aligned} \tag{4.6}$$

The second term on the left hand side of (4.2) converges to

$$\mathbf{E}^{x_0}\big[u(X_{U_j});\, U_j<S_M=T_{K_0\cup A_0}\big]$$

as $i \to \infty$ by Corollary 4.5(ii) and (iii). Let $R(M, i, j)$ denote the third term on the left hand side of (4.2). Note that

$$\big|R(M,i,j)\big| \leqslant \big(\sup_k \|F_k\|_\infty\big)\mathbf{P}^{x_0}(S_M < T_{K_0\cup A_0}). \tag{4.7}$$

Finally the fourth term on the left hand side converges to $-u(x_0)$ as $i \to \infty$.

The limit $\widetilde{R}(M, j) = \lim_{i\to\infty} R(M, i, j)$ exists because all the other terms in (4.2) converge. Taking the limit as $i \to \infty$ in (4.2) we have

$$\mathbf{E}^{x_0}\big[u(X_{U_j}); U_j < S_M = T_{K_0\cup A_0}\big] + \widetilde{R}(M, j) - u(x_0) = -\mathbf{E}^{x_0}\left[\int_0^{S_M\wedge U_j} H_0(X_s)\,ds\right]. \tag{4.8}$$

Next we see what happens as we let $j \to \infty$. Note that u is continuous on D_0, $u = 0$ on K_0, and so $\lim_{j\to\infty} u(X_{U_j})\mathbf{1}_{\{U_j<S_M=T_{K_0\cup A_0}\}} = 0$. Hence, by the bounded convergence theorem, the first term on the left hand side of (4.8) converges to 0. The right hand side of (4.8) clearly converges as $j \to \infty$, so $\widetilde{R}(M, j)$ must converge to some limit $\widehat{R}(M)$ and we obtain

$$\widehat{R}(M) - u(x_0) = -\mathbf{E}^{x_0}\left[\int_0^{S_M} H_0(X_s)\,ds\right]. \tag{4.9}$$

Finally we let $M \to \infty$. Since $\mathbf{P}^{x_0}(S_M < T_{K_0\cup A_0}) \to 0$, we conclude using (4.7) that

$$-u(x_0) = -\mathbf{E}^{x_0}\left[\int_0^{T_{K_0\cup A_0}} H_0(X_s)\,ds\right]. \tag{4.10}$$

This is true for any weak solution to (2.1). In particular, since $\mathbb{P}^{x_0}$ is the law of a weak solution,

$$-u(x_0) = -\mathbb{E}^{x_0}\left[\int_0^{T_{K_0\cup A_0}} H_0(X_s)\,ds\right]. \tag{4.11}$$

Combining this with (4.10) yields our result. □

The argument from here on is very similar to the argument in [5]. We would like the conclusion of Theorem 4.7 to hold even for $x_0 \in \partial D_0$.

Proposition 4.8. *If $x_0 \in \overline{D}_0$ and H is a bounded Borel measurable function,*

$$\mathbf{E}^{x_0}\left[\int_0^{T_{K_0\cup A_0}} H(X_s)\,ds\right] = \mathbb{E}^{x_0}\left[\int_0^{T_{K_0\cup A_0}} H(X_s)\,ds\right].$$

Proof. If $x_0 \in K_0 \cup A_0$, both sides are zero, and the result holds in this case. So we suppose $x_0 \in \partial_r D_0 \cap D_0$. Since X spends zero time in $\partial_r D_0$ under both $\mathbb{P}^{x_0}$ and $\mathbf{P}^{x_0}$, it suffices to prove the proposition for H in C^∞ with support in $D_0 \setminus K_0$; we make this additional assumption on H until the end of the proof. We can then extend the result first to continuous functions and then bounded functions H by a limit argument. If $x \in D_0$, the result follows by Theorem 4.7, so we suppose $x_0 \in \partial_r D_0$. Choose $t_n \downarrow 0$ so that $\mathbf{P}^{x_0}(X_{t_n} \in \partial_r D_0) = 0$; this is possible since X spends zero time in $\partial_r D$. Let $S_1(n) = t_n \wedge T_{K_0\cup A_0}$.

By Proposition 3.4 the law of the process $X_\cdot \circ \theta_{S_1(n)}$ under a regular conditional probability for $\mathbf{P}^{x_0}(\cdot \mid \mathcal{F}_{S_1(n)})$ is a solution to (2.1) started at $X_{S_1(n)}$. This, Theorem 4.7, and the facts that the result holds for $x_0 \in K_0 \cup A_0$ and that $X_{S_1(n)} \notin \partial_r D_0$ with probability one, imply that

$$\mathbf{E}^{x_0}\left[\int_0^{T_{K_0\cup A_0}} H(X_s)\,ds \circ \theta_{S_1(n)} \mid \mathcal{F}_{S_1(n)}\right] = v(X_{S_1(n)}), \tag{4.12}$$

where

$$v(x) = \mathbb{E}^x\left[\int_0^{T_{K_0\cup A_0}} H(X_s)\,ds\right]. \tag{4.13}$$

Taking expectations,

$$\mathbf{E}^{x_0}\left[\int_{S_1(n)}^{T_{K_0\cup A_0}} H(X_s)\,ds\right] = \mathbf{E}^{x_0}\big[v(X_{S_1(n)})\big]. \tag{4.14}$$

Letting $t_n \downarrow 0$ and using Lemma 4.6, the left hand side of (4.14) converges to $\mathbf{E}^{x_0}[\int_0^{T_{K_0\cup A_0}} H(X_s)\,ds]$.

Let I be the support of H and $S_2 = \inf\{t\colon X_t \in I\}$. We have assumed that H is C^∞ with support in $D_0 \setminus K_0$ and $x_0 \in \partial_r D_0$, so $S_2 > 0$ a.s. If $B(x,r) \subset \overline{D}_0 \setminus (K_0 \cup A_0 \cup I)$, then by the strong Markov property of standard RBM

$$v(x) = \mathbb{E}^x\big[v(X_{\tau_{B(x,r)\cap\overline{D}_0}})\big],$$

so the function $v(x)$ is harmonic with respect to standard RBM X in $\overline{D}_0 \setminus (K_0 \cup A_0 \cup I)$ and hence v is continuous there (Properties 3.1(vi) and (vii)). We write

$$\mathbf{E}^{x_0}\big[v(X_{S_1(n)})\big] = \mathbf{E}^{x_0}\big[v(X_{S_1(n)\wedge S_2})\big] + \mathbf{E}^{x_0}\big[v(X_{S_1(n)}) - v(X_{S_1(n)\wedge S_2})\big].$$

The first term on the right converges to $v(x_0)$ as $n \to \infty$ by the continuity of v, while the second term on the right is bounded in absolute value by $2\|v\|_\infty \mathbf{P}^{x_0}(S_1(n) > S_2)$, which goes to zero as $n \to \infty$. Letting $n \to \infty$ in (4.14), we then have

$$\mathbf{E}^{x_0}\left[\int_0^{T_{K_0\cup A_0}} H(X_s)\,ds\right] = v(x_0).$$

Using (4.13) this proves the proposition. □

Theorem 4.9. *Let H be a bounded Borel measurable function with support in $\overline{D}_0$ and $x_0 \in \overline{D}_0$. If $0 < \lambda < (2\sup_{y\in\overline{D}_0} \mathbb{E}^y[T_{K_0\cup A_0}])^{-1}$, then*

$$\mathbf{E}^{x_0}\left[\int_0^{T_{K_0\cup A_0}} e^{-\lambda t} H(X_t)\,dt\right] = \mathbb{E}^{x_0}\left[\int_0^{T_{K_0\cup A_0}} e^{-\lambda t} H(X_t)\,dt\right].$$

Proof. The result is obvious if $x_0 \in K_0$, so we suppose $x_0 \in \overline{D}_0 \setminus K_0$. We start with an observation similar to the one in the previous proof, that since X spends zero time on the boundary of D under both $\mathbb{P}^{x_0}$ and $\mathbf{P}^{x_0}$, it is enough to consider H that are C^∞ with support in $D_0 \setminus K_0$. Let v be defined by (4.13). By Proposition 3.4, under a regular

conditional probability for $\mathbf{P}^{x_0}(\cdot \mid \mathcal{F}_t)$ the law of $X_\cdot \circ \theta_t$ is a weak solution to (2.1) started at X_t. This, together with Proposition 4.8, implies that

$$\mathbf{1}_{\{t<T_{K_0\cup A_0}\}}\mathbf{E}^{x_0}\left[\int_0^{T_{K_0\cup A_0}} H(X_s\circ\theta_t)\,ds \mid \mathcal{F}_t\right] = \mathbf{1}_{\{t<T_{K_0\cup A_0}\}}\mathbb{E}^{X_t}\left[\int_0^{T_{K_0\cup A_0}} H(X_s)\,ds\right] = \mathbf{1}_{\{t<T_{K_0\cup A_0}\}}\,v(X_t).$$

For f a bounded Borel measurable function define

$$S_\lambda f(x_0) = \mathbf{E}^{x_0}\left[\int_0^{T_{K_0\cup A_0}} e^{-\lambda t} f(X_t)\,dt\right].$$

We then have

$$\begin{aligned} S_\lambda v(x_0) &= \mathbf{E}^{x_0}\left[\int_0^{T_{K_0\cup A_0}} e^{-\lambda t} v(X_t)\,dt\right] = \mathbf{E}^{x_0}\left[\int_0^{T_{K_0\cup A_0}} e^{-\lambda t}\,\mathbf{E}^{x_0}\left[\int_0^{T_{K_0\cup A_0}} H(X_{s+t})\,ds \mid \mathcal{F}_t\right] dt\right] \\ &= \mathbf{E}^{x_0}\left[\int_0^{T_{K_0\cup A_0}} e^{-\lambda t}\int_t^{T_{K_0\cup A_0}} H(X_s)\,ds\,dt\right] = \mathbf{E}^{x_0}\left[\int_0^{T_{K_0\cup A_0}} H(X_s)\int_0^s e^{-\lambda t}\,dt\,ds\right] \\ &= \mathbf{E}^{x_0}\left[\int_0^{T_{K_0\cup A_0}} H(X_s)\frac{1-e^{-\lambda s}}{\lambda}\,ds\right] = \frac{1}{\lambda}v(x_0) - \frac{1}{\lambda}S_\lambda H(x_0), \end{aligned} \tag{4.15}$$

or $S_\lambda H(x_0) = v(x_0) - \lambda S_\lambda v(x_0)$. Define the operator R_λ on bounded Borel measurable functions by

$$R_\lambda f(x) = \mathbb{E}^x\left[\int_0^{T_{K_0\cup A_0}} e^{-\lambda t} f(X_t)\,dt\right]. \tag{4.16}$$

Then $v = R_0 H$ and so

$$S_\lambda H(x_0) = R_0 H(x_0) - \lambda S_\lambda R_0 H(x_0). \tag{4.17}$$

By using a standard limit argument, we have (4.17) holding if H is bounded and Borel measurable.

Let

$$\Theta = \sup_{\|H\|_\infty\leqslant 1} \big|S_\lambda H(x_0) - R_\lambda H(x_0)\big|,$$

where $\|H\|_\infty$ is the usual supremum norm, and note that

$$\Theta \leqslant \sup_{\|H\|_\infty\leqslant 1} \big(\big|S_\lambda H(x_0)\big| + \big|R_\lambda H(x_0)\big|\big) \leqslant 2/\lambda < \infty.$$

We have $\|R_\lambda H\|_\infty \leqslant \lambda^{-1}\|H\|_\infty$ and $\|R_0 H\|_\infty \leqslant c_1\|H\|_\infty$, where

$$c_1 = \sup_{y\in\overline{D}_0} \mathbb{E}^y[T_{K_0\cup A_0}].$$

Note that by Lemma 4.6, $c_1 < \infty$. From the semigroup property of $\mathbb{P}^x$ (cf. [4, p. 19]),

$$R_\lambda H(x_0) = R_0 H(x_0) - \lambda R_\lambda R_0 H(x_0). \tag{4.18}$$

Subtracting (4.18) from (4.17),

$$\bigl|S_\lambda H(x_0) - R_\lambda H(x_0)\bigr| = \bigl|\lambda\bigl(S_\lambda R_0 H(x_0) - R_\lambda R_0 H(x_0)\bigr)\bigr| \leqslant \lambda\Theta\|R_0 H\|_\infty \leqslant \lambda\Theta c_1\|H\|_\infty.$$

Taking the supremum over H with $\|H\|_\infty \leqslant 1$, if $\lambda \leqslant 1/(2c_1)$,

$$\Theta \leqslant \lambda\Theta c_1 \leqslant \Theta/2,$$

so $\Theta = 0$ because $\Theta < \infty$. In other words $S_\lambda H(x_0) = R_\lambda H(x_0)$ for all bounded and Borel measurable functions H. This is equivalent to the assertion of the theorem. □

Proof of Theorem 2.3. First suppose that $d \geqslant 3$ and D has the same form as D_0 described before Proposition 4.3. Recall the notation from Theorem 4.9 and its proof and the fact that $S_\lambda H(x_0) = R_\lambda H(x_0)$. Suppose H is continuous in $\overline{D}$. By the uniqueness of the Laplace transform and the continuity of $H(X_t)$, we see that $\mathbf{E}^{x_0}\bigl[H(X_{t\wedge T_{K_0\cup A_0}})\bigr] = \mathbb{E}^{x_0}\bigl[H(X_{t\wedge T_{K_0\cup A_0}})\bigr]$. As $\rho_0 \to 0$, then $T_{K_0\cup A_0} \to T_{\{x_0\}\cup A_0}$. Since X behaves like a Brownian motion when away from ∂D_0, then $T_{\{x_0\}}$ is infinite with probability one. So $\mathbf{E}^{x_0}\bigl[H(X_{t\wedge T_{A_0}})\bigr] = \mathbb{E}^{x_0}\bigl[H(X_{t\wedge T_{A_0}})\bigr]$. Since the above is true for every arbitrary but fixed $x_0 \in \overline{D}\setminus A_0$, it implies (see [42, Chapter 6]) that the finite dimensional distributions of $X_{t\wedge T_{A_0}}$ under $\mathbf{P}^{x_0}$ and under $\mathbb{P}^{x_0}$ agree (this is where Proposition 3.5 is needed). Therefore $\mathbf{P}^{x_0} = \mathbb{P}^{x_0}$ on $\mathcal{F}_{T_{A_0}}$.

Now let D be an arbitrary bounded Lipschitz domain. By standard piecing-together arguments (see [42]), it suffices to show that for each $x_0 \in \overline{D}$, any solution $\mathbf{P}^{x_0}$ agrees with $\mathbb{P}^{x_0}$ locally. That is, if $x_0 \in \overline{D}$, there exists $r > 0$ (depending on x_0) such that $\mathbf{P}^{x_0}$ and $\mathbb{P}^{x_0}$ agree on $\mathcal{F}_{T_{\partial B(x_0,r)}}$. Inside D, X under both $\mathbf{P}^{x_0}$ and $\mathbb{P}^{x_0}$ behaves like ordinary Brownian motion, so we need only consider $x_0 \in \partial D$. Let a coordinate system and a domain D_0 be chosen so that D_0 agrees with D in a neighborhood $B(x_0, r)\cap D$ of x_0 and D_0 is of the form described preceding Proposition 4.3. If r is small enough, $T_{\partial B(x_0,r)}$ will be less than T_{A_0}, and we can therefore apply the preceding paragraph.

Finally we consider the case $d = 2$. Suppose that X has law $\mathbf{P}^{x_0}$ and state space $\overline{D}$, where D is a two-dimensional Lipschitz domain. Let B be a one-dimensional Brownian motion reflecting at -1 and 1 and independent of X. Then the law of (B, X) is a weak solution to (2.1) for the Lipschitz domain $(-1, 1)\times D$, and so is unique. The uniqueness of the law of X follows easily. □

Corollary 4.10 below is presented with a view toward possible future applications. In the proof of Theorem 4.7 we applied (2.3) once for the function

$$f = \sup_i |\nabla F_i|^2 \tag{4.19}$$

and (2.2) once for the set

$$C_1 = \left\{x \in \partial_r D_0\colon \frac{\partial F_i}{\partial \mathbf{n}}(x) \not\to 0 \text{ as } j \to \infty\right\}. \tag{4.20}$$

Let $\{H_0^j\}_{j=1}^\infty$ be a countable collection of C^∞ functions with support in $D_0\setminus K_0$ whose closure under the supremum norm contains the collection of all continuous functions with support in $D_0\setminus K_0$. Let f^j and C_1^j be defined analogously to (4.19) and (4.20), but with H_0^j in place of H_0.

Corollary 4.10. *Suppose that* $\mathbf{P}^{x_0}$ *is the law of a process* X *which satisfies* (2.1) *for some Brownian motion* W, *such that* (2.4) *and* (2.5) *hold, and*

$$\int_\varepsilon^t f^j(X_s)\,dL_s < \infty, \quad \text{a.s., for all } \varepsilon > 0, \quad t < \infty, \quad j = 1, 2, \ldots \tag{4.21}$$

and

$$\int_0^\infty \mathbf{1}_{C_1^j}(X_s)\,dL_s = 0, \quad \text{a.s., for all } j = 1, 2, \ldots. \tag{4.22}$$

Then $\mathbf{P}^{x_0} = \mathbb{P}^{x_0}$.

Proof. By the proof of Theorem 4.7, using (4.21) and (4.22) in place of (2.3) and (2.2), we have

$$\mathbf{E}^{x_0}\left[\int_0^{T_{K_0\cup A_0}} H_0^j(X_s)\,ds\right] = \mathbb{E}^{x_0}\left[\int_0^{T_{K_0\cup A_0}} H_0^j(X_s)\,ds\right]$$

if $x_0 \in D_0$. By taking limits, we have the conclusion of Theorem 4.7 for all H_0 in C^∞ with support in $D_0 \setminus K_0$. With this change, we can now follow the argument given by Proposition 4.8, Theorem 4.9, and the proof of Theorem 2.3. □

The same piecing-together argument as that in the proof of Theorem 2.3 yields the following result. We leave the proof to the reader.

Theorem 4.11. *Weak uniqueness holds for* (2.1) *in special Lipschitz domains* $D \subset \mathbb{R}^d$.

5. Strong solutions in planar lip domains

In this section, we focus on strong solutions to (2.1) on "special" planar lip domains, to be defined below. We will explain at the end of this section how a strong solution can be constructed for a general lip domain from those on special lip domains.

We will say that D is a *special* planar lip domain if $D = \{(\tilde{x}, \hat{x}) \in \mathbb{R}^2\colon\ \hat{x} > \Phi(\tilde{x})\}$ where Φ is a Lipschitz function with Lipschitz constant κ strictly less than 1.

Fix a special planar lip domain D and suppose that $\mathbf{v}$ is a vector field on ∂D. We will assume that all vector fields on ∂D considered in the rest of the paper are Lebesgue measurable and satisfy $0 < c_1 < |\mathbf{v}(x)| < c_2 < \infty$, for all x, where the constants c_1 and c_2 may depend on $\mathbf{v}$. To simplify the notation, we write $\mathbf{v}(\tilde{x})$ for $\mathbf{v}(\tilde{x}, \Phi(\tilde{x}))$. Suppose W is a given two-dimensional Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, \mathbf{P})$. Let $\mathcal{F}_t^0 = \sigma(W_s\colon\ s \leqslant t)$ and let $\mathcal{F}_t$ be the usual augmentation of $\mathcal{F}_t^0$. It is well known that $\{\mathcal{F}_t\}$ is right continuous (see, e.g., [37]). We will say that X is reflecting Brownian motion in $\overline{D}$ with oblique direction of reflection $\mathbf{v}$, relative to $\mathbf{P}$ and W, starting at x_0, if X is continuous and adapted to $\{\mathcal{F}_t\}$, $X_0 = x_0$, X takes values in $\overline{D}$, and there exists a nondecreasing continuous process L (a "local time of X on the boundary of D") which is adapted to $\{\mathcal{F}_t\}$ such that $\mathbf{P}$-almost surely,

$$X_t = x_0 + W_t + \int_0^t \mathbf{v}(X_s)\,dL_s, \quad t \geqslant 0, \tag{5.1}$$

and

$$\int_0^\infty \mathbf{1}_{\partial D}(X_s)\,ds = 0. \tag{5.2}$$

Note that the above is a strong solution definition. The present definition of RBM with oblique direction of reflection is less stringent than that given in Definition 2.1 in the case of the normal reflection: here we do not assume

conditions (2.2) and (2.3) on the local time L. In this section, we will use only this definition of RBM. We will say "a strong solution to (2.1)" when we want to emphasize that we mean a strong solution for RBM with normal reflection in the sense of Definition 2.1.

We will identify points in $\mathbb{R}^2$, points in $\mathbb{C}$, and two-dimensional vectors in the obvious way. For any vector $\mathbf{u}$, let $\angle(\mathbf{u})$ be the angle formed by $\mathbf{u}$ with the positive half-line. We introduce a partial order in $\mathbb{R}^2$ by declaring that

$$x \prec y \quad \text{if and only if} \quad x = y \text{ or } \angle(y-x) \in [-\alpha_0, \alpha_0],$$

where $\alpha_0 = (\pi/4 + \arctan\kappa)/2$. Note that $\pi/8 < \alpha_0 < \pi/4$. Let $[\alpha_1, \alpha_2] = [\pi/2 - \alpha_0, \pi/2 + \alpha_0]$. Throughout this section, we will consider only those vector fields $\mathbf{v}$ which satisfy $\angle(\mathbf{v}(\tilde{x})) \in [\alpha_1, \alpha_2]$ for all $\tilde{x}$. Fix some base point $x_0 \in D$.

Let ν denote Lebesgue surface measure on ∂D. If $u : D \to \mathbb{R}$ is a bounded harmonic function, then, by Fatou's theorem for bounded harmonic functions in D, the non-tangential limit of u exists and is finite at ν-a.e. $x \in \partial D$. For $x \in \partial D$, we define $u(x)$ as the non-tangential limit of u at x whenever the limit exists and is finite.

We will say that a sequence of open sets $K_n \subset \mathbb{C}$ converges to an open set $K \subset \mathbb{C}$ if for every compact set $M \subset K$ and every compact set $L \subset (\overline{K})^c$, there is an integer $N \geqslant 1$ such that $M \subset K_n$ and $L \subset (\overline{K}_n)^c$ for every $n \geqslant N$.

Lemma 5.1. *For every vector field* $\mathbf{v}$ *that satisfies* $\angle(\mathbf{v}(\tilde{x})) \in [\alpha_1, \alpha_2]$ *for all* $\tilde{x}$*, one can find a domain* $D_{\mathbf{v}}$ *and a univalent analytic function* $f_{\mathbf{v}} : D \to D_{\mathbf{v}}$ *such that the following properties hold.*

(i) $f_{\mathbf{v}}(x_0) = 0$ *and* $\arg f'_{\mathbf{v}}(x) = \pi/2 - \angle(\mathbf{v}(\tilde{x}))$ *for* ν*-a.e.* $x = (\tilde{x}, \hat{x}) \in \partial D$.

(ii) *If* $\mathbf{v}_n$ *converges to* $\mathbf{v}$ *pointwise, then* $D_{\mathbf{v}_n}$ *converges to* $D_{\mathbf{v}}$.

(iii) *Let* $\mathcal{V}_c$ *be the family of all continuous vector fields* $\mathbf{v}$ *on* ∂D *satisfying* $\angle(\mathbf{v}(\tilde{x})) \in [\alpha_1, \alpha_2]$*, let* $\mathcal{V}_{c,b}$ *be the subfamily of* $\mathcal{V}_c$ *consisting of those vector fields that satisfy* $\angle(\mathbf{v}(\tilde{x})) = \alpha_1$ *for* $\tilde{x}$ *outside a compact interval* (*depending on* $\mathbf{v}$)*, and finally let* $\mathcal{V} \subset \mathcal{V}_c$ *be the class of* C^2*-smooth vector fields whose elements* $\mathbf{v} \in \mathcal{V}$ *correspond to functions* $f_{\mathbf{v}}$ *which are* C^2 *on* $\overline{D}$*. For every* $\mathbf{v} \in \mathcal{V}_c$*, there exists a sequence of vector fields* $\mathbf{v}_n \in \mathcal{V}$*, converging to* $\mathbf{v}$ *uniformly on bounded intervals. For every* $\mathbf{v} \in \mathcal{V}_{c,b}$*, there exists a sequence of vector fields* $\mathbf{v}_n \in \mathcal{V}$*, such that* $\angle(\mathbf{v}_n(\tilde{x}))$ *converges to* $\angle(\mathbf{v}(\tilde{x}))$ *uniformly over* $\mathbb{R}$.

(iv) *Let* $N(\varepsilon) = \{(\tilde{x}, \hat{x}) \in D\colon\ \hat{x} < \Phi(\tilde{x}) + \varepsilon\}$*. Then for every* $r > 0$,

$$\limsup_{\varepsilon \to 0} \sup_{\mathbf{v}} \sup_{\substack{x \in f_{\mathbf{v}}(N(\varepsilon)) \\ |x| < r}} \operatorname{dist}(x, \partial D_{\mathbf{v}}) = 0, \tag{5.3}$$

where the first supremum is taken over all vector fields $\mathbf{v}(\tilde{x})$ *which are Lebesgue measurable and satisfy* $\angle(\mathbf{v}(\tilde{x})) \in [\alpha_1, \alpha_2]$ *for all* $\tilde{x}$.

(v) *Let* $K^r_{\mathbf{v}}$ *be the closure of* $B(0, r) \cap D_{\mathbf{v}}$*. Suppose that vector fields* $\mathbf{v}_n$ *converge to* $\mathbf{v}$ *pointwise,* $\angle(\mathbf{v}(\tilde{x})) \in [\alpha_1, \alpha_2]$ *and* $\angle(\mathbf{v}_n(\tilde{x})) \in [\alpha_1, \alpha_2]$ *for all* $\tilde{x}$*. Let* K^*_r *be the closure of* $f_{\mathbf{v}}^{-1}(K^r_{\mathbf{v}}) \cap \bigcap_n f_{\mathbf{v}_n}^{-1}(K^r_{\mathbf{v}_n})$*. Then for any* $0 < r_1 < \infty$ *there exists* $0 < r_2 < \infty$ *such that* $B(0, r_1) \cap D \subset K^*_r$ *for* $r \geqslant r_2$.

Proof. We will use the approach of [18, Lemma 2.2]. Let $\theta(x) = \pi/2 - \angle(\mathbf{v}(\tilde{x}))$ for $x = (\tilde{x}, \hat{x}) \in \partial D$. We will denote the harmonic extension of θ to D by θ also. Let $\tilde{\theta}(x)$ be the conjugate harmonic function of $\theta(x)$ with $\tilde{\theta}(x_0) = 0$ and define $f_{\mathbf{v}} : D \to \mathbb{C}$ by setting $f_{\mathbf{v}}(x_0) = 0$ and

$$f'_{\mathbf{v}}(x) = \exp\bigl(i\bigl(\theta(x) + i\tilde{\theta}(x)\bigr)\bigr). \tag{5.4}$$

Let $D_{\mathbf{v}} = f_{\mathbf{v}}(D)$. Parts (i) and (ii) of the lemma can be proved using ideas from Lemmas 2.2 and 2.3 of [18] – we leave the details to the reader. However, we will outline the geometric idea of the construction. The function θ represents the desired amount of twisting at the boundary of D, that is, it represents $\arg f'_{\mathbf{v}}$. The boundary values of $\arg f'_{\mathbf{v}}$ uniquely determine the values of this function inside the domain, via the harmonic extension. This in turn determines the harmonic conjugate of $\arg f'_{\mathbf{v}}$, up to a constant. In this sense, $f_{\mathbf{v}}$ is uniquely determined by the

boundary values of $\arg f'_{\mathbf{v}}$, up to a few normalizing constants. Informally speaking, the function $f_{\mathbf{v}}$ is chosen to map $\mathbf{v}$ onto a vector field pointing up at almost every boundary point of $D_{\mathbf{v}}$.

We turn to part (iii). Assume that $\mathbf{v} \in \mathcal{V}_c$ and let $f_{\mathbf{v}}$ be the corresponding analytic function. We define $|\mathbf{v}(x)|$ for $x \in D$ as a harmonic extension of $\{|\mathbf{v}(x)|, x \in \partial D\}$ (recall that $|\mathbf{v}(x)|$ is assumed to be bounded). For $\varepsilon > 0$, let $f^{\varepsilon}_{\mathbf{v}}$ be the mapping constructed in a similar way to the construction of $f_{\mathbf{v}}$ but relative to the base point $x_0 + i\varepsilon$ rather than x_0. Let $\mathbf{v}_\varepsilon(\tilde{x}) = |\mathbf{v}(\tilde{x}, \hat{x}+\varepsilon)|i/(f^{\varepsilon}_{\mathbf{v}})'(\tilde{x}, \Phi(\tilde{x})+\varepsilon)$ and note that $\mathbf{v}_\varepsilon$ is analytic on ∂D because both $|\mathbf{v}(\tilde{x}, \hat{x}+\varepsilon)|$ and $(f^{\varepsilon}_{\mathbf{v}})'(\tilde{x}, \Phi(\tilde{x})+\varepsilon)$ are. Since the definition of $f_{\mathbf{v}}$ is based only on $\angle(\mathbf{v})$, we see that $f_{\mathbf{v}_\varepsilon}(x) = f^{\varepsilon}_{\mathbf{v}}(x + i\varepsilon)$. This implies that $\mathbf{v}_\varepsilon$ corresponds to a mapping $f_{\mathbf{v}_\varepsilon}$ which is analytic on $\overline{D}$. It is not hard to show that for a given continuous $\mathbf{v}$, the vector fields $\mathbf{v}_\varepsilon$ converge to $\mathbf{v}$ uniformly on bounded intervals.

Suppose that $\mathbf{v} \in \mathcal{V}_{c,b}$. Let $a < \infty$ be such that $\angle(\mathbf{v}(\tilde{x})) = \alpha_1$ for $|\tilde{x}| \geqslant a$. It follows from the definition of $\mathbf{v}_\varepsilon$ that $\angle(\mathbf{v}_\varepsilon(\tilde{x}))$ is the value of the harmonic extension h of $\angle(\mathbf{v})$ to D, evaluated at $(\tilde{x}, \hat{x}+\varepsilon)$. It will suffice to show that for any $\delta > 0$ there exists $\varepsilon_0 > 0$ such that for all $x = (\tilde{x}, \hat{x}+\varepsilon)$ with $|\tilde{x}| \geqslant a$ and $\varepsilon \leqslant \varepsilon_0$ we have $|h(x) - \alpha_1| \leqslant \delta$. Find $\varepsilon_0 > 0$ so small that Brownian motion starting from 0 makes a closed loop around $B(0, \varepsilon_0)$ before leaving $B(0, a)$ with probability greater than $1 - \delta/\alpha_2$. Then for any $x \in D$ with $\hat{x} - \Phi(\tilde{x}) \leqslant \varepsilon_0$, the harmonic measure of $B(x, a)^c \cap \partial D$ in D, relative to x, is bounded by δ/α_2. Hence, for points $x \in D$ satisfying $|\tilde{x}| \geqslant a$ and $\varepsilon \leqslant \varepsilon_0$, the value of $h(x)$ is bounded below by α_1 and bounded above by $\alpha_1 + (\delta/\alpha_2)\alpha_2 = \alpha_1 + \delta$.

(iv) By the maximum principle, $\theta(x) \in (-\alpha_0, \alpha_0)$ for all $x \in D$. Hence, $\operatorname{Re} f'_{\mathbf{v}}(x) = e^{-\widetilde{\theta}(x)} \cos\theta(x) > 0$ for all x. Let $r_0 > 0$ be such that $B(x_0, r_0) \subset D$. Since $\widetilde{\theta}(x_0) = 0$, we have $\sqrt{2}/2 \leqslant \Re f'_{\mathbf{v}}(x_0) \leqslant 1$. By the Harnack principle and the fact that the real part of an analytic function is harmonic, $0 < c_1 < \operatorname{Re} f'_{\mathbf{v}}(x) < c_2 < \infty$ in $B(x_0, r_0/2)$, where c_1 and c_2 do not depend on $\mathbf{v}$. These bounds on $\operatorname{Re} f'_{\mathbf{v}}(x)$ and the fact that $\theta(x) \in (-\alpha_0, \alpha_0)$ easily imply that for some $r_1 > 0$ independent of $\mathbf{v}$, we have $B(0, r_1) \subset f_{\mathbf{v}}(B(x_0, r_0/2))$.

Suppose that (5.3) fails. Then there exist $a, r > 0$, a sequence of vector fields $\{\mathbf{v}_n\}$ and points $\{x_n\}$ such that $\angle(\mathbf{v}_n(\tilde{x})) \in [\alpha_1, \alpha_2]$, $|x_n| < r$, $x_n \in f_{\mathbf{v}_n}(N(1/n))$, and $\operatorname{dist}(x_n, \partial D_{\mathbf{v}_n}) > a$ for all n. Note that for every vector field $\mathbf{v}$ satisfying $\angle(\mathbf{v}(\tilde{x})) \in [\alpha_1, \alpha_2]$, the corresponding domain $D_{\mathbf{v}}$ lies above the graph of a Lipschitz function $\Phi_{\mathbf{v}}$ whose Lipschitz constant is bounded by κ. Hence we can assume, passing to a subsequence, if necessary, that $\Phi_{\mathbf{v}_n}$ converge uniformly on bounded intervals to a function Φ_{v_∞}, $x_n \to x_\infty$, and the functions $s \to \int_0^s \theta_n(\tilde{x})\, d\tilde{x}$ converge uniformly on bounded intervals to a function $s \to \int_0^s \theta_\infty(\tilde{x})\, d\tilde{x}$. We have $\operatorname{dist}(x_\infty, \partial D_{\mathbf{v}_\infty}) \geqslant a > 0$ and $\operatorname{dist}(0, \partial D_{\mathbf{v}_\infty}) \geqslant r_1 > 0$. There exists $p > 0$ such that the Brownian motion starting from $0 \in D_{\mathbf{v}_\infty}$ can make a loop around x_∞ and return to $B(0, r_1/2)$ without hitting $\partial D_{\mathbf{v}_\infty}$, with probability p. This easily implies that for large n, Brownian motion starting from $0 \in D_{\mathbf{v}_n}$ can make a loop around x_n and return to $B(0, 3r_1/4)$ without hitting $\partial D_{\mathbf{v}_n}$, with probability $p/2$ or higher. By the conformal invariance of Brownian motion, Brownian motion can start from $x_0 \in D$, make a loop around $f^{-1}_{\mathbf{v}_n}(x_n) \in N(1/n)$, and return to $B(x_0, r_0/2)$ before hitting ∂D, with probability equal to or greater than $p/2$. Since the distance from $f^{-1}_{\mathbf{v}_n}(x_n)$ to ∂D goes to 0 as $n \to \infty$, this uniform bound on the probability of such a loop cannot hold as $n \to \infty$. We have obtained a contradiction, which completes the proof of part (iv).

(v) It follows from (5.4) and from the assumption that $\angle(\mathbf{v}(\tilde{x})) \in [\alpha_1, \alpha_2]$, that $\arg f'_{\mathbf{v}}(x) \in [\pi/2 - \alpha_2, \pi/2 - \alpha_1] = [-\alpha_0, \alpha_0]$ for all $x \in D$. It is elementary to see that one can find an increasing sequence of domains D_n such that $D_n \subset D$, $\bigcup_n D_n = D$, ∂D_n is the graph of a C^2-smooth function Φ_n, all Φ_n are Lipschitz with the same Lipschitz constant κ as Φ, and $|\Phi_n(\tilde{x}) - \Phi(\tilde{x})| \in (0, 1/n)$ for all $\tilde{x}$. The bound on $\arg f'_{\mathbf{v}}(x)$, the Lipschitz character of Φ_n, and elementary geometry show that $f_{\mathbf{v}}(\partial D_n)$ is the graph of a Lipschitz function with Lipschitz constant $\kappa_1 < \infty$, where κ_1 depends only on κ. Since $f_{\mathbf{v}}(D_n)$ increases to $D_{\mathbf{v}}$, and the limit of any sequence of Lipschitz functions with constant κ_1 is a Lipschitz function with constant κ_1, we see that $\partial D_{\mathbf{v}}$ is the graph of a Lipschitz function with constant κ_1. Since ∂D and $\partial D_{\mathbf{v}}$ have Lipschitz boundaries, the functions $f_{\mathbf{v}}$ and $f^{-1}_{\mathbf{v}}$ have continuous extensions to $\overline{D}$ and $\overline{D}_{\mathbf{v}}$. Hence, $f_{\mathbf{v}}(\overline{B(0, r_1) \cap D})$ is bounded and it follows that for every $r_1 < \infty$ there exists $r_2 < \infty$ such that $B(0, r_1) \cap D \subset f^{-1}_{\mathbf{v}}(K^r_{\mathbf{v}})$ for $r \geqslant r_2$. Thus, we may ignore $f^{-1}_{\mathbf{v}}(K^r_{\mathbf{v}})$ in the rest of the proof.

We will argue by contradiction. Suppose that there exists $r_1 < \infty$, $r_m < \infty$, $x_m \in B(0, r_1) \cap D$, and $\mathbf{v}_{n_m}$ such that $r_m \to \infty$ and $x_m \notin f^{-1}_{\mathbf{v}_{n_m}}(K^{r_m}_{\mathbf{v}_{n_m}})$. By compactness, we may assume that $x_m \to x_\infty \in \overline{B(0, r_1) \cap D}$. It is a

straightforward consequence of the fact that $\mathbf{v}_n \to \mathbf{v}$ and (5.4) that $f_{\mathbf{v}_n}$ converges to $f_{\mathbf{v}}$ uniformly on compact subsets of D. This easily implies that $x_\infty \notin D$.

Recall the base point $x_0 \in D$ such that $f_{\mathbf{v}}(x_0) = 0$ and fix some other point $y_0 \in D$ whose distance from $\partial D \cup \{x_0\}$ is $\rho > 0$. By the uniform convergence of $f_{\mathbf{v}_n}$ to $f_{\mathbf{v}}$ on compact subsets of D, $f_{\mathbf{v}_n}(y_0)$ converges to y_∞, and for some $\rho_0 \in (0, \rho)$, $\rho_1 > 0$ and large n, we have $f_{\mathbf{v}_n}(B(y_0, \rho_0)) \subset B(y_\infty, \rho_1) \subset D_{\mathbf{v}_n}$. Planar Brownian motion starting from x_0 hits ∂D before it makes a closed loop around $B(y_0, \rho_0)$ with probability $p > 0$, so, by conformal invariance, Brownian motion starting from 0 hits $\partial D_{\mathbf{v}_n}$ before making a closed loop around $B(y_\infty, \rho_1)$ with probability bounded below by p. This implies that for any n, the distance from $\partial D_{\mathbf{v}_n}$ to 0 is bounded above by a constant $\rho_2 < \infty$ depending only on D. It follows that a vertical half-line in $D^c_{\mathbf{v}_n}$, extending to infinity in the downward direction, has its endpoint not further than ρ_2 from 0.

Let $\partial_1 D$ and $\partial_2 D$ be the left and right connected components of $\partial D \setminus B(0, r_1)$ and let $p_0 > 0$ be such that the harmonic measure of each of the sets $\partial_1 D$ and $\partial_2 D$ in D, relative to x_0, is greater than p_0. We find $\rho_3 < \infty$ so large that Brownian motion starting from 0 will make a closed loop in the annulus $B(0, \rho_3) \setminus B(0, \rho_2)$ with probability greater than $1 - p_0$. It follows that $f_{\mathbf{v}_n}(\partial_j D) \cap B(0, \rho_3) \neq \emptyset$ for $j = 1, 2$. Since all boundaries $\partial D_{\mathbf{v}_n}$ are represented by Lipschitz functions with the same constant κ_1, this implies that for some $\rho_4 < \infty$, not depending on n, $f_{\mathbf{v}_n}(B(0, r_1) \cap D) \subset B(0, \rho_4)$.

We can find Jordan arcs $\gamma_n \subset \overline{D}$ with endpoints x_n and x_∞ such that Brownian motion starting from x_0 hits γ_n with probability q_n before hitting ∂D, and $q_n \to 0$. Since $x_\infty \in \overline{B(0, r_1) \cap D}$ and $x_m \notin f^{-1}_{\mathbf{v}_{nm}}(K^{r_m}_{\mathbf{v}_{nm}})$, one of the endpoints of the Jordan arc $f_{\mathbf{v}_n}(\gamma_n)$ is at the distance less than ρ_4 from 0 and the other one is at the distance greater than r_n. This and the facts that $r_n \to \infty$, all boundaries $\partial D_{\mathbf{v}_n}$ are represented by Lipschitz functions with the same constant κ_1, and all of them are at the distance not greater than ρ_2 from 0 easily imply that the Brownian motion starting from 0 must hit $f_{\mathbf{v}_n}(\gamma_n)$ before hitting $\partial D_{\mathbf{v}_n}$ with a probability greater than some $\hat{q} > 0$, not depending on n. This contradicts the assertion that $q_n \to 0$ and finishes our proof. □

Lemma 5.2. *Suppose that the vector field* $\mathbf{v}$ *belongs to* $\mathcal{V}$. *Then, given a Brownian motion* W *and* $x_0 \in \overline{D}$, *there exists a* (*pathwise unique*) *reflecting Brownian motion* X *in* $\overline{D}$ *with reflection field* $\mathbf{v}$ *and starting point* $X_0 = x_0$, *relative to* W. *Let* $\sigma^{\mathbf{v}}_t = \int_0^t |f'_{\mathbf{v}}(X_s)|^2\, ds$ *and* $\tau^{\mathbf{v}}_t = \inf\{s > 0\colon \sigma^{\mathbf{v}}_s > t\}$. *The process* $Y^{\mathbf{v}}_t = f_{\mathbf{v}}(X_{\tau^{\mathbf{v}}_t})$ *is reflecting Brownian motion in* $\overline{D}_{\mathbf{v}}$ *with vertical direction of reflection, relative to some Brownian motion* B.

Proof. For $n \geqslant 1$, let $D_n = \{x \in D\colon |\tilde{x}| < n, \hat{x} < n\}$ and let $\mathbf{v}_n$ be a vector field on ∂D_n which is of class C^2 and such that $\mathbf{v}_n = \mathbf{v}$ on $\partial D_n \cap \partial D_{n/2}$. It is elementary to check that our assumptions that the Lipschitz constant κ is less than 1 and $\angle(\mathbf{v}(\tilde{x})) \in [\alpha_1, \alpha_2]$ imply that the exterior cone condition (3.2) in [24] is satisfied for $\mathbf{v}$ on ∂D, and, therefore, for $\mathbf{v}_n$ on $\partial D_n \cap \partial D_{n/2}$. We choose $\mathbf{v}_n$ so that the exterior cone condition is satisfied on the whole boundary of D_n. Hence, by Corollary 5.2 of [24] (Case 1), we have strong existence and uniqueness for reflecting Brownian motion in D_n with reflection field $\mathbf{v}_n$. If n is so large that $x_0 \in \overline{D}_{n/2}$, we let X^n be reflecting Brownian motion in $\overline{D}_n$ with reflection field $\mathbf{v}_n$, relative to W, starting from x_0, and stopped at the hitting time τ_n of $D^c_{n/2}$. Let L^n denote the local time of X^n on ∂D_n and note that L^n does not increase after time ρ_n. By the strong uniqueness, $X^n_t = X^m_t$ for all integers $n \geqslant m \geqslant 1$ and all $t \leqslant \rho_m$, a.s.

We will prove that all the stopped RBMs X^n are equicontinuous. We will show that the modulus of equicontinuity of any such process is controlled by the modulus of continuity of the driving Brownian motion; see formula (5.7) below. Let $c_1 = \sin(\pi/4 - \alpha_0)$ and note that if $x \in \overline{D}$ and $\angle(y - x) \in [\alpha_1, \alpha_2]$ then

$$\operatorname{dist}(y, D^c) \geqslant c_1 |y - x|. \tag{5.5}$$

We will show that for all $t > s > 0$,

$$\left| \int_s^t \mathbf{v}_n(X^n_u)\, dL^n_u \right| \leqslant 2 \sup_{u,v \in [s,t]} |W_u - W_v|/c_1. \tag{5.6}$$

We will assume that s and t are rational. We will not incur any loss of generality because both sides of (5.6) are continuous in s and t, a.s. We always have

$$\angle\left(\int_s^t \mathbf{v}_n(X_u^n)\,dL_u^n\right) \in [\alpha_1, \alpha_2].$$

Suppose that (5.6) is not true for some s, t, and note that this implies that $X_u^n \in \partial D$ for some $u \in (s,t)$. Let t_1 be the supremum of $u < t$ such that $X_u^n \in \partial D$ and note that $t_1 > s$, a.s. We have

$$X_{t_1}^n = X_s^n + \int_s^{t_1} \mathbf{v}_n(X_u^n)\,dL_u^n + (W_{t_1} - W_s) = X_s^n + \int_s^t \mathbf{v}_n(X_u^n)\,dL_u^n + (W_{t_1} - W_s).$$

Note that $X_s^n \in \overline{D}$ and $\angle(\int_s^t \mathbf{v}_n(X_u^n)\,dL_u^n) \in [\alpha_1, \alpha_2]$. Using (5.5), we see that the distance from $x_* \stackrel{\text{def}}{=} X_s^n + \int_s^t \mathbf{v}_n(X_u^n)\,dL_u^n$ to the boundary of D is bounded below by $c_1|\int_s^t \mathbf{v}_n(X_u^n)\,dL_u^n|$. We have assumed that (5.6) is false, so the distance from x_* to ∂D is greater than $2\sup_{u,v\in[s,t]}|W_u - W_v| \geqslant 2|W_{t_1} - W_s|$. It follows that the distance from $X_{t_1}^n = x_* + W_{t_1} - W_s$ to ∂D is greater than $|W_{t_1} - W_s|$. The last quantity is non-zero because $t_1 > s$, and for all rational s and all real $u > s$, we have $W_u \neq W_s$, a.s. We have shown that $X_{t_1}^n \notin \partial D$, which is a contradiction. We conclude that (5.6) holds and so

$$|X_s^n - X_t^n| \leqslant (1 + 2/c_1) \sup_{u,v\in[s,t]} |W_u - W_v|. \tag{5.7}$$

This estimate holds with probability one simultaneously for all X^n, for integer $n \geqslant 1$.

Recall the times ρ_n when the X^n are stopped. Clearly, the ρ_n are nondecreasing in n. Let $\rho_\infty = \lim_{n\to\infty} \rho_n$. We will show that $\rho_\infty = \infty$, a.s. Suppose otherwise and let

$$M = (1 + 2/c_1) \sup_{u,v\in[0,\rho_\infty]} |W_u - W_v|.$$

By assumption, M is finite with a positive probability. By (5.7), for all sufficiently large n,

$$|X_0^n - X_{\rho_n}^n| \leqslant |X_0^n - X_{\rho_\infty}^n| \leqslant M.$$

It is easy to see that this contradicts the definition of ρ_n and we conclude that $\lim_{n\to\infty} \rho_n = \infty$. We define X_t to be X_t^n for n and t such that $t \leqslant \rho_n$. It is clear that X is reflecting Brownian motion in $\overline{D}$ with reflection field $\mathbf{v}$, relative to W, starting from x_0.

Under the assumptions of the lemma, $f_{\mathbf{v}}$ is C^2 on $\overline{D}$. Let $u(x)$ and $v(x)$ be the real and imaginary parts of $f_{\mathbf{v}}(x)$ with $x = (x_1, x_2)$. Then u and v are harmonic functions in D with $u_{x_1} + iv_{x_1} = v_{x_2} - iu_{x_2} = f'_{\mathbf{v}}(x)$ for $x \in \overline{D}$. So $|\nabla u(x)| = |\nabla v(x)| = |f'_{\mathbf{v}}(x)|$ and $\nabla u \cdot \nabla v = 0$ on $\overline{D}$. By Ito's formula, for $t \geqslant 0$,

$$u(X_t) - u(X_0) = \int_0^t \nabla u(X_s)\,dX_s = \int_0^t \nabla u(X_s)\,dW_s + \int_0^t \nabla u(X_s)\cdot\mathbf{v}(X_s)\,dL_s$$

and

$$v(X_t) - u(X_0) = \int_0^t \nabla v(X_s)\,dX_s = \int_0^t \nabla v(X_s)\,dW_s + \int_0^t \nabla v(X_s)\cdot\mathbf{v}(X_s)\,dL_s.$$

For $x \in \partial D$, $\nabla u(x)$ is the boundary value of the complex conjugate of $f'_{\mathbf{v}}(x)$, which equals $\exp(-\tilde{\theta}(x) - i\theta(x))$, where $\theta(x) = \pi/2 - \angle(\mathbf{v}(x))$. So $\nabla u(x)\cdot\mathbf{v}(x) = 0$. On the other hand, for $x \in \partial D$, $\nabla v(x) = (-u_{x_2}, u_{x_1})$ cor-

responds to $\exp(-\tilde{\theta}(x) + i\angle(\mathbf{v}(x)))$ and so $\nabla v(x) \cdot \mathbf{v}(x) > 0$. So after the time change, $Y_t^{\mathbf{v}} = f_{\mathbf{v}}(X_{\tau_t^{\mathbf{v}}})$ has the decomposition

$$Y_t^{\mathbf{v}} = Y_0^{\mathbf{v}} + B_t + \int_0^t \mathbf{e}_2(Y_s^{\mathbf{v}})\, d\widetilde{L}_s \quad \text{for } t \geqslant 0, \tag{5.8}$$

where B is a standard Brownian motion on $\mathbb{R}^2$, $\mathbf{e}_2 = (0,1)$ is the unit vertical vector in $\mathbb{R}^2$, and $\widetilde{L}$ is a continuous increasing process that increases only when $Y^{\mathbf{v}}$ is on the boundary. So $Y^{\mathbf{v}}$ is RBM in $\overline{D}_{\mathbf{v}}$ with vertical direction of reflection. The process $Y^{\mathbf{v}}$ is the unique pathwise solution to (5.8) and weak uniqueness for (5.8) holds. To see this, write $Y_t^{\mathbf{v}} = (Y_t^1, Y_t^2)$ and $B_t = (B_t^1, B_t^2)$, and denote the Lipschitz function representing $\partial D_{\mathbf{v}}$ by $\Phi_{\mathbf{v}}$. Then clearly $Y_t^1 = Y_0^1 + B_t^1$ and Y^2 can be viewed as the reflection of $Y_0^2 + B_t^2$ on the function $\Phi_{\mathbf{v}}(Y_t^1)$. By the Skorokhod lemma established in Lemma 3.13 of [16], such a reflection is pathwise unique. In particular, this Skorokhod lemma implies that $Y^{\mathbf{v}}$ is adapted to the filtration generated by B. □

Theorem 5.3. *Suppose that $\mathbf{v}_1, \mathbf{v}_2 \in \mathcal{V}$ and $\angle(\mathbf{v}_1(\tilde{x})) < \angle(\mathbf{v}_2(\tilde{x}))$ for all $\tilde{x}$. Let X^1 and X^2 be reflecting Brownian motions in $\overline{D}$ with the same driving Brownian motion W, starting from the same point $x_0 \in \overline{D}$, and with reflection directions given by $\mathbf{v}_1$ and $\mathbf{v}_2$, resp. Then $X_t^2 \prec X_t^1$ for all $t \geqslant 0$, a.s.*

Proof. Let $t_0 = \inf\{t\colon\ X_t^2 \not\prec X_t^1\}$. That is, if $\mathcal{T} = \{t\colon\ X_t^2 \prec X_t^1\}$, then $t_0 = \inf\{t\colon\ t \notin \mathcal{T}\}$. We will assume that $t_0 < \infty$ and show that this leads to a contradiction.

Step 1. We will use the argument of this first step twice in this proof. Note that in this step, we are using only two facts about t_0; the first fact is that $X_{t_0}^2 \prec X_{t_0}^1$ and the second one is that the inequality fails for some times in every right neighborhood of t_0.

First suppose that $X_{t_0}^1, X_{t_0}^2 \in D$. Then, by the continuity of paths of RBM, there exists $t_1 > t_0$ such that X_t^1, $X_t^2 \in D$ for $t \in [t_0, t_1]$. Hence, the boundary local times do not increase on $[t_0, t_1]$ for either of the two processes and so $X_t^1 - X_t^2 = X_{t_0}^1 - X_{t_0}^2$ for $t \in [t_0, t_1]$. Since $X_{t_0}^2 \prec X_{t_0}^1$, we obtain $X_t^2 \prec X_t^1$ for all $t \in [t_0, t_1]$. This contradicts the definition of t_0.

Next assume that one and only one of the processes is on the boundary at time t_0. Without loss of generality assume that $X_{t_0}^1 \in \partial D$ and $X_{t_0}^2 \in D$. Find $t_2 > t_0$ such that $X_t^2 \in D$ for $t \in [t_0, t_2]$. Let $V_t^1 = \int_0^t \mathbf{v}_1(X_s^1)\, dL_s^1$ and note that $t \to V_t^1 - V_{t_0}^1$ is a continuous vector function which satisfies $\angle(V_t^1 - V_{t_0}^1) \in [\alpha_1, \alpha_2]$ for $t \geqslant t_0$, by our assumption on $\mathbf{v}_1$. Observe that $\angle(X_{t_0}^1 - X_{t_0}^2) \in [-\alpha_0, \alpha_0)$ and $X_t^1 - X_t^2 = (X_{t_0}^1 - X_{t_0}^2) + (V_t^1 - V_{t_0}^1)$ for all $t > t_0$ which are sufficiently close to t_0, because X^2 cannot hit ∂D in some right neighborhood of t_0. If $\angle(X_{t_0}^1 - X_{t_0}^2) \in (-\alpha_0, \alpha_0)$ then, by the continuity of the trajectories of X^1 and X^2, for some $t_3 \in (t_0, t_2)$ and all $t \in (t_0, t_3)$,

$$\angle(X_t^1 - X_t^2) = \angle\big((X_{t_0}^1 - X_{t_0}^2) + (V_t^1 - V_{t_0}^1)\big) \in (-\alpha_0, \alpha_0).$$

This cannot be true, in view of the definition of t_0. If $\angle(X_{t_0}^1 - X_{t_0}^2) = -\alpha_0$ then every sufficiently short vector $\mathbf{w}$ with $\angle\mathbf{w} \in [\alpha_1, \alpha_2]$, whose starting point is at $X_{t_0}^1$ must have its endpoint inside the cone with vertex $X_{t_0}^2$ and edges inclined at angles $-\alpha_0$ and α_0. Hence, for some $t_3 \in (t_0, t_2)$ and all $t \in (t_0, t_3)$,

$$\angle(X_t^1 - X_t^2) = \angle\big((X_{t_0}^1 - X_{t_0}^2) + (V_t^1 - V_{t_0}^1)\big) \in [-\alpha_0, \alpha_0].$$

This contradicts the definition of t_0.

If $X_{t_0}^1, X_{t_0}^2 \in \partial D$ and $X_{t_0}^1 \neq X_{t_0}^2$ then $\angle(X_{t_0}^1 - X_{t_0}^2) \in (-\alpha_0, \alpha_0)$, so by the continuity of paths, the same is true for all $t \in (t_0, t_4)$, where t_4 is some time strictly greater than t_0. Once again, we have obtained a contradiction to the definition of t_0.

Step 2. It remains to consider the case when $X_{t_0}^1 = X_{t_0}^2 \in \partial D$. First we claim that for every $t_5 > t_0$ there exist $s, t \in (t_0, t_5)$ such that $X_t^1 \in \partial D$ and $X_s^2 \in \partial D$. If neither process visits ∂D during (t_0, t_5) then $X_t^1 = X_t^2$ for $t \in (t_0, t_5)$, which contradicts the definition of t_0. Suppose that for some $t_5 > t_0$ there exists $t_6 \in (t_0, t_5)$ such that

$X^1_{t_6} \in \partial D$ but $X^2_s \in D$ for all $s \in (t_0, t_5)$. Then $\angle(X^1_{t_6} - X^2_{t_6}) = \angle(V^1_{t_6} - V^1_{t_0}) \in [\alpha_1, \alpha_2]$. This implies that $X^2_{t_6} \notin D$, a contradiction. A similar argument applies when the roles of the processes are reversed. This completes the proof of our claim that for every $t_5 > t_0$ there exist $s, t \in (t_0, t_5)$ such that $X^1_t \in \partial D$ and $X^2_s \in \partial D$. If $X^1_t \in \partial D$, then $\angle(X^2_t - X^1_t) \in [-\arctan\kappa, \pi + \arctan\kappa]$, and similarly, if $X^2_s \in \partial D$ then $\angle(X^1_s - X^2_s) \in [-\arctan\kappa, \pi + \arctan\kappa]$. Assuming $X^1_t \in \partial D$ and $X^2_s \in \partial D$, the continuity of the function $t \to \angle(X^2_t - X^1_t)$ implies that there must be a time u between t and s with $X^1_u = X^2_u$, or $\angle(X^2_u - X^1_u) \in [-\alpha_0, \alpha_0]$, or $\angle(X^1_u - X^2_u) \in [-\alpha_0, \alpha_0]$. Therefore for every $t_5 > t_0$ there exists $u \in (t_0, t_5)$ such that $X^2_u \prec X^1_u$ or $X^1_u \prec X^2_u$.

We will show that for some $t \in (t_0, t_5)$, $X^1_t \prec X^2_t$. Suppose there is no such t. Consider the open set of all $t \in (t_0, t_5)$ where the condition $X^2_t \prec X^1_t$ fails and let t_7 be the midpoint of the longest interval in this set. Note that $t_0 < t_7 < t_5$. Let t_8 be the supremum of $t \prec t_7$ such that $X^2_t \prec X^1_t$ or $X^1_t \prec X^2_t$, and note that $t_8 \in (t_0, t_7)$. If $X^1_{t_8} \prec X^2_{t_8}$, then we are done. Otherwise $X^2_{t_8} \prec X^1_{t_8}$ but this inequality fails in every right neighborhood of t_8. The argument of Step 1 applied with t_8 in place of t_0 implies that $X^1_{t_8} = X^2_{t_8}$, so $X^1_{t_8} \prec X^2_{t_8}$. We have proved that for every $t_5 > t_0$, there exists $t \in (t_0, t_5)$ with $X^1_t \prec X^2_t$.

Let $V^2_t = \int_0^t \mathbf{v}_2(X^2_s)\, dL^2_s$ and recall that $\mathbf{v}_1$ and $\mathbf{v}_2$ are C^2 and $\angle(\mathbf{v}_1(\tilde{x})) < \angle(\mathbf{v}_2(\tilde{x}))$. By the continuity of the trajectories of X^1 and X^2, there exist $t_9 > t_0$ and α_3, α_4 such that

$$\alpha_1 < \angle(V^1_t - V^1_{t_0}) < \alpha_3 < \alpha_4 < \angle(V^2_t - V^2_{t_0}) < \alpha_2, \tag{5.9}$$

for all $t \in (t_0, t_9)$. Find $t_{10} \in (t_0, t_9)$ such that $X^1_{t_{10}} \prec X^2_{t_{10}}$. Since $X^1_{t_0} = X^2_{t_0}$,

$$(V^2_{t_{10}} - V^2_{t_0}) - (V^1_{t_{10}} - V^1_{t_0}) = X^2_{t_{10}} - X^1_{t_{10}},$$

so

$$\angle\big((V^2_{t_{10}} - V^2_{t_0}) - (V^1_{t_{10}} - V^1_{t_0})\big) = \angle(X^2_{t_{10}} - X^1_{t_{10}}) \in [-\alpha_0, \alpha_0].$$

This condition and (5.9) applied with $t = t_{10}$ cannot hold simultaneously for any triplet of vectors $(V^1_{t_{10}} - V^1_{t_0}, V^1_{t_{10}} - V^1_{t_0}, (V^2_{t_{10}} - V^2_{t_0}) - (V^1_{t_{10}} - V^1_{t_0}))$. □

Suppose that Y is reflecting Brownian motion in $\overline{D}_{\mathbf{v}}$ with vertical direction of reflection, $\tau^{\mathbf{v}}_t = \int_0^t |(f^{-1}_{\mathbf{v}})'(Y_s)|^2\, ds$, $\sigma^{\mathbf{v}}_t = \inf\{s > 0\colon \tau^{\mathbf{v}}_s > t\}$, and $X_t = f^{-1}_{\mathbf{v}}(Y(\sigma^{\mathbf{v}}_t))$. Then we will call X a *conformally invariant* reflecting Brownian motion (CIRBM) in $\overline{D}$ with direction of reflection $\mathbf{v}$, and the same term will be applied to any process with the same distribution.

Recall the following. A function ϕ is called lower semicontinuous on $\mathbb{R}$ if for every $x \in \mathbb{R}$, $\phi(x) \leqslant \liminf_{y\to x} \phi(y)$. A function ϕ is lower semicontinuous if and only if for each $a \in \mathbb{R}$, $\{x\colon \phi(x) > a\}$ is open. A function ψ is called upper semicontinuous if $-\psi$ is lower semicontinuous. Hence if G is an open subset of $\mathbb{R}$, then $\mathbf{1}_G$ is lower semicontinuous. It is well known that ϕ is lower semicontinuous if and only if there is a strictly increasing sequence of continuous functions ϕ_n that converge to ϕ pointwise on $\mathbb{R}$.

Lemma 5.4. (i) *Suppose that $\delta > 0$ and the vector field $\mathbf{v}$ on ∂D is such that $\angle\mathbf{v}(\tilde{x})$ is lower semicontinuous and takes values in $[\alpha_1 + \delta, \alpha_2 - \delta]$. Then, given a Brownian motion W, there exists a CIRBM X in $\overline{D}$ with reflection field $\mathbf{v}$, adapted to the filtration of W, and such that $X_t = X_0 + W_t + U_t$ for every $t \geqslant 0$ where $\angle(U_t - U_s) \in [\alpha_1, \alpha_2]$ for all $t > s$, and U does not change when $X \in D$, i.e., $U_t = U_s$ if $X_u \in D$ for all $u \in [s, t]$.*

(ii) *If $x_0 \in \overline{D}$, $\mathbf{v}_1$ and $\mathbf{v}_2$ satisfy the assumptions of part* (i), *and $\angle(\mathbf{v}_1(\tilde{x})) \leqslant \angle(\mathbf{v}_2(\tilde{x}))$ for all $\tilde{x}$, then one can construct the corresponding CIRBM's X^1 and X^2 as in* (i), *relative to the same Brownian motion W, starting from $x_0 \in \overline{D}$, and such that $X^2_t \prec X^1_t$ for all $t \geqslant 0$, a.s.*

Proof. (i) It follows from the first part of the proof for Lemma 5.2 (see the paragraph containing (5.5)–(5.7)) that all RBMs with reflection field $\mathbf{v} \in \mathcal{V}$ satisfy the same condition (5.7). Hence, they are equicontinuous.

We will prove that there is sequence of vector fields $\mathbf{v}_n \in \mathcal{V}$, such that $\angle(\mathbf{v}_n(\tilde{x})) < \angle(\mathbf{v}_m(\tilde{x}))$ for all $n < m$ and $\tilde{x}$, and $\lim_{n\to\infty} \angle(\mathbf{v}_n(\tilde{x})) = \angle(\mathbf{v}(\tilde{x}))$ for all $\tilde{x}$.

Recall that $\alpha_1 \in (\frac{\pi}{4}, \frac{3\pi}{8})$ is the angle specified in the fourth paragraph in this section. By the remark given before the lemma, in view of the assumption that $\angle(\mathbf{v})$ is lower semicontinuous, there exists a strictly increasing sequence of continuous functions $\tilde{\psi}_n$ converging to $\angle(\mathbf{v})$ pointwise on $\mathbb{R}$. Let $\psi_n(\tilde{x}) = \max\{\tilde{\psi}_n(\tilde{x}), \alpha_1 + \delta\}$ and we define for $\tilde{x} \in \mathbb{R}$,

$$\phi_n(\tilde{x}) = \begin{cases} \psi_n(\tilde{x}), & |\tilde{x}| \leqslant n, \\ \alpha_1 + \delta, & |\tilde{x}| \geqslant n+1, \\ \min\{\psi_n(\tilde{x}),\ \psi_n(-n)(\tilde{x}+n+1) - (\alpha_1+\delta)(\tilde{x}+n)\}, & \tilde{x} \in (-n-1,\ -n), \\ \min\{\psi_n(\tilde{x}),\ \psi_n(n)(n-\tilde{x}+1) + (\alpha_1+\delta)(\tilde{x}-n)\}, & \tilde{x} \in (n,\ n+1). \end{cases}$$

It is easy to see that every function ϕ_n is continuous and takes values in $[\alpha_1 + \delta, \alpha_2 - \delta]$. The sequence $\{\phi_n, n \geqslant 1\}$ is nondecreasing and converges to $\angle(\mathbf{v})$ pointwise on $\mathbb{R}$. For each $n \geqslant 1$, since $\phi_n(\tilde{x}) = \alpha_1 + \delta$ for $|\tilde{x}| \geqslant n+1$, by Lemma 5.1(iii), there exist $\mathbf{v}_n \in \mathcal{V}$ such that

$$\left| \angle\big(\mathbf{v}_n(\tilde{x})\big) - \left(\phi_n(\tilde{x}) - \frac{\delta}{2^n}\right) \right| < \frac{\delta}{2^{n+2}} \quad \text{for all } \tilde{x} \in \mathbb{R}.$$

It is clear that $\angle(\mathbf{v}_n(\tilde{x})) < \angle(\mathbf{v}_m(\tilde{x}))$ for all $n < m$ and $\tilde{x} \in \mathbb{R}$, and $\lim_{n\to\infty} \angle(\mathbf{v}_n(\tilde{x})) = \angle(\mathbf{v}(\tilde{x}))$ for all $\tilde{x} \in \mathbb{R}$.

Let $X^n = (X^{n,1}, X^{n,2})$ be RBM in $\overline{D}$ with reflection direction $\mathbf{v}_n$. By Theorem 5.3, $X_t^n \succ X_t^m$ for every $n < m$ and every $t \geqslant 0$ a.s., so $X_t^{n,1} \geqslant X_t^{m,1}$, and, therefore, $\lim_{n\to\infty} X_t^{n,1} = X_t^{\infty,1}$ exists and is finite for all $t \geqslant 0$, a.s., by (5.7). Next we will show that a.s., $\lim_{n\to\infty} X_t^{n,2}$ exists for all $t \geqslant 0$. Suppose that this is not the case for some ω. Let

$$M_t = \sup\big\{|W_u - W_s|\colon\ 0 \leqslant s \leqslant u \leqslant t\big\} + \operatorname{dist}(x_0, \partial D);$$

this quantity is finite for every $t \geqslant 0$, a.s. Since X^n is the sum of W and a process which does not change when X^n is inside the domain, the distance from X_t^n to the boundary of D does not exceed M_s for all $0 \leqslant t \leqslant s$. Using compactness and the diagonalization method, we can extract a subsequence n_k (depending on ω) such that $X_t^{n_k,2}$ converges for every rational $t > 0$. Let $\widehat{X}_t^{\infty,2}$ be the limit of $X_t^{n_k,2}$. The function $\widehat{X}_t^{\infty,2}$ can be extended in a continuous way to all real $t \geqslant 0$ and, moreover, $X_t^{n_k,2} \to \widehat{X}_t^{\infty,2}$ for all real $t \geqslant 0$, because (5.7) shows that the processes $X_t^{n_k,2}$ are equicontinuous. We claim that if $(X_t^{\infty,1}, \widehat{X}_t^{\infty,2}) \in D$ for all $t \in [t_1, t_2]$ where $0 \leqslant t_1 < t_2 < \infty$ then $(X_t^{\infty,1}, \widehat{X}_t^{\infty,2}) - (X_s^{\infty,1}, \widehat{X}_s^{\infty,2}) = W_t - W_s$ for all $s, t \in [t_1, t_2]$. To see this, note that $\delta_0 \stackrel{\text{def}}{=} \inf_{t\in[t_1,t_2]} \operatorname{dist}((X_t^{\infty,1}, \widehat{X}_t^{\infty,2}), \partial D) > 0$. Since the $X_t^{n_k}$ are equicontinuous, they converge uniformly to $(X_t^{\infty,1}, \widehat{X}_t^{\infty,2})$ on $[t_1, t_2]$, and so for large n_k, $\inf_{t\in[t_1,t_2]} \operatorname{dist}(X_t^{n_k}, \partial D) \geqslant \delta_0/2$. This implies that the local time term in the Skorokhod decomposition for X^{n_k} does not change on the interval $[t_1, t_2]$, for any large n_k, and so $X_t^{n_k} - X_s^{n_k} = W_t - W_s$ for large n_k and all $s, t \in [t_1, t_2]$. Our claim now follows by taking the limit.

Suppose that for some other subsequence m_k, $\lim_{k\to\infty} X_t^{m_k,2} = \widetilde{X}_t^{\infty,2}$ for all $t \geqslant 0$, where $\widetilde{X}_t^{\infty,2}$ is not identically equal to $\widehat{X}_t^{\infty,2}$. Since both functions are continuous, there exist $0 \leqslant t_3 < t_4 < \infty$ such that at least one of the following must hold: (I) $(X_t^{\infty,1}, \widehat{X}_t^{\infty,2}) \in \partial D$ for $t = t_3, t_4$ but not for any $t \in (t_3, t_4)$, and $(X_{t_5}^{\infty,1}, \widetilde{X}_{t_5}^{\infty,2}) \in \partial D$ for some $t_5 \in (t_3, t_4)$, or (II) $(X_t^{\infty,1}, \widetilde{X}_t^{\infty,2}) \in \partial D$ for $t = t_3, t_4$ but not for any $t \in (t_3, t_4)$, and $(X_{t_5}^{\infty,1}, \widehat{X}_{t_5}^{\infty,2}) \in \partial D$ for some $t_5 \in (t_3, t_4)$. We will only discuss (I) as (II) can be treated in an analogous way. Let $t_6 = \sup\{t < t_5\colon\ \widetilde{X}_t^{\infty,2} = \widehat{X}_t^{\infty,2}\}$ and note that $t_6 \geqslant t_3$. The second component of every vector $\mathbf{v}(\tilde{x})$ is positive, so the second component of $\int_s^t \mathbf{v}_n(X_u^n)\, dL_u^n$ is non-negative for all n and $0 \leqslant s < t$, and we obtain $\widetilde{X}_{t_5}^{n,2} - \widetilde{X}_{t_6}^{n,2} \geqslant W_{t_5}^2 - W_{t_6}^2$. Passing to the limit along m_k,

$$\widetilde{X}_{t_5}^{\infty,2} - \widetilde{X}_{t_6}^{\infty,2} \geqslant W_{t_5}^2 - W_{t_6}^2 = \widehat{X}_{t_5}^{\infty,2} - \widehat{X}_{t_6}^{\infty,2}.$$

But this contradicts the fact that $\widetilde{X}_{t_5}^{\infty,2} < \widehat{X}_{t_5}^{\infty,2}$. This completes the proof that the limit $\lim_{n\to\infty} X_t^{n,2}$ exists. Hence, $X_t = \lim_{n\to\infty} X_t^n$ exists for all $t \geqslant 0$, a.s.

Next we will show that X is a CIRBM with reflection direction $\mathbf{v}$. Let $\sigma_t^n = \int_0^t |f'_{\mathbf{v}_n}(X_s^n)|^2\,ds$, $\tau_t^n = \inf\{s > 0\colon \sigma_s^n \geqslant t\}$, and recall from Lemma 5.2 that $Y_t^n = f_{\mathbf{v}_n}(X^n(\tau_t^n))$ is RBM in $\overline{D}_{\mathbf{v}_n}$ with vertical direction of reflection (always pointing up).

The domain $D_{\mathbf{v}}$ has the representation $\{(\tilde{x}, \hat{x}) : \hat{x} > \Phi_{\mathbf{v}}(\tilde{x})\}$ for some continuous (in fact, Lipschitz) function $\Phi_{\mathbf{v}}$. Recall that $f_{\mathbf{v}}(x_0) = 0$. If a two-dimensional Brownian motion $Z = (Z^1, Z^2)$ is given, we let $\widetilde{Y}^1 = Z^1$ and we define $\widetilde{Y}^2$ as the reflection of Z^2 on the function $\Phi_{\mathbf{v}}(Z^1)$, using the deterministic Skorokhod lemma (see Lemma 3.13 of [16]). Then $\widetilde{Y} = (\widetilde{Y}^1, \widetilde{Y}^2)$ is the (pathwise unique) RBM in $\overline{D}_{\mathbf{v}}$ with vertical direction of reflection, starting from 0 and driven by Z. The boundary functions $\Phi_{\mathbf{v}_n}$ of the domains $D_{\mathbf{v}_n}$ converge, by Lemma 5.1(ii). Note that $Y_0^n = f_{\mathbf{v}_n}(x_0) = 0$ for all n. Our explicit construction of the RBM with vertical direction of reflection together with Corollary 3.16 of [16] show that if $\widetilde{Y}^n$ is a sequence of RBMs in $\overline{D}_{\mathbf{v}_n}$ with vertical direction of reflection, starting from 0, and driven by the same Brownian motion Z, then $\widetilde{Y}^n$ converges a.s. to a RBM $\widetilde{Y}$ in $\overline{D}_{\mathbf{v}}$ with vertical direction of reflection, starting from 0, with respect to the uniform topology on bounded intervals. This, of course, implies that $\widetilde{Y}^n$ converges in distribution to $\widetilde{Y}$. By Lemma 5.2 each Y^n has the same distribution as $\widetilde{Y}^n$. Hence, Y^n converges in distribution to $\widetilde{Y}$. We will later show that a subsequence of $\{Y^n,\ n \geqslant 1\}$ converges a.s. to a process Y.

Next we will show that the processes $\widetilde{Y}$ and X spend zero time on the boundary. Recall the construction of $\widetilde{Y}$ using the ideas of [16] given in the previous paragraph. Conditioning on $\{Z_t^1, t \geqslant 0\}$, and using Corollary 4.7 of [16], we see that $\widetilde{Y}$ spends zero time on the boundary of the domain. The argument for the process X is different. Let $\alpha_3 = \pi/4 + (\pi/4 - \alpha_0)/2$. We will say that W has an α_3-cone point at time $t_1 \geqslant 0$ if for some $t_2 > t_1$ and all $t \in (t_1, t_2)$, $\angle(W_t - W_{t_1}) \in (\pi/2 - \alpha_3, \pi/2 + \alpha_3)$. Since $\alpha_3 > \pi/4$, Brownian paths contain α_3-cone points a.s., by the results of [14] or [39]. We need the following stronger version of this result. For every $t_1 > 0$, with probability 1, for every $t_2 \in (0, t_1)$, there exists $t_3 \in (t_2, t_1)$, depending on ω, such that $\angle(W_t - W_{t_3}) \in (\pi/2 - \alpha_3, \pi/2 + \alpha_3)$ for all $t \in (t_3, t_1)$. This stronger version follows easily from an interpretation of cone points as the times when an obliquely reflecting Brownian motion in a wedge hits the vertex (see [35]) and the fact that the vertex is a regular point for such a process. Fix any $t_1 > 0$ and find a time $t_3 \in (0, t_1)$ with the property stated above. Define the process $\overline{X}_t^n$ by $\overline{X}_t^n = X_t^n$ for $t \leqslant t_3$ and $\overline{X}_t^n = X_{t_3}^n + W_t - W_{t_3}$ for $t \in (t_3, t_1]$. By the definition of t_3, $\overline{X}_t^n$ stays in the cone $C = \{x \in \mathbb{C}\colon \angle(x - X_{t_3}^n) \in (\pi/2 - \alpha_3, \pi/2 + \alpha_3)\}$ for $t \in (t_3, t_1]$. Any open cone with vertex in $\overline{D}$ and the edges forming angles $\pi/2 - \alpha_3$ and $\pi/2 + \alpha_3$ with the horizontal is a subset of D, by our assumptions on the boundary of D. We conclude that $\overline{X}_t^n$ stays in D for $t \in (t_3, t_1]$ so it is a solution to (2.1) on $[0, t_1]$. By the pathwise uniqueness of the solution [24], we see that $\overline{X}_t^n = X_t^n$ for $t \in [0, t_1]$ (strictly speaking, the results in [24] are only proved for bounded domains but a simple stopping time argument can be combined with those results to draw the needed conclusion; we leave the details to the reader). It follows that for a fixed t_1, $X_{t_1}^n \in D$ with probability 1. By Fubini's theorem, X^n spends zero time on the boundary of D. We can strengthen this result as follows. Let $\rho(t_1) > 0$ be the distance from $\overline{X}_{t_1}^n = X_{t_1}^n$ to the boundary of the open cone C. The quantity $\rho(t_1)$ depends only on the trajectory of W and by Fubini's theorem, $\rho(t) > 0$ for almost all $t > 0$, a.s. Note that $\rho(t_1)$ is a lower bound for the distance from $X_{t_1}^n$ to ∂D. This property is invariant under limits in the following sense. If V^n is any sequence of processes such that the distance from V_t^n to ∂D is bounded below by $\rho(t)$ for every $t > 0$, then the same holds for $V^\infty = \lim_{n\to\infty} V^n$, if the limit exists. By Fubini's theorem, V^∞ spends zero time on ∂D, a.s. The last conclusion applies, in particular, to all processes appearing in Lemma 5.6 and its proof.

Recall the definition of $N(\varepsilon)$ from Lemma 5.1(iv) and let $N^c(\varepsilon)$ be the complement of this set. Let $\chi_\varepsilon : \mathbb{R}^2 \to [0, 1]$ be a smooth function that is equal to 1 on $N(\varepsilon/2)$ and equal to 0 on $N^c(\varepsilon)$. We will denote $1 - \chi_\varepsilon$ by χ_ε^c. For $t \geqslant 0$, let

$$\bar{\sigma}_t^{n,\varepsilon} = \int_0^t \left|f'_{\mathbf{v}_n}(X_s^n)\right|^2 \chi_\varepsilon(X_s^n)\,ds, \quad \overline{\sigma}_t^{\varepsilon} = \int_0^t \left|f'_{\mathbf{v}}(X_s)\right|^2 \chi_\varepsilon(X_s)\,ds, \quad \tilde{\sigma}_t^{n,\varepsilon} = \int_0^t \left|f'_{\mathbf{v}_n}(X_s^n)\right|^2 \chi_\varepsilon^c(X_s^n)\,ds,$$

$$\tilde{\sigma}_t^{\varepsilon} = \int_0^t \left|f'_{\mathbf{v}}(X_s)\right|^2 \chi_\varepsilon^c(X_s)\,ds, \quad \sigma_t^n = \int_0^t \left|f'_{\mathbf{v}}(X_s)\right|^2 (X_s^n)\,ds, \quad \sigma_t = \int_0^t \left|f'_{\mathbf{v}}(X_s)\right|^2 (X_s)\,ds,$$

$$\tau_t = \inf\{s > 0\colon \sigma_s > t\}, \quad N_{\mathbf{v}}(\delta) = \big\{x \in D_{\mathbf{v}}\colon \operatorname{dist}(x, \partial D_{\mathbf{v}}) < \delta\big\},$$

and note that $\sigma_t^n = \bar{\sigma}\sigma_t^{n,\varepsilon} + \tilde{\sigma}_t^{n,\varepsilon}$ and $\sigma_t = \bar{\sigma}_t^{\varepsilon} + \tilde{\sigma}_t^{\varepsilon}$. It follows from (5.4) that $f'_{\mathbf{v}_n} \to f'_{\mathbf{v}}$ uniformly on compact subsets of D. This and the convergence of X^n to X imply that for fixed $t, \varepsilon > 0$, we have a.s.,

$$\lim_{n\to\infty} \tilde{\sigma}_t^{n,\varepsilon} = \tilde{\sigma}_t^{\varepsilon} \leqslant \sigma_t. \tag{5.10}$$

Let $K_{\mathbf{v}}^r$ be the closure of $B(0,r) \cap D_{\mathbf{v}}$ and let K_r^* be the closure of $f_{\mathbf{v}}^{-1}(K_{\mathbf{v}}^r) \cap \bigcap_n f_{\mathbf{v}_n}^{-1}(K_{\mathbf{v}_n}^r)$. By Lemma 5.1(v), for any $r_1 < \infty$ there exists $r_2 < \infty$ such that $B(x_0, r_1) \cap D \subset K_r^*$ for $r \geqslant r_2$. Let $T(Y, r)$ be the first exit time from K_r for the process Y and let $T^*(X^n, r)$ be the first exit time from K_r^* for X^n. A similar notation will apply to other processes.

Fix some $t > 0$ and arbitrarily small $p, \eta > 0$. Find $r_1 > 0$ so large that for some $r_2 > 0$, $B(x_0, r_2) \cap D \subset K_{r_1}^*$ and

$$\mathbf{P}\big(\big(1 + (2/c_1)\big) \sup_{u,v\in[0,t]} |W_u - W_v| \geqslant r_2\big) \leqslant p.$$

Then, by (5.7), for all n,

$$\mathbf{P}\big(T^*(X^n, r_2) \leqslant t\big) \leqslant p. \tag{5.11}$$

Note that

$$\bar{\sigma}_{T^*(X^n,r_2)}^{n,\varepsilon} \leqslant \int_0^{T(Y^n,r_1)} \mathbf{1}_{f_{\mathbf{v}_n}(N(\varepsilon))}(Y_s^n)\, ds.$$

The second component $\widetilde{Y}^2$ of $\widetilde{Y}$ is a Brownian motion plus a non-decreasing process. If the vertical component Z^2 of the driving Brownian motion Z has an increment greater than $3r_1$ over some interval $[s,t]$ then so does $\widetilde{Y}^2$ and then $T(\widetilde{Y}, r_1) \leqslant t$. The probability that there are no $s, t \in [m, m+1]$ with $Z_t^2 - Z_s^2 \geqslant 3r_1$ is less than $q_1 < 1$, not depending on m. Hence, the probability that there are no $s, t \in [0, m]$ with $Z_t^2 - Z_s^2 \geqslant 3r_1$ is less than q_1^m. This shows that $\mathbf{P}(T(\widetilde{Y}, r_1) \geqslant m) \leqslant q_1^m$, so the expectation of $T(\widetilde{Y}, r_1)$ is finite. Note that the same argument applies to $\widetilde{Y}^n$ in place of $\widetilde{Y}$, so there is a uniform upper bound for expectations of stopping times $T(\widetilde{Y}^n, r_1)$.

Since $\widetilde{Y}$ spends zero time on the boundary, we can find $\delta > 0$ small such that

$$\mathbf{P}\left(\int_0^{T(\widetilde{Y},r_1)} \mathbf{1}_{N_{\mathbf{v}}(\delta)}(\widetilde{Y}_s)\, ds > \eta\right) \leqslant p.$$

Since $\widetilde{Y}_t^n$ converges to $\widetilde{Y}_t$ a.s.,

$$\mathbf{P}\left(\int_0^{T(\widetilde{Y}^n,r_1)} \mathbf{1}_{N_{\mathbf{v}}(\delta/2)}(\widetilde{Y}_s^n)\, ds > 2\eta\right) \leqslant 2p,$$

for large n. By Lemma 5.1(iv), we may choose $\varepsilon > 0$ so small that

$$\mathbf{P}\left(\int_0^{T(\widetilde{Y}^n,r_1)} \mathbf{1}_{f_{\mathbf{v}_n}(N(\varepsilon))}(\widetilde{Y}_s^n)\, ds > 2\eta\right) \leqslant 2p,$$

for large n. Since the processes Y^n and $\widetilde{Y}^n$ have the same distribution,

$$\mathbf{P}(\bar{\sigma}_{T^*(X^n,r_2)}^{n,\varepsilon} > 2\eta) \leqslant \mathbf{P}\left(\int_0^{T(Y^n,r_1)} \mathbf{1}_{f_{\mathbf{v}_n}(N(\varepsilon))}(Y_s^n)\, ds > 2\eta\right) \leqslant 2p,$$

for large n. This and (5.11) imply that for each fixed $t > 0$, $\mathbf{P}(\bar{\sigma}_t^{n,\varepsilon} > 2\eta) \leqslant 3p$, for large n. By (5.10), we obtain $\mathbf{P}(\sigma_t^n > \sigma_t + 3\eta) \leqslant 4p$ for large n. It follows from $\mathbf{P}(\bar{\sigma}_t^{n,\varepsilon} > 2\eta) \leqslant 3p$ that $\mathbf{P}(\liminf_{n\to\infty} \bar{\sigma}_t^{n,\varepsilon} > 2\eta) \leqslant 3p$ for every $t > 0$. By Fatou's lemma, $\mathbf{P}(\bar{\sigma}_t^{\varepsilon} > 2\eta) \leqslant 3p$. This holds for arbitrarily small $p > 0$ and an appropriate choice of $\varepsilon = \varepsilon(p) > 0$. It is clear that $\tilde{\sigma}_t^{\varepsilon} < \infty$, so $\sigma_t < \infty$ a.s. Since $\lim_{\varepsilon\downarrow 0} \tilde{\sigma}_t^{\varepsilon} = \sigma_t$, we can choose $\varepsilon > 0$ so small that $\mathbf{P}(\sigma_t - \tilde{\sigma}_t^{\varepsilon} > \eta) < p$. For sufficiently large n, by (5.10), $\mathbf{P}(\sigma_t^{n,\varepsilon} < \sigma_t - 2\eta) \leqslant p$. We see that $\sigma_t^n \to \sigma_t$ in probability. By passing to a subsequence, if necessary, and using the diagonal selection method, we conclude that $\sigma_t^n \to \sigma_t$ a.s. simultaneously for all rational $t \geqslant 0$.

For a fixed $\varepsilon > 0$, $s > 0$ and all large n, the functions $t \mapsto \tilde{\sigma}_t^{n,\varepsilon}$ are Lipschitz on $[0, s]$ with a constant depending on ω, but not on n, because the integrands in their definitions are uniformly bounded. On an event A of probability greater than $1 - 3p$, the functions $t \mapsto \bar{\sigma}_t^{n,\varepsilon}$ are bounded by 2η on the interval $[0, s]$, for large n. It is elementary to show that the limit of the functions $t \mapsto \tilde{\sigma}_t^{n,\varepsilon} + \bar{\sigma}_t^{n,\varepsilon}$ cannot have jumps grater than 2η on A. We can make p and η arbitrarily small, so the function $t \mapsto \sigma_t$ is continuous a.s.

Since $t \mapsto \sigma_t^n$ and $t \mapsto \sigma_t$ are strictly increasing functions, the convergence of σ_t^n to σ_t holds for all real $t \geqslant 0$ and is uniform on compact time intervals. It is elementary to show that all this implies the convergence of the inverse functions, i.e., $\tau_t^n \to \tau_t$ a.s. for all real $t \geqslant 0$. The processes X^n converge a.s. in the uniform topology on finite time intervals so for a fixed $t \geqslant 0$, $X^n(\tau_t) \to X(\tau_t)$, a.s. By (5.4), $f_{\mathbf{v}_n}$ converges to $f_{\mathbf{v}}$, uniformly on compact subsets of D. Since X spends zero time on the boundary, $f_{\mathbf{v}_n}(X^n(\tau_t)) \to f_{\mathbf{v}}(X(\tau_t))$, a.s. We conclude that for a fixed t, Y_t^n converges a.s. to $Y_t = f_{\mathbf{v}}(X(\tau_t))$. Since Y^n also converges in distribution to $\widetilde{Y}$, the finite dimensional distributions of Y are the same as those of $\widetilde{Y}$. Both processes are continuous so the distribution of Y is the same as $\widetilde{Y}$.

It remains to show that $X_t = f_{\mathbf{v}}^{-1}(Y(\sigma_t))$. This follows immediately from the fact that τ_t and σ_t are inverse functions of each other, and from the fact that X spends zero time on the boundary. This proves that X is the CIRBM with the reflection direction $\mathbf{v}$.

The last claim of part (i) is elementary and left to the reader.

(ii) We will prove that if $\mathbf{v}_1$ and $\mathbf{v}_2$ satisfy the assumptions of (i) and $\angle(\mathbf{v}_1(\tilde{x})) \leqslant \angle(\mathbf{v}_2(\tilde{x}))$ then one can find vector fields $\mathbf{v}_n^1, \mathbf{v}_n^2 \in \mathcal{V}$ such that $\lim_{n\to\infty} \angle(\mathbf{v}_n^1(\tilde{x})) = \angle(\mathbf{v}_1(\tilde{x}))$, $\lim_{n\to\infty} \angle(\mathbf{v}_n^2(\tilde{x})) = \angle(\mathbf{v}_2(\tilde{x}))$, $\angle(\mathbf{v}_n^1)$ and $\angle(\mathbf{v}_n^2)$ are strictly increasing in n, and $\angle(\mathbf{v}_n^1(\tilde{x})) < \angle(\mathbf{v}_n^2(\tilde{x}))$ for all n and $\tilde{x}$. The argument will be a modification of the argument given at the beginning of part (i) of the proof.

Let ϕ_n^1 and ϕ_n^2 be defined relative to $\mathbf{v}_1$ and $\mathbf{v}_2$, respectively, in the same way as ϕ_n was defined relative to $\mathbf{v}$ at the beginning of part (i) of the proof. Each function ϕ_n^1 and ϕ_n^2 is continuous and takes values in $[\alpha_1 + \delta, \alpha_2 - \delta]$. The sequences $\{\phi_n^1, n \geqslant 1\}$ and $\{\phi_n^2, n \geqslant 1\}$ are nondecreasing and converge to $\angle(\mathbf{v}_1)$ and $\angle(\mathbf{v}_2)$ pointwise on $\mathbb{R}$, respectively. Since $\angle(\mathbf{v}_1(\tilde{x})) \leqslant \angle(\mathbf{v}_2(\tilde{x}))$ for every $\tilde{x} \in \mathbb{R}$, $\{\max\{\phi_n^1, \phi_n^2\}, n \geqslant 1\}$ is a nondecreasing sequence of continuous functions taking values in $[\alpha_1 + \delta, \alpha_2 - \delta]$ that converges to $\angle(\mathbf{v}_2)$ pointwise on $\mathbb{R}$. Recall that for each $n \geqslant 1$ and $i = 1, 2$, $\phi_n^i(\tilde{x}) = \alpha_1 + \delta$ for $|\tilde{x}| \geqslant n + 1$. By Lemma 5.1(iii), there exist $\mathbf{v}_n^1 \in \mathcal{V}$ and $\mathbf{v}_n^2 \in \mathcal{V}$ such that

$$\left| \angle(\mathbf{v}_n^1(\tilde{x})) - \left(\phi_n^1(\tilde{x}) - \frac{\delta}{2^n} \right) \right| < \frac{\delta}{2^{n+2}} \quad \text{for all } \tilde{x} \in \mathbb{R},$$

and

$$\left| \angle(\mathbf{v}_n^2(\tilde{x})) - \left(\max\{\phi_n^1(\tilde{x}), \phi_n^2(\tilde{x})\} - \frac{\delta}{2^{n+2}} \right) \right| < \frac{\delta}{2^{n+4}} \quad \text{for all } \tilde{x} \in \mathbb{R}.$$

It is clear then that $\lim_{n\to\infty} \angle(\mathbf{v}_n^1(\tilde{x})) = \angle(\mathbf{v}_1(\tilde{x}))$, $\lim_{n\to\infty} \angle(\mathbf{v}_n^2(\tilde{x})) = \angle(\mathbf{v}_2(\tilde{x}))$, $\angle(\mathbf{v}_n^1)$ and $\angle(\mathbf{v}_n^2)$ are strictly increasing in n, and $\angle(\mathbf{v}_n^1(\tilde{x})) < \angle(\mathbf{v}_n^2(\tilde{x}))$ for all n and $\tilde{x}$.

Part (ii) of the lemma follows from part (i), Theorem 5.3 and the existence of vector fields $\mathbf{v}_n^1, \mathbf{v}_n^2 \in \mathcal{V}$ with the properties listed above. □

Lemma 5.5. *For every positive bounded Lebesgue measurable function u on $\mathbb{R}$, there is a strictly decreasing sequence of lower semicontinuous functions u_n such that $u = \lim_{n\to\infty} u_n$ a.e. on $\mathbb{R}$.*

Proof. Without loss of generality, we may assume u is Borel measurable. Since u is bounded, there is a decreasing sequence of simple Borel functions u_n converging to u pointwise. For each Borel set A of finite Lebesgue measure, there is a decreasing sequence of open sets U_n such that $\mathbf{1}_{U_n} \to \mathbf{1}_A$ a.e. on $\mathbb{R}$. Note that $\mathbf{1}_{U_n}$ is lower semicontinuous and $(1+\frac{1}{n})\mathbf{1}_{U_n}$ is strictly decreasing to $\mathbf{1}_A$ a.e. on $\mathbb{R}$. It follows that there is a decreasing sequence of simple functions u_n that are lower semicontinuous such that $u = \lim_{n\to\infty} u_n$ a.e. on $\mathbb{R}$. □

Lemma 5.6. *Suppose that* $\mathbf{v}$ *is a vector field such that for some* $\delta > 0$, $\angle(\mathbf{v}(\tilde{x})) \in [\alpha_1 + \delta, \alpha_2 - \delta]$ *for all* $\tilde{x}$. *Then given a Brownian motion* W, *there exists a conformally invariant reflecting Brownian motion* X *in* $\overline{D}$ *with direction of reflection* $\mathbf{v}$, *adapted to the filtration of* W, *and such that* $X_t = X_0 + W_t + U_t$ *where* $\angle(U_t - U_s) \in [\alpha_1, \alpha_2]$ *for all* $t > s$, *and* U *does not change when* $X \in D$.

Proof. By Lemma 5.5, there is a sequence of vector fields $\mathbf{v}_n$ such that $\angle(\mathbf{v}_n(\tilde{x})) \in [\alpha_1 + \frac{\delta}{2}, \alpha_2 - \frac{\delta}{2}]$ for all n and $\tilde{x}$, $\angle(\mathbf{v}_n(\tilde{x}))$ is lower semicontinuous, is strictly decreasing, and converges to $\angle(\mathbf{v}(\tilde{x}))$ for a.e. $\tilde{x} \in \mathbb{R}$. Let $\mathbf{v}'$ be the vector field with $\angle(\mathbf{v}'(\tilde{x})) = \lim_{n\to\infty} \angle(\mathbf{v}_n(\tilde{x}))$ for every $\tilde{x} \in \mathbb{R}$. The argument from the proof of Lemma 5.4 shows the existence of a conformally invariant reflecting Brownian motion X corresponding to $\mathbf{v}'$. The definitions of $f_{\mathbf{v}}$ and $D_{\mathbf{v}}$ given in the proof of Lemma 5.1 are based on the harmonic extension θ of $\pi/2 - \angle(\mathbf{v})$ to D. Since $\mathbf{v}$ and $\mathbf{v}'$ are equal almost everywhere, the corresponding harmonic extensions are equal everywhere in D, and so $D_{\mathbf{v}} = D_{\mathbf{v}'}$. This implies that the process X is a CIRBM in $\overline{D}$ with direction of reflection $\mathbf{v}$. □

Theorem 5.7. *Let* $\mathbf{v}(\tilde{x}) = \mathbf{n}(\tilde{x})$ *if* $\mathbf{n}(\tilde{x})$ *exists in the classical sense and let* $\mathbf{v}(\tilde{x}) = (0,1)$ *otherwise. Let* X *be the CIRBM constructed in Lemma* 5.6. *Then* X *has the same distribution as standard RBM in* $\overline{D}$ *and* X *is a strong solution to* (2.1) *driven by the given Brownian motion* W.

Proof. It is standard to check that $\angle\mathbf{v}$ is Lebesgue measurable. Note that $\angle(\mathbf{v}(\tilde{x})) \in [\pi/2 - \arctan\kappa, \pi/2 + \arctan\kappa] \subset (\alpha_1, \alpha_2)$, so Lemma 5.6 applies. What remains to be shown is that the CIRBM X in $\overline{D}$ with normal reflection $\mathbf{v}$ is indeed standard RBM with normal reflection in $\overline{D}$.

Standard RBM with normal reflection in a Lipschitz domain U_1 can be characterized (see [10] and [31]) as the continuous strong Markov process Y on $\overline{U}_1$ whose Dirichlet form is $(\mathcal{E}, W^{1,2}(U_1))$, where $W^{1,2}(U_1)$ consists of all L^2-integrable functions in U_1 whose first order distributional derivatives are L^2-integrable and

$$\mathcal{E}(f,g) = \frac{1}{2}\int_{U_1} \nabla f(x)\cdot\nabla g(x)\,dx.$$

Suppose ϕ is a conformal map from U_1 onto another Lipschitz domain U_2. Then the Dirichlet form for the image process $\phi(Y)$ under the symmetrizing measure $|\phi'(x)|^2\,dx$ is $(\mathcal{E}, W^{1,2}(U_2))$, where $\mathcal{E}$ is defined as above except that U_2 is in place of U_1. Therefore $\phi(Y)$ is a time changed standard reflecting Brownian motion in $\overline{U}_2$ with normal reflection. The last assertion follows from the Dirichlet form characterization of time-changed processes due to Silverstein and Fitzsimmons (see Theorems 8.2 and 8.5 of [40] – the proofs contained a gap, but a new proof was given later by [28]).

Recall from the proof of Lemma 5.1(v) that $D_{\mathbf{v}}$ is a Lipschitz domain. Standard results on angular derivatives for conformal mappings (see, e.g., Section V.5 in [4]) can be used to prove that for almost every $x \in \partial D$, the half-line with the endpoint at x along the direction of $\mathbf{n}(\tilde{x})$ is mapped by $f_{\mathbf{v}}$ onto a smooth curve whose tangent line at its endpoint on $\partial D_{\mathbf{v}}$ exists, is vertical and is perpendicular to $\partial D_{\mathbf{v}}$. This implies that $D_{\mathbf{v}}$ is the upper half-plane. Now we apply the previous paragraph with U_1 being the upper half space in $\mathbb{R}^2$ and $U_2 = D$ to see that the CIRBM X in $\overline{D}$ obtained through Lemma 5.6 is a standard reflecting Brownian motion in $\overline{D}$. We have mentioned in Section 3 that standard reflecting Brownian motion in $\overline{D}$ has the following Skorokhod decomposition:

$$X_t = X_0 + B_t + \int_0^t \mathbf{v}(X_s)\,dL_s \quad \text{for } t \geqslant 0, \tag{5.12}$$

where B is a Brownian motion that is adapted to the filtration generated by X and L is a positive continuous additive functional of X that increases only when $X \in \partial D$.

The process X is adapted to the filtration of W. Since the law of X is that of standard RBM, then its law is equal to $\mathbb{P}^{x_0}$ and is hence a weak solution to (2.1) as defined in Definition 2.1(1). On the other hand, by Lemma 5.6,

$$X_t = X_0 + W_t + U_t \quad \text{for } t \geqslant 0, \tag{5.13}$$

where U is a continuous $\mathbb{R}^2$-valued process with $U_0 = (0,0)$ and $\angle(U_t - U_s) \in [\alpha_1, \alpha_2]$ for all $t > s$, and U does not change when X is in the interior of D. Since $[\alpha_1, \alpha_2] \subset (\pi/4,\ 3\pi/4)$, if we write $U_t = (U_t^{(1)}, U_t^{(2)})$, then $t \mapsto U_t^{(2)}$ is increasing and $t \mapsto U_t^{(1)}$ is of bounded variation whose total variation process is dominated by $U_t^{(2)}$. Since $W_t = \int_0^t \mathbf{1}_D(X_s)\, dX_s$, W is adapted to the filtration generated by X. Now by the uniqueness of the Doob–Meyer decomposition for the semimartingale X, we have from (5.12) and (5.13) that $B_t = W_t$ and $U_t = \int_0^t \mathbf{n}(X_s)\, dL_s$. Hence X is a strong solution to (2.1) driven by the Brownian motion W as described in Definition 2.1. □

Theorem 5.8. *Let D be a special planar lip domain and $x_0 \in \overline{D}$. Then* (2.1) *has a strong solution and the solution is pathwise unique.*

Proof. The existence of strong solution is proved in Theorem 5.7. Suppose that X and X' are two weak solutions to (2.1) starting from the same point x_0 with the same driving Brownian motion W but possibly adapted to two different filtrations (we ignore L in the spirit of Remark 2.2(ii)). Let X'' be the strong solution to (2.1), starting from x_0, with W as the driving Brownian motion that we constructed in Theorem 5.8; in particular this means that X'' is adapted to the Brownian filtration $\{\mathcal{F}_t^W\}_{t\geqslant 0}$. Let $C([0,\infty), \mathbb{R}^2)$ be the collection of the continuous functions from $[0,\infty)$ to $\mathbb{R}^2$. It follows that there is a Borel measurable map $\mathcal{A}$ from $C([0,\infty), \mathbb{R}^2)$ to itself such that $X'' = \mathcal{A}(W)$. Since X, X' and X'' are weak solutions to (2.1), we have for all $t \geqslant 0$,

$$W_t = \int_0^t \mathbf{1}_D(X_s)\, dX_s, \quad W_t = \int_0^t \mathbf{1}_D(X'_s)\, dX'_s, \quad \text{and} \quad W_t = \int_0^t \mathbf{1}_D(X''_s)\, dX''_s. \tag{5.14}$$

Each of the processes X, X' and X'' has the same law $\mathbb{P}^{x_0}$ by Theorem 2.3. Using (5.14), we conclude that the joint laws of the pairs (X, W), (X', W) and (X'', W) are equal. So we have

$$\mathbf{P}\big(X \neq \mathcal{A}(W)\big) = \mathbf{P}\big(X' \neq \mathcal{A}(W)\big) = \mathbf{P}\big(X'' \neq \mathcal{A}(W)\big) = 0.$$

But then $X' = \mathcal{A}(W) = X$ a.s. This proves pathwise uniqueness. □

Note that in the above proof we use weak uniqueness for the law of the pair (X, W), not just weak uniqueness for the law of X.

Theorem 5.9. *Let D be a bounded planar lip domain and let W_t be a two-dimensional Brownian motion. There exists a strong solution to* (2.1) *driven by W_t.*

Proof. First we will describe how Definition 2.1 can be generalized to allow for a random starting point. When it comes to weak solutions, Definition 2.1 can be extended in a straightforward way to allow for a random initial starting distribution rather than a deterministic starting point.

A strong solution to (2.1) with a random starting point X_0 is defined as follows. Suppose that $W = \{W_t, t \geqslant 0\}$ is a two-dimensional Brownian motion and ξ is a random variable that takes values in $\overline{D}$ and is independent of W, both defined on the same probability space $(\Omega, \mathcal{F}, \mathbf{P})$. Let $\{\mathcal{F}_t\}_{t\geqslant 0}$ be the augmentation of the filtration generated by the natural filtration of W and ξ. A strong solution to (2.1) with $X_0 = \xi$, relative to W, is a pair of continuous processes (X, L) such that $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geqslant 0}, \mathbf{P}, (X, W, L))$ is a weak solution to (2.1) with initial distribution of ξ.

By Theorem 4.11 and Theorem 5.8, weak uniqueness and pathwise uniqueness hold for (2.1) for every deterministic $x_0 \in \overline{D}$. Given any probability distribution μ on $\overline{D}$, there exists a *standard* reflecting Brownian motion in $\overline{D}$ with initial distribution μ; it is a weak solution to (2.1) with initial distribution μ. Hence by Corollary 3 of [45], there is a universally measurable function

$$F : \overline{D} \times C\big([0,\infty), \mathbb{R}^d\big) \to C\big([0,\infty), \overline{D}\big) \tag{5.15}$$

such that for every $t \geqslant 0$, it is universally measurable as a map from $\overline{D} \times C([0,t], \mathbb{R}^d) \to C([0,t], \overline{D})$ and every solution (X, W) of (2.1) with (random) starting point X_0 satisfies $X = F(X_0, W)$. Moreover, since a weak solution to (2.1) exists for any initial random distribution, it follows that for any given Brownian motion W with $W_0 = 0$ and random variable ξ that takes values in $\overline{D}$ and is independent of W, $X = F(\xi, W)$ solves (2.1) with $X_0 = \xi$. That is, a strong solution to (2.1) exists for any initial random distribution.

Fix any $x_0 \in \overline{D}$. For each $x \in \partial\overline{D}$ there exist r_x and a rotated special planar lip domain U_x such that $D \cap B(x, 2r_x) = U_x \cap B(x, 2r_x)$. By compactness of ∂D, we can find a finite subfamily $B_1, \ldots, B_n$ of $\{B(x, r_x)\}_{x\in\partial\overline{D}}$ with $\partial D \subset \bigcup_{k=1}^n B_k$. Let B_0 be a smooth domain such that

$$D \setminus \left(\bigcup_{k=1}^n B_k \right) \subset B_0 \subset \overline{B}_0 \subset D.$$

Define $U_0 = B_0$ and for $1 \leqslant i \leqslant n$, if $B_i = B(x_i, r_{x_i})$, we will write $U_i = U_{x_i}$. For $1 \leqslant i \leqslant n$, let F_i be the function defined in (5.15) with U_i in place of D. Let $W = \{W_t, t \geqslant 0\}$ be a given d-dimensional Brownian motion with $W_0 = 0$. Let $S_0 = T_0 = 0$, $X_0^0 = x_0$, and let $N_0 = \inf\{j\colon\ x_0 \in B_j\}$. For $i \geqslant 0$, define inductively

(i) $W_t^i = W_{t+S_i} - W_{S_i}$ for $t \geqslant 0$ (a Brownian motion),
(ii) (a) if $N_i = 0$, let $X_t^{i+1} = X_{T_i}^i + W_t^i$ for $t \geqslant 0$,
(b) if $N_i \geqslant 1$, let $X^{i+1} = F_{N_i}(X_{T_i}^i, W^i)$,
(iii) $T_{i+1} = \inf\{t > 0\colon\ X_t^{i+1} \notin B_{N_i}\}$,
(iv) $N_{i+1} = \inf\{j\colon\ X_{T_{i+1}}^{i+1} \in B_j\}$,
(vii) $S_{i+1} = T_0 + \cdots + T_{i+1}$.

Let $X_t = X_{t-S_i}^{i+1}$ for $S_i \leqslant t < S_{i+1}$. When $N_i = 0$, X^{i+1} is a Brownian motion in B_0, and when $N_i \neq 0$, each X^{i+1} is a strong solution to (2.1) in U_{N_i} with initial (random) position $X_{T_i}^i$. The law of X^{i+1} is the same as the standard RBM in U_{N_i} with initial starting point $X_{T_i}^i$. Property 3.1(iii) and a standard argument show that $S_i \to \infty$ a.s. as $i \to \infty$ (cf. [6, Theorem VI.3.4]). It is now routine to check that X is indeed a strong solution to (2.1) in $\overline{D}$ with W as the driving Brownian motion. □

Remark 5.10. R. Atar pointed out to the first author that there is a gap in the proof of [12]. The proof that the strong solution constructed in that paper is adapted is faulty. An attempt to correct this error in [7] was unsuccessful. A further discussion may be found in [9].

It is still an open question as to whether a strong solution to the Skorokhod equation exists in $C^{1+\alpha}$ domains in dimension three and higher. In [12] weak uniqueness for RBM in $C^{1+\alpha}$ domains was proved under weaker assumptions than those in Definition 2.1; the conditions (2.2) and (2.3) are unnecessary. Therefore to show strong existence in the $C^{1+\alpha}$ situation, one needs only to find an adapted solution to (2.1), where L is a local time of X on the boundary of D.

6. Pathwise uniqueness for RBM in planar lip domains

In this section we will first prove Theorem 2.4, and then we will show that it also holds for the type of planar lip domains introduced in Burdzy and Chen [15], which are a variant of the ones considered above. Then we will apply it to synchronous couplings of RBMs in both types of planar lip domains.

Proof of Theorem 2.4. The proof is the same as that for Theorem 5.8. □

In [15], Burdzy and Chen studied the behavior of "synchronous couplings" in polygonal and Lipschitz domains. A synchronous coupling is a pair of reflecting Brownian motions X and Y in the same domain D, driven by the same Brownian motion W_t. Lacking a strong existence result for RBM in general Lipschitz domains, the synchronous coupling of RBMs in a Lipschitz domain D is constructed in a weak sense in [15] as a limit of synchronous couplings of RBMs in a sequence of smooth domains that increase to D. Atar and Burdzy [2] similarly circumvented the problem of constructing a "mirror coupling" in a lip domain (we call reflecting Brownian motions X and Y in D a *mirror* coupling if the line of symmetry for X and Y does not change whenever both processes stay away from ∂D).

Using Theorem 2.4, we can derive the following for planar lip domains.

Theorem 6.1. *Given a planar lip domain D and a Brownian motion W, there exists a synchronous coupling (X, Y) of reflecting Brownian motions in $\overline{D}$ driven by W such that $\{(X_t, Y_t), t \geqslant 0\}$ is a strong Markov process with respect to the filtration generated by W.*

Although Theorem 6.1 does not immediately prove the existence of a "mirror coupling" (X, Y) with the strong Markov property in a lip domain, one could try to apply the method of this paper to answer this open question.

Theorem 6.1 derives its main interest from possible applications in the context of the research presented in Burdzy and Chen [15], where the definition of a lip domain is slightly different from this paper. For this reason, we will prove Theorem 6.1 using the following alternative definition of a lip domain that was used in [15]. There a lip domain was defined to be a Lipschitz domain D that is bounded between two Lipschitz functions f_1 and f_2:

$$D = \big\{(x_1, x_2)\colon\ f_1(x_1) < x_2 < f_2(x_1),\ z_1 \leqslant x_1 \leqslant z_2\big\} \tag{6.1}$$

such that $f_1(z_1) = f_2(z_1)$, $f_1(z_2) = f_2(z_2)$, $f_1(x_1) < f_2(x_1)$ for $-\infty < z_1 < x_1 < z_2 < \infty$, and the functions f_1 and f_2 are Lipschitz with Lipschitz constant $\kappa \in (0, 1)$: for $k = 1, 2$,

$$\big|f_k(x_1) - f_k(y_1)\big| \leqslant \kappa|x_1 - y_1|, \quad \text{for all } z_1 \leqslant x_1, y_1 \leqslant z_2. \tag{6.2}$$

Note that the assumption that D is a Lipschitz domain puts additional constraints on the functions f_k in addition to (6.2). In a neighborhood of the left or right endpoint, the boundary of D is the graph of a Lipschitz function in some coordinate system, but the Lipschitz constant of that function may be larger than 1. This makes it impossible to construct solutions in D of (6.1) using a piecing-together procedure – our main theorem does not apply near the left and right endpoints of the domain defined by (6.1).

Proof of Theorem 6.1. It suffices to show that the conclusion of Theorem 2.4 holds for any planar lip domain in the sense of (6.1). Since standard RBM in a Lipschitz domain does not hit points, our result follows from Theorem 2.4 by a piecing-together procedure unless the starting point is either the left-most or right-most point of the domain. Suppose that x_0 is one of the extreme points, say, x_0 is the left-most point of the domain. Weak uniqueness follows from Theorem 2.3. In particular, we may use $\mathbb{P}$ in place of $\mathbf{P}$ in the remainder of the proof. We therefore turn to strong existence. Define

$$R_+ = \big\{(s, t)\colon\ s \geqslant 0, |t| < \kappa s\big\}.$$

Take a sequence of points $x_n \in D$ converging to x_0 so that $x_m - x_n \in R_+$ for every $n > m$ and let X^n be the strong solution for (2.1) starting from x_n. By step 3 in the proof of Theorem 1.1(ii) in [15] as well as Theorem 2.3 in [15], almost surely $X_t^m - X_t^n \in R_+$ for all $t \geqslant 0$. By step 1 in the proof of Theorem 1.1(ii) in [15], there is an eigenfunction ϕ corresponding to the second eigenvalue $\mu_2 < 0$ of the half Laplacian in D with Neumann boundary conditions such that $\nabla\phi(x) \in R_+$ for every $x \in D$. Thus for $n > m$, almost surely $\phi(X_t^m) - \phi(X_t^n) \geqslant 0$ for every $t \geqslant 0$; in other words, $\phi(X_t^n)$ is decreasing in n. On the other hand, for each $n \geqslant 1$, $\phi(X_t^n)e^{-\mu_2 t}$ is a martingale. Thus for each fixed $t > 0$

$$e^{-\mu_2 t}\mathbb{E}\big[\phi(X_t^m) - \phi(X_t^n)\big] = \phi(x_m) - \phi(x_n).$$

The right hand side goes to zero as $n, m \to \infty$ by the continuity of ϕ. Hence almost surely, $\phi(X_t^n) - \phi(X_t^m)$ converges to zero for every $t \geqslant 0$ when $n > m$ both go to infinity. By step 2 in the proof of Theorem 1.1(ii) in [15] and the fact that the amount of time X^n spends on the boundary has zero Lebesgue measure, one concludes that $|X_t^n - X_t^m|$ goes to zero as $n, m \to \infty$. Therefore almost surely $X_t = \lim_{n\to\infty} X_t^n$ exists for every $t \geqslant 0$. Clearly X has the same distribution as standard RBM in $\overline{D}$ starting from x_0 (cf. Lemma 4.3 of [10]). Note that for each n,

$$X_t^n = x_n + W_t + \int_0^t \mathbf{n}(X_s^n)\, dL_s^n \quad \text{for every } t \geqslant 0,$$

where L^n is the local time of X^n on the boundary of ∂D corresponding to the measure $\frac{1}{2}\nu$ and ν denotes surface measure on ∂D. Since $X_t = \lim_{n\to\infty} X_t^n$, we see that the limit $A_t = \lim_{n\to\infty} \int_0^t \mathbf{n}(X_s^n)\, dL_s^n$ exists and

$$X_t = x_0 + W_t + A_t \quad \text{for every } t \geqslant 0.$$

Let $p(t, x, y)$ be the transition density function for standard RBM in $\overline{D}$. We have for $t \geqslant 0$,

$$\mathbb{E}[L_t^n] = \frac{1}{2}\int_0^t \int_{\partial D} p(s, x_n, y)\nu(dy)\, ds \leqslant c_1\, \nu(\partial D)$$

in view of Theorem 3.1 of [10], where $c_1 > 0$ is a constant that depends only on t. It follows that A is a process of bounded variation. By an argument that is similar to but simpler than that from (5.12) to the end of the proof of Theorem 5.7, we conclude that $A_t = \int_0^t \mathbf{n}(X_s)\, dL_s$ for $t \geqslant 0$, where L is the boundary local time of X corresponding to the measure $\frac{1}{2}\nu$. This shows that X is a strong solution to (2.1) driven by W. □

7. Counterexamples

We will present two examples in this section. The first one will provide a proof for Theorem 2.5. The second one will illustrate the importance of the choice of the definition of $\mathbf{N}_0(x)$.

Proof of Theorem 2.5. Fix some small $\kappa_1 > 0$ and let $C_1 \subset [0, 1]$ be the classical Cantor set. Let $\phi : \mathbb{R} \to \mathbb{R}$ be a continuous function which is equal to 0 on $C_2 = (-\infty, 0] \cup C_1 \cup [1, \infty)$ and is defined elsewhere as follows. The set C_2 is closed so its complement consists of a countable union of disjoint open intervals. For every such interval, say, (a, b), we let the function ϕ be linear on $(a, (a + b)/2)$ with slope $-\kappa_1$, and linear on $((a + b)/2, b)$ with slope κ_1.

Let $D_1 = \{(x_1, x_2) \in \mathbb{R}^2 \colon\ x_2 > \phi(x_1)\}$ and $C_3 = \{(x_1, x_2) \in \partial D_1 \colon\ x_1 \in C_1\}$. As ϕ is Lipschitz, it follows that a subset of $\overline{D}_1$ is not hit by standard RBM in $\overline{D}_1$ if and only if it is not hit by standard Brownian motion in $\mathbb{R}^2$ (see, for example, Remark 2.2(3) in [13]). As C_3 has positive $\log 2/\log 3$-Hausdorff measure, it is hit by standard Brownian motion in $\mathbb{R}^2$. Therefore C_3 will be hit by standard RBM in $\overline{D}_1$ with positive probability.

Fix $\kappa > 1$ arbitrarily close to 1. Let $D_3 = \{(x_1, x_2, x_3) \in \mathbb{R}^3\colon\ x_3 > \phi(x_1)\}$ and let D_4 be obtained from D_3 by a rotation $\mathcal{R}$ around the x_1-axis by an angle α with the following properties. First, the line $\{(x_1, x_2, x_3) \in \partial D_4\colon\ x_1 = 0\}$ should form an angle with the (x_1, x_2)-plane strictly greater than $\pi/4$. Second, D_4 should be a domain above the graph of a Lipschitz function Φ with Lipschitz constant $\kappa_2 = (\kappa + 1)/2$ (see Section 2). It is elementary to see that we can find an angle α with the above properties if $\kappa_1 > 0$ is sufficiently small. Let $C_5 = \{(x_1, x_2, x_3) \in \partial D_3\colon\ x_1 \in C_1\}$, $C_6 = \mathcal{R}(C_5)$, and $\mathbf{m} = \mathcal{R}((0, 1, 0))$. The component of standard RBM in $\overline{D}_4$ in the direction of $\mathbf{m}$ is a standard 1-dimensional (non-reflecting) Brownian motion, independent of the other two components, so the fact that C_3 is non-polar in D_1 implies that C_6 is non-polar in D_4.

Since C_6 is non-polar, it supports a measure which does not charge polar sets and which has positive mass. Such a measure will be the Revuz measure of a continuous additive functional M of standard RBM X in $\overline{D}_4$ Assume without loss of generality that $\mathbf{m} = (0, m_2, m_3)$ with $m_2, m_3 > 0$. Suppose X is standard RBM in $\overline{D}_4$ starting from $(0, 0, 0)$, and let

$$Y_t = X_t + M_t\mathbf{m}, \quad t \geqslant 0. \tag{7.1}$$

Find $t_0 \in (0, \infty)$ such that $\mathbb{P}(M_{t_0} > 1) > 3/4$. Let $\kappa_3 = (\kappa_2 + \kappa)/2$,

$$A = (0, 0, 1) + \big\{(x_1, x_2, x_3) \in \mathbb{R}^3\colon\ x_3 \leqslant -\kappa_3\sqrt{x_1^2 + x_2^2},\ (x_1, x_2, x_3 + 1) \cdot \mathbf{m} \leqslant 0\big\},$$

and $A_\varepsilon = \varepsilon A$. Note that every set A_ε lies below the graph of a Lipschitz function with Lipschitz constant κ_3. Let $\{(y_1^j, y_2^j)\}_{j\geqslant 1}$ be a sequence consisting of all points in the plane whose coordinates are both rational and such that in some neighborhood of $y^j = (y_1^j, y_2^j, \Phi(y_1^j, y_2^j))$, the boundary of D_4 is a piece of a plane. Since $\kappa_3 > \kappa_2$, elementary geometry shows that for any fixed j, the domains $D_4 \setminus (y^j + A_\varepsilon)$ converge to D_4 as $\varepsilon \to 0$. Any single point is polar for X, so it is not hard to see that every point y^j is polar for Y, i.e., Y does not hit y^j, with probability 1. Let $\varepsilon_j > 0$ be so small that the probability that Y hits $y^j + A_{\varepsilon_j}$ before time t_0 is less than 2^{-j-1}, and $y^j + A_{\varepsilon_j}$ does not intersect $C_6 \cup (D_4 \cap \bigcup_{k<j}(y^k + A_{\varepsilon_k}))$. If the last condition cannot be satisfied for any $\varepsilon_j > 0$, we take $\varepsilon_j = 0$. Let $D_5 = D_4 \setminus \bigcup_{j\geqslant 1}(y^j + A_{\varepsilon_j})$ and let T be the first hitting time of $\bigcup_{j\geqslant 1}(y^j + A_{\varepsilon_j})$ by Y. Note that $\mathbb{P}(T \leqslant t_0) \leqslant 1/2$ and so $\mathbb{P}(M_T > 1) \geqslant 1/4$.

Let $Z_t = Y_t$ for $t \leqslant T$. We continue the process Z_t for $t \geqslant T$ as standard RBM in $\overline{D}_5$, starting from Y_T but otherwise independent of $\{Y_t, t \in [0, T]\}$. It follows from (7.1) that Z_t satisfies

$$Z_t = W_t + \int_0^t \mathbf{n}(Z_t)\,dL_s + \int_0^t \mathbf{m}1_{C_6}(Z_s)\,dM_{s\wedge T}, \quad t \geqslant 0,$$

for some Brownian motion W, where L is the usual local time on the boundary of D_5 and $\mathbf{n}$ is the unit inward normal vector field on ∂D_5. We claim that this is a solution to (2.1) in D_5 (without the conditions (2.2) and (2.3)). Clearly, the processes L and M do not increase when Z is in the interior of D_5. Recall the definition of $\mathbf{N}_0(x)$ from Section 2. For points $x \in C_6$, the normal vector $\mathbf{n}(x)$ is not well defined so we have to use the alternative definition of $\mathbf{N}_0(x)$ as the intersection of $\{\mathbf{N}_\varepsilon(x)\}_{\varepsilon>0}$ (see Section 2). We have $\mathbf{m} \in \mathbf{N}_0(x)$ for $x \in C_6$ because by construction, every point of C_6 is a cluster point of the sets $y^j + A_{\varepsilon_j}$. This completes the proof of the claim that Z satisfies (2.1).

Our construction generated a domain D_5 lying above the graph of a Lipschitz function with Lipschitz constant $\kappa_3 < \kappa$. Since $\{Z_t; t < T\}$ is clearly different from standard RBM X in $\overline{D}_4$ (or equivalently, standard RBM in $\overline{D}_5$) before hitting $\bigcup_{j\geqslant 1}(y^j + A_{\varepsilon_j})$, we do not have pathwise uniqueness in the Lipschitz domain D_5. □

Our next example will illustrate the importance of the definition of the "set of direction constraints" $\mathbf{N}_0(x)$. On the technical side, the construction given below is similar to that in the proof of Theorem 2.5, but simpler.

Example 7.1. Fix an arbitrarily small $\kappa > 0$, let $D_1 = \{(x_1, x_2, x_3) \in \mathbb{R}^3\colon\ x_3 > 0\}$, and let $\mathbf{m} = (\kappa, 0, 1)$. Let X be RBM in $\overline{D}_1$ with oblique direction of reflection $\mathbf{m}$, starting from $(0, 0, 0)$. The process X satisfies the equation

$$X_t = W_t + \int_0^t \mathbf{m}\, dL_s \quad \text{for } t \geqslant 0,$$

where W is a Brownian motion and L does not increase when X is in D_1. Let

$$A = \left\{(x_1, x_2, x_3) \in \mathbb{R}^3\colon\ x_3 \leqslant -\kappa\sqrt{x_1^2 + x_2^2}\right\},$$

and let $\{(y_1^j, y_2^j)\}_{j\geqslant 1}$ be a sequence consisting of all points in the plane with rational coordinates. For any fixed j, the domains $D_1 \setminus ((y_1^j, y_2^j, \varepsilon) + A)$ converge to D_1 as $\varepsilon \to 0$. A fixed point is polar for X, so one can find $\varepsilon_j > 0$ so small that the probability that X hits $(y_1^j, y_2^j, \varepsilon_j) + A$ before time 1 is less than 2^{-j-1}. We make $\varepsilon_j > 0$ even smaller, if necessary, so that $(y_1^j, y_2^j, \varepsilon_j) + A$ does not intersect $D_1 \cap \bigcup_{k<j}((y_1^k, y_2^k, \varepsilon_k) + A)$. If the last condition cannot be satisfied for any $\varepsilon_j > 0$, we take $\varepsilon_j = 0$. Let $D_2 = D_1 \setminus \bigcup_{j\geqslant 1}((y_1^j, y_2^j, \varepsilon_j) + A)$ and let T be the first hitting time of $\bigcup_{j\geqslant 1}((y_1^j, y_2^j, \varepsilon_j) + A)$ by X_t. Note that $\mathbb{P}(T \leqslant 1) \leqslant 1/2$ and so $\mathbb{P}(L_T > c_1) \geqslant 1/4$ for some $c_1 > 0$.

Let $Z_t = X_t$ for $t \in [0,\ T]$. We continue the process Z for $t \geqslant T$ as standard RBM in $\overline{D}_2$, starting from X_T but otherwise independent of $\{X_t, t \in [0, T]\}$. It follows that Z satisfies

$$Z_t = W_t + \int_0^{t\wedge T} \mathbf{m}\, dL_s + \int_{t\wedge T}^t \mathbf{n}(Z_t)\, dL_s, \quad t \geqslant 0, \tag{7.2}$$

for some Brownian motion W, and a process L which does not increase when Z is inside D_2. Since $\mathbb{P}(L_T > c_1) \geqslant 1/4$, the first integral gives a non-trivial contribution with positive probability.

The domain D_2 is a Lipschitz domain with boundary function Φ having Lipschitz constant κ. The process satisfying (7.2) is a solution to (2.1) in D_2 if we define $\mathbf{N}_0(x)$ as $\bigcap_{\varepsilon>0}\mathbf{N}_\varepsilon(x)$ for every $x \in \partial D_2$. This is because every point $x \in \partial D_1 \cap \partial D_2$ is a cluster point of the sets $(y_1^j, y_2^j, \varepsilon_j) + A$ and so $\mathbf{m} \in \bigcap_{\varepsilon>0}\mathbf{N}_\varepsilon(x)$.

We will argue that Z is not standard RBM in $\overline{D}_2$. Note that for $x \in \partial D_1 \cap \partial D_2$, the normal vector $\mathbf{n}(x)$ is equal to $(0, 0, 1)$ or it is not well-defined. For standard RBM, dL_t does not charge sets of measure 0, so if Z were standard RBM, we would have to have

$$\int_0^{t\wedge T} \mathbf{m}\, dL_s = \int_0^{t\wedge T} \mathbf{n}(Z_t)\, dL_s = \int_0^{t\wedge T} (0, 0, 1)\, dL_s \quad \text{for } t \geqslant 0,$$

a contradiction.

This proves that uniqueness for (2.1) does not hold even in Lipschitz domains with arbitrarily small Lipschitz constant κ if we adopt $\bigcap_{\varepsilon>0}\mathbf{N}_\varepsilon(x)$ as the definition of $\mathbf{N}_0(x)$ for all $x \in \partial D_2$.

We note that both examples in this section prove the lack of weak uniqueness, not just strong uniqueness. The examples indicate that when there is ambiguity about the choice of a normal reflection direction at a non-polar set of boundary points for standard RBM, weak uniqueness of solutions of (2.1) might fail unless one specifies some extra conditions, such as (2.2)–(2.3). Consequently, this shows that the deterministic version of the Skorokhod equation in such a Lipschitz domain in $\mathbb{R}^3$ might not have a unique solution, for otherwise we would have pathwise uniqueness and therefore weak uniqueness for (2.1).

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