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Ann. I. H. Poincaré – PR 41 (2005) 151–178

ANNALES DE L'INSTITUT HENRI POINCARÉ
PROBABILITÉS ET STATISTIQUES

www.elsevier.com/locate/anihpb

Gaussian fluctuations of eigenvalues in the GUE

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Received 2 February 2004; accepted 23 April 2004

Available online 11 September 2004

Abstract

Under certain conditions on k we calculate the limit distribution of the kth largest eigenvalue, x_k, of the Gaussian Unitary Ensemble (GUE). More specifically, if n is the dimension of a random matrix from the GUE and k is such that both $n-k$ and k tends to infinity as $n \to \infty$ then x_k is normally distributed in the limit. We also consider the joint limit distribution of $x_{k_1} < \cdots < x_{k_m}$ where we require that $n-k_i$ and k_i, $1 \leqslant i \leqslant m$, tends to infinity with n. The result is an m-dimensional normal distribution.

Résumé

Sous certaines conditions sur k, nous calculons la distribution limite de la $k$ième valeur propre, x_k, du GUE (Ensemble Unitaire Gaussien). Plus spécifiquement, si n est la dimension de la matrice aléatoire du GUE et k est tel que k et $n-k$ tendent vers l'infini quand $n \to \infty$, alors x_k est distribué normalement à la limite. Nous considérons aussi la distribution limite jointe de $x_{k_1} < \cdots < x_{k_m}$ où $n-k_i$ et k_i, $1 \leqslant i \leqslant m$, tendent vers l'infini en même temps que n. Le résultat est une distribution normale de dimension m.

MSC: 15A52; 60F05

Keywords: Random matrices; Limit distribution

1. Introduction and formulation of results

The Gaussian Unitary Ensemble (GUE) is a classical random matrix ensemble. It is defined by the probability distribution on the space of $n \times n$ Hermitian matrices given by

$$P(\mathrm{d}H) = C_n \cdot \mathrm{e}^{-\operatorname{Trace} H^2}\,\mathrm{d}H.$$

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0246-0203/$ – see front matter
doi:10.1016/j.anihpb.2004.04.002

By $\mathrm{d}H$ we mean the Lebesgue measure on the n^2 essentially different members of the matrix, namely

$$\{\operatorname{Re} H_{ij};\, 1 \leqslant i \leqslant j \leqslant n,\, \operatorname{Im} H_{ij};\, 1 \leqslant i < j \leqslant n\}. \tag{1.1}$$

In other words this means that the entries in (1.1) are independent $N(0, \frac{1+\delta_{ij}}{4})$ random variables. The measure on the matrices naturally induces a measure on the corresponding n real eigenvalues x_i. This induced measure can be explicitly calculated and it's density is given by

$$p_n(x_1,\dots,x_n) = \frac{1}{Z_n^{(2)}} \prod_{1\leqslant i<j\leqslant n} |x_i - x_j|^2 \cdot \exp\left[-\sum_{i=1}^{n} x_i^2\right].$$

The normalization constant $Z_n^{(2)}$ is called the partition function. It is often convenient to work with the eigenvalues being ordered. Naming the eigenvalues so that $x_1 < \cdots < x_n$, gives that the probability density $\rho_{n,n}(x_1,\dots,x_n)$ of the ordered eigenvalues defined on the space

$$\{x_1,\dots,x_n;\, x_1 < \cdots < x_n\}$$

is given by

$$\rho_{n,n}(x_1,\dots,x_n) = n!\, p_n(x_1,\dots,x_n).$$

This density ρ is a member of a family of functions called the correlation functions. These functions are defined by

$$\rho_{n,k}(x_1,\dots,x_k) = \frac{n!}{(n-k)!} \int_{\mathbb{R}^{n-k}} p_n(x_1,\dots,x_n)\,\mathrm{d}x_{k+1}\dots\mathrm{d}x_n = \det\bigl(K_n(x_i,x_j)\bigr)_{i,j=1}^{k}.$$

Here $K_n(x,y)$ is given by

$$K_n(x,y) = \sum_{i=0}^{n-1} h_i(x)h_i(y)\mathrm{e}^{-\frac{1}{2}(x^2+y^2)},$$

where $\{h_i\}$ are the orthonormalized Hermite polynomials, that is

$$\int_{-\infty}^{\infty} h_i(x)h_j(x)\mathrm{e}^{-x^2}\,\mathrm{d}x = \delta_{ij}.$$

The kernel $K_n(x,y)$ can also be represented by the so called Christoffel–Darboux identity. For $x \neq y$ it holds that

$$K_n(x,y) = \left(\frac{n}{2}\right)^{1/2} \frac{h_n(x)h_{n-1}(y) - h_n(y)h_{n-1}(x)}{x-y}\mathrm{e}^{-\frac{1}{2}(x^2+y^2)}$$

and for $x = y$ one has

$$K_n(x,y) = \bigl(nh_n^2(x) - \sqrt{n(n+1)}h_{n-1}(x)h_{n+1}(x)\bigr)\mathrm{e}^{-x^2}.$$

The correlation function $\rho_{n,1}$ describes the overall density of the eigenvalues. Wigner's semi-circle law states that

$$\lim_{n\to\infty} \sqrt{\frac{2}{n}}\rho_{n,1}(\sqrt{2n}x) = \begin{cases} \frac{2}{\pi}\sqrt{1-x^2} & \text{if } |x| \leqslant 1,\\ 0 & \text{if } |x| > 1. \end{cases} \tag{1.2}$$

All the results above and more can be found in the book of Mehta [7].

This paper deals with the distribution of eigenvalue number k of the GUE. More specifically we look at the distribution of the kth largest eigenvalue as n and k tends to infinity. For example if

$$k = k(n) = n - \log n$$

then as n becomes large, k is very close (relatively) to the right edge of the spectrum. Another example is when $k = n/2$. In this case we are in the middle of the bulk of the spectrum. In both cases one ends up with a normal distribution in the limit. The following theorems generalize and specify this statement.

Theorem 1.1 (The bulk). *Set*

$$G(t)=\frac{2}{\pi}\int_{-1}^{t}\sqrt{1-x^2}\,\mathrm{d}x,\quad -1\leqslant t\leqslant 1,$$

and $t=t(k,n)=G^{-1}(k/n)$ *where* $k=k(n)$ *is such that* $k/n\to a\in(0,1)$ *as* $n\to\infty$. *If* x_k *denotes the* k*-th eigenvalue in the GUE then it holds that as* $n\to\infty$

$$\frac{x_k-t\sqrt{2n}}{(\frac{\log n}{4(1-t^2)n})^{1/2}}\longrightarrow \mathrm{N}(0,1)$$

in distribution.

Theorem 1.2 (The edge). *Let* k *be such that* $k\to\infty$ *but* $\frac{k}{n}\to 0$ *as* $n\to\infty$ *and define* x_{n-k} *as eigenvalue number* $n-k$ *in the GUE. Then it holds that as* $n\to\infty$,

$$\frac{x_{n-k}-\sqrt{2n}(1-(\frac{3\pi k}{4\sqrt{2}n})^{2/3})}{((\frac{1}{12\pi})^{2/3}\frac{\log k}{n^{1/3}k^{2/3}})^{1/2}}\longrightarrow \mathrm{N}(0,1)$$

in distribution.

Remark 1. The theorems deal with the bulk and the right spectrum edge. One gets the equivalent for the left edge with some obvious modifications.

Remark 2. In [12] the distribution of the largest eigenvalue was studied. Also eigenvalue number k with k fixed has been studied.

Remark 3. Set $I=(-\infty,s]$. In [1] it is shown that

$$P\big(x_k\in(s+\mathrm{d}s)\big)=\left(\frac{1}{(k-1)!}\int_{I^{k-1}}J_k(x_1,\ldots,x_{k-1},s)\mu(\mathrm{d}x_1)\cdots\mu(\mathrm{d}x_{k-1})\right)\mu(\mathrm{d}s).\tag{1.3}$$

Here J_k is the so called Janossy density and

$$\mu(\mathrm{d}x)=\mathrm{Const}\cdot \mathrm{e}^{-x^2/2}\,\mathrm{d}x.$$

In [1] it is also proven that J_k can be expressed explicitly by a determinantal formula. For k and n as in (1.1) or (1.2) we thus have that (1.3) is for large n approximately equal to the probability density function of the Normal Distribution $N(\mu,\sigma)$. The parameters μ and σ should of course be taken to be those indicated from the relevant theorem above.

Remark 4. The zero number k of the Hermite polynomial of degree n is close to the expected value of eigenvalue number k of GUE_n. This can be shown directly by the following result [4]:

There are constants k_0 and C such that for $k_0\leqslant k\leqslant n-k_0$ and $\alpha=k/n$ it holds that

$$\left|\frac{z_{k,n}}{\sqrt{2n}}-G^{-1}\left[\frac{k}{n}-\frac{1}{2\pi n}\arcsin\big(G^{-1}(k/n)\big)+\frac{1}{2n}\right]\right|\leqslant\frac{\mathrm{C}}{n^2(\alpha(1-\alpha))^{4/3}}.$$

Here $z_{1,n} < \cdots < z_{n,n}$ are the zeros of the Hermite polynomial of degree n. When we're in the Bulk this translates into

$$\left| z_{k,n} - \sqrt{2n} G^{-1}\left(\frac{k}{n}\right) \right| \leqslant \frac{C}{\sqrt{n}}.$$

This means that one can replace $t\sqrt{2n}$ by $z_{k,n}$ in Theorem 1.1. Close to the edge this replacement is not allowed. The zeros and the expected values are not close enough there.

A motivation for this approximate equality between the locations of zeros and eigenvalues goes as follows. Set

$$W = \frac{1}{2}\sum_{i=1}^{n} x_i^2 - \sum_{1 \leqslant i < j \leqslant n} \log|x_i - x_j|$$

and note that

$$\rho_{n,n}(x_1, \ldots, x_n) = \mathrm{Const} \cdot \mathrm{e}^{-2W}.$$

It is a fact [7] that W obtains its minimum exactly when $x_i = z_{i,n}$, $1 \leqslant i \leqslant n$. This configuration is hence the most "probable" for the eigenvalues. Expanding around this minimum we see that it is reasonable that x_k should have Gaussian fluctuations around $z_{k,n}$.

Remark 5. If one is interested in the distribution of the eigenvalues of some other ensemble one should in many cases be able to apply the same methodology that has been used here.

It is also interesting to see what happens when looking at two eigenvalues at the same time. With $k(n) \sim n^{\theta}$ is meant that $k(n) = h(n) \cdot n^{\theta}$ where h is any function satisfying

$$\frac{h(n)}{n^{\varepsilon}} \to 0 \quad \text{and} \quad h(n)n^{\varepsilon} \to \infty \tag{1.4}$$

as $n \to \infty$ for all $\varepsilon > 0$. We have the following results:

Theorem 1.3 (The bulk). *Let $\{x_{k_i}\}_1^m$ be eigenvalues of the GUE such that $0 < k_i - k_{i+1} \sim n^{\theta_i}$, $0 < \theta_i \leqslant 1$, and $k_i/n \to a_i$, where $a_i \in (0,1)$ as $n \to \infty$. Define $s_i = s_i(k_i, n) = G^{-1}(k_i/n)$ and set*

$$X_i = \frac{x_{k_i} - s_i\sqrt{2n}}{(\frac{\log n}{4(1-s_i^2)n})^{1/2}}, \quad i = 1, \ldots, m.$$

Then as $n \to \infty$

$$\mathbb{P}[X_1 \leqslant x_1, \ldots, X_m \leqslant x_m] \to \Phi_{\Lambda}(x_1, \ldots, x_m)$$

where Λ is the $m \times m$ correlation matrix with $\Lambda_{i,j} = 1 - \max_{i \leqslant k < j < m} \theta_k$, and Φ_{Λ} is the cdf[1] *for the normalized m-dimensional Normal Distribution with correlation matrix Λ.*

Theorem 1.4 (The edge). *Let $\{x_{n-k_i}\}_1^m$ be eigenvalues of the GUE such that $k_1 \sim n^{\gamma}$ where $0 < \gamma < 1$ and $0 < k_{i+1} - k_i \sim n^{\theta_i}$, $0 < \theta_i < \gamma$. Set*

$$X_i = \frac{x_{n-k_i} - \sqrt{2n}(1 - (\frac{3\pi k_i}{4\sqrt{2}n})^{2/3})}{((\frac{1}{12\pi})^{2/3} \frac{\log k_i}{n^{1/3} k_i^{2/3}})^{1/2}}, \quad i = 1, \ldots, m,$$

[1] Cumulative Distribution Function.

then as $n \to \infty$

$$\mathbb{P}[X_1 \leqslant x_1, \ldots, X_m \leqslant x_m] \to \Phi_\Lambda(x_1, \ldots, x_m),$$

where Λ *is the* $m \times m$ *correlation matrix with* $\Lambda_{i,j} = 1 - \frac{1}{\gamma} \max_{i \leqslant k < j < m} \theta_k$, *and* Φ_Λ *is the cdf for the normalized* m*-dimensional Normal Distribution with correlation matrix* Λ.

Remark 1. As one would expect the eigenvalues get less correlated as they get closer to the edge.

Remark 2. The eigenvalues are quite correlated in the bulk. In order for x_k and x_m to be independent in the limit it must hold that $|k - m| \sim n$. It is interesting to compare with the following result by Mosteller[2] [3, p. 201]:

Let X_i $(i = 1, \ldots, n)$, be an independent random sample from the Uniform Distribution on $(0, 1)$. Consider the asymptotic joint distribution of the m sample quantiles X_{n_j} $(j = 1, \ldots, m)$, where $n_j = [\lambda_j n] + 1$ and $0 < \lambda_1 < \cdots < \lambda_m$.

Theorem 1.5 (Mosteller). *As* $n \to \infty$ *the joint distribution of* $X_{n_1}, \ldots, X_{n_m}$ *tends to an* m*-dimensional Normal Distribution with means* λ_j, *variances* $n^{-1}\lambda_j(1 - \lambda_j)$ *and correlations*

$$\rho(X_{n_j} X_{n_{j'}}) = \sqrt{\frac{\lambda_j(1 - \lambda_{j'})}{\lambda_{j'}(1 - \lambda_j)}}, \quad j \leqslant j'.$$

Hence in this case $\{X_{n_j}\}_1^m$ are in the limit globally correlated.

2. Proofs of Theorems 1.1 and 1.2

The proofs of Theorems 1.1 and 1.2 relies on a theorem by Costin, Lebowitz and Soshnikov [2,10]. Before presenting it we need some notation.

Let $\{P_t\}$, $t \in \mathbb{R}_+$, be a family of random point fields [9], on the real line such that their correlation functions have a determinantal form.[3] Call the determinant kernels $K_t(x, y)$ and let $\{I_t\}$ be a set of intervals. A_t denotes an integral operator on I_t with kernel $K_t(x, y)$, $A_t : L^2(I_t) \to L^2(I_t)$. By E_t and Var_t is meant expectation and variance with respect to the probability distribution P_t. Finally, let $\#I_t$ stand for the number of particles in I_t.

Theorem 2.1 (Costin, Lebowitz, Soshnikov). *Let* $A_t = K_t \cdot \chi_{I_t}$ *be a family of trace class operators associated with determinantal random point fields* $\{P_t\}$ *such that* $\mathrm{Var}_t(\#I_t) = \mathrm{Trace}(A_t - A_t^2)$ *goes to infinity as* $t \to \infty$. *Then*

$$\frac{\#I_t - E[\#I_t]}{\sqrt{\mathrm{Var}(\#I_t)}} \longrightarrow N(0, 1)$$

in distribution with respect to the random point field P_t.

The following lemmas will be proven in Sections 4 and 5:

Lemma 2.1. *Let* $t = t(k, n)$ *be the solution to the equation*

$$n \frac{2}{\pi} \int_{-1}^{t} \sqrt{1 - x^2}\, \mathrm{d}x = k,$$

2 Mosteller actually allowed for X_i to come from more general distributions.
3 An example is the GUE.

where $k=k(n)$ is such that $k/n\to a\in(0,1)$ as $n\to\infty$. The expected number of eigenvalues in the interval

$$I_n=\left[\sqrt{2n}t+x\sqrt{\frac{\log n}{2n}},\infty\right)$$

is given by

$$\mathbb{E}[\#I_n]=n-k-\frac{x}{\pi}\sqrt{(1-t^2)\log n}+\mathcal{O}\left(\frac{\log n}{n}\right).$$

Lemma 2.2. *The expected number of eigenvalues in the interval $I_n=[\sqrt{2n}t,\infty]$ where $t\to 1^-$ as $n\to\infty$, is given by*

$$\mathbb{E}[\#I_n]=g(t)=\frac{4\sqrt{2}}{3\pi}n(1-t)^{3/2}+\mathcal{O}(1).$$

Lemma 2.3. *The variance of the number of eigenvalues in the interval $[t\sqrt{2n},\infty)$ is equal to $\frac{1}{2\pi^2}\log[n(1-t)^{3/2}](1+\eta(n))$ where $\lim_{n\to\infty}\eta(n)=0$.*

Using the lemmas and Theorem 2.1 we are now ready to prove Theorems 1.1 and 1.2.

Proof of Theorem 1.1. Set

$$I_n=\left[t\sqrt{2n}+\xi\left(\frac{\log n}{4(1-t^2)n}\right)^{1/2},\infty\right).$$

Using Lemmas 2.1 and 2.3 we get

$$\begin{aligned}\mathbb{P}_n\left[\frac{x_k-t\sqrt{2n}}{(\frac{\log n}{4(1-t^2)n})^{1/2}}\leqslant\xi\right]&=\mathbb{P}_n\left[x_k\leqslant t\sqrt{2n}+\xi\left(\frac{\log n}{4(1-t^2)n}\right)^{1/2}\right]\\&=\mathbb{P}_n[\#I_n\leqslant n-k]=\mathbb{P}_n\left[\frac{\#I_n-\mathbb{E}_n[\#I_n]}{(\mathrm{Var}(\#I_n))^{1/2}}\leqslant\frac{n-k-\mathbb{E}_n[\#I_n]}{(\mathrm{Var}(\#I_n))^{1/2}}\right]\\&=\mathbb{P}_n\left[\frac{\#I_n-\mathbb{E}_n[\#I_n]}{(\mathrm{Var}(\#I_n))^{1/2}}\leqslant\xi+\varepsilon(n)\right],\end{aligned}$$

where $\varepsilon(n)\to 0$ as $n\to\infty$. By the Costin–Lebowitz–Soshnikov theorem the conclusion follows. □

Proof of Theorem 1.2. Let $g(t)$ be the expected number of eigenvalues in the interval $I_n=[t\sqrt{2n},\infty)$. We have

$$\mathbb{P}_n\big[x_{n-k}\leqslant t\sqrt{2n}\big]=\mathbb{P}_n[\#I_n\leqslant k]=\mathbb{P}_n\left[\frac{\#I_n-g(t)}{(\mathrm{Var}_n(\#I_n))^{1/2}}\leqslant\frac{k-g(t)}{(\mathrm{Var}_n(\#I_n))^{1/2}}\right].$$

If we can find t such that

$$\frac{k-g(t)}{(\mathrm{Var}_n(\#I_n))^{1/2}}\to\xi \tag{2.1}$$

as $n\to\infty$, then by the Costin–Lebowitz–Soshnikov theorem it holds that

$$\mathbb{P}_n\big[x_{n-k}\leqslant t\sqrt{2n}\big]\longrightarrow\int_{-\infty}^{\xi}\frac{1}{\sqrt{2\pi}}\exp\left[-\frac{x^2}{2}\right]\mathrm{d}x.$$

The idea now is therefore to find a candidate for t. We will then insert this t in the equation above to see if it is satisfied. Set for simplicity $h(t) = (\mathrm{Var}_n(\#I_n))^{1/2}$. We have from Lemmas 2.2 and 2.3 that

$$g(t) = a_1 n(1-t)^{3/2} + \mathcal{O}(1),$$
$$h(t) = a_2 \log^{1/2}\bigl[n(1-t)^{3/2}\bigr] + \mathrm{o}\bigl(\log^{1/2}\bigl[n(1-t)^{3/2}\bigr]\bigr),$$

where a_i are known constants. We have the equation

$$k = g(t) + \xi h(t)$$

or, since g is a strictly decreasing function,

$$t = g^{-1}\bigl(k - \xi h(t)\bigr) \approx g^{-1}(k) - (g^{-1})'(k) \cdot \xi h(t).$$

Since

$$(g^{-1})'(k) = \frac{1}{g'(g^{-1}(k))}$$

we need to study $g^{-1}(k)$.

$$k \approx a_1 n(1-t)^{3/2} \quad \Rightarrow \quad t \approx 1 - \left(\frac{k}{a_1 n}\right)^{2/3}.$$

A reasonable guess for the derivative of g is that

$$g'(t) \approx -\frac{3a_1}{2} n\sqrt{1-t}.$$

We now get

$$g'\bigl(g^{-1}(k)\bigr) \approx -\frac{3a_1}{2} n\left(\left(\frac{k}{a_1 n}\right)^{2/3}\right)^{1/2} = -\frac{3a_1^{2/3}}{2} k^{1/3} n^{2/3}$$

and

$$h(t) \approx h\bigl(g^{-1}(k)\bigr) \approx a_2 \log^{1/2}\left[n\frac{k}{a_1 n}\right] \approx a_2 \log^{1/2} k.$$

When gluing the pieces together one gets

$$t \approx 1 - \left(\frac{k}{a_1 n}\right)^{2/3} + \xi \frac{2a_2}{3a_1^{2/3}} \frac{\log^{1/2} k}{k^{1/3} n^{2/3}}.$$

When inserting this expression in (2.1) it turns out that it all works out. Some rearranging finally yields the result. □

3. Proof of Theorems 1.3 and 1.4

We shall use the following theorem [11]:

Theorem 3.1 (Soshnikov). *Let $(X, \mathcal{F}, P_L)$ be a family of determinantal random point fields with Hermitian locally trace class kernels K_L and $\{I_L^{(1)}, \ldots, I_L^{(k)}\}_{L\geqslant 0}$ be a family of Borel subsets of $\mathbb{R}$, disjoint for any fixed L, with compact closure. Then if for some $\alpha_1, \ldots, \alpha_k \in \mathbb{R}$, the variance of the linear statistics $\sum_{i=-\infty}^{\infty} f_L(x_i)$ with $f_L(x) =$*

$\sum_{j=1}^{k}\alpha_j\cdot\chi_{I_L^{(j)}}(x)$, *grows to infinity in such a way that* $\mathrm{Var}_L(\#I_L^{(j)})=\mathcal{O}(\mathrm{Var}_L(\sum_{i=-\infty}^{\infty}f_L(x_i)))$ *for any* $1\leqslant j\leqslant k$, *the Central Limit Theorem holds*:

$$\frac{\sum_{j=1}^{k}\alpha_j^{(L)}\#I_L^{(j)}-E_L[\sum_{j=1}^{k}\alpha_j^{(L)}\#I_L^{(j)}]}{\sqrt{\mathrm{Var}_L(\sum_{j=1}^{k}\alpha_j^{(L)}\#I_L^{(j)})}}\longrightarrow \mathrm{N}(0,1)$$

in distribution.

Remark 1. The theorem in [11] is actually more general than the theorem stated here.

Remark 2. If the prerequisites in the theorem holds for any $\alpha_1,\dots,\alpha_k$ then $\#I_L^{(1)},\dots,\#I_L^{(k)}$ are jointly normally distributed in the limit [5].

Proof of Theorem 1.3. Take $\{k_i\}$, s_i and X_i as in the formulation of Theorem 1.3. If $k_i-k_{i+1}\sim n^{\theta_i}$ then $s_i-s_{i+1}\sim n^{\theta_i-1}$ and for any real numbers x_i we therefore have the identity (for n large enough)

$$\begin{aligned}\mathbb{P}[X_1\leqslant x_1,\dots,X_m\leqslant x_m]=\mathbb{P}\Bigg[&\frac{\#I_1-E[\#I_1]}{(\mathrm{Var}(\#I_1))^{1/2}}\leqslant\frac{n-k_1-E[\#I_1]}{(\mathrm{Var}(\#I_1))^{1/2}},\\&\frac{\#I_1+\#I_2-E[\#I_1+\#I_2]}{(\mathrm{Var}(\#I_1+\#I_2))^{1/2}}\leqslant\frac{n-k_2-E[\#I_1+\#I_2]}{(\mathrm{Var}(\#I_1+\#I_2))^{1/2}},\dots,\\&\frac{\sum_{i=1}^{m}\#I_i-E[\sum_{i=1}^{m}\#I_i]}{(\mathrm{Var}(\sum_{i=1}^{m}\#I_i))^{1/2}}\leqslant\frac{n-k_m-E[\sum_{i=1}^{m}\#I_i]}{(\mathrm{Var}(\sum_{i=1}^{m}\#I_i))^{1/2}}\Bigg].\end{aligned}$$

Here the intervals I_i are given by

$$\begin{aligned}I_1&=\left(s_1\sqrt{2n}+x_1\left(\frac{\log n}{4(1-s_1^2)n}\right)^{1/2},\infty\right),\\I_i&=\left(s_i\sqrt{2n}+x_i\left(\frac{\log n}{4(1-s_i^2)n}\right)^{1/2},s_{i-1}\sqrt{2n}+x_{i-1}\left(\frac{\log n}{4(1-s_{i-1}^2)n}\right)^{1/2}\right],\end{aligned}$$

where $2\leqslant i\leqslant m$. We would now like to investigate the joint normality of

$$\#I_1,\#I_1+\#I_2,\dots,\sum_{i=1}^{m}\#I_i.$$

To do this we shall consider linear combinations of the variables and show that they are normally distributed in the limit. Since

$$\alpha_1\#I_1+\alpha_2(\#I_1+\#I_2)=(\alpha_1+\alpha_2)\#I_1+\alpha_2\#I_2$$

and so forth it is clear that one can instead look at all linear combinations of $\{\#I_i\}_1^m$. Hence, by the theorem above,[4] we must calculate (Appendix)

$$\mathrm{Var}(\alpha_1\#I_1+\alpha_2\#I_2+\dots+\alpha_m\#I_m)=\sum_{i=1}^{m}\alpha_i^2\iint\limits_{I_i\times I_i^c}K_n^2(x,y)\,\mathrm{d}x\,\mathrm{d}y-\sum_{i\neq j}^{m}\alpha_i\alpha_j\iint\limits_{I_i\times I_j}K_n^2(x,y)\,\mathrm{d}x\,\mathrm{d}y$$

[4] The theorem by Soshnikov does not apply directly to this situation since I_1 does not have compact closure. This is however easily overcome simply by chopping of the interval far out where the probability of finding any eigenvalue is exponentially small in n.

to see that it is of magnitude $\log n$. First define the set M by $k \in M \iff \theta_k = 1$. Hence

$$M = \{k_1, \ldots, k_j\}; \quad 1 \leqslant k_1 < k_2 < \cdots < k_j \leqslant m - 1$$

for some j such that $0 \leqslant j \leqslant m - 1$.

Suppose first that $j = 0$ which means that $\theta_i < 1$ for all i. If $\alpha_1 \neq 0$ then by using the inequality $xy \leqslant \frac{1}{2}(x^2 + y^2)$ we get

$$\begin{aligned}\operatorname{Var}(\alpha_1 \# I_1 + \alpha_2 \# I_2 + \cdots + \alpha_m \# I_m) &\geqslant \sum_{i=1}^m \alpha_i^2 \iint_{I_i \times I_i^c} K_n^2(x, y)\,\mathrm{d}x\,\mathrm{d}y - \sum_{i \neq j}^m \frac{1}{2}(\alpha_i^2 + \alpha_j^2) \iint_{I_i \times I_j} K_n^2(x, y)\,\mathrm{d}x\,\mathrm{d}y \\ &= \sum_{i=1}^m \alpha_i^2 \left(\iint_{I_i \times I_i^c} K_n^2(x, y)\,\mathrm{d}x\,\mathrm{d}y - \sum_{j \neq i}^m \iint_{I_i \times I_j} K_n^2(x, y)\,\mathrm{d}x\,\mathrm{d}y \right). \end{aligned} \tag{3.1}$$

All the terms in the sum are non-negative and the first term can be calculated as in the proof of Lemma 2.3. It was shown in the lemma that in the domain

$$\Omega = \left\{ (x, y); s \leqslant x \leqslant s + \frac{1}{\log n}, s - \frac{1}{\log n} \leqslant y \leqslant s \right\}$$

it holds that

$$2n K_n(\sqrt{2n}x, \sqrt{2n}y) = \frac{1}{2\pi^2 (x - y)^2} + \mathcal{O}\left(\frac{1}{\log n} \right).$$

It was also shown that if

$$\Omega' = \left\{ (x, y); \sqrt{2n}s \leqslant x \leqslant \infty, -\infty < y \leqslant \sqrt{2n}s \right\} / \sqrt{2n} \cdot \Omega$$

then[5]

$$\iint_{\Omega'} K_n^2(x, y)\,\mathrm{d}x\,\mathrm{d}y = \mathcal{O}(\log \log n).$$

In what follows we shall often make use of these facts without mentioning it. The main contribution to the first term in (3.1) can now be calculated to be (disregarding α_1^2)

$$\int_{s_1}^{s_1 + \frac{1}{\log n}} \int_{s_1 - \frac{1}{\log n}}^{s_1 - n^{\theta^* - 1}} \frac{1}{(x - y)^2}\,\mathrm{d}y\,\mathrm{d}x = \frac{1 - \theta^*}{2\pi^2} \log n + \mathcal{O}(\log \log n)$$

where $\theta^* = \max_i \theta_i < 1$. By our definition of $\sim$ above the integration in the y-variable should have been over the interval $(s_1 - 1/\log n, s_1 - h(n) n^{\theta^* - 1})$ where $h(n)$ satisfies (1.4). However, because of the logarithmic answer this h will only produce lower order terms.

Now suppose that $j = 0$ as before, $\alpha_1 = \cdots = \alpha_{k-1} = 0$ but $\alpha_k \neq 0$. In this case we get

$$\operatorname{Var}(\alpha_k \# I_k + \cdots + \alpha_m \# I_m) \geqslant \sum_{i=k}^m \alpha_i^2 \left(\iint_{I_i \times I_i^c} K_n^2(x, y)\,\mathrm{d}x\,\mathrm{d}y - \sum_{k \leqslant j \neq i}^m \iint_{I_i \times I_j} K_n^2(x, y)\,\mathrm{d}x\,\mathrm{d}y \right).$$

Using the estimates above it is straightforward to verify that the k-term is of order $\log n$.

[5] Since $K_n(x, y) = K_n(y, x)$ it is clear that the same estimates hold in the domains obtained from reflection with respect to the $x = y$-line.

When $j \geqslant 1$ meaning that there is at least one k with $\theta_k = 1$, things are only slightly more complicated. Let k^* be the largest integer i such that $\theta_i = 1$. It is sufficient to consider the case when there exists $i \geqslant k^* + 1$ such that $\alpha_i \neq 0$. On the other hand if this is the case then we are in a situation very similar to when $j = 0$. Either $\alpha_{k^*+1} \neq 0$ or $\alpha_{k^*+1} = \cdots = \alpha_{l-1} = 0$ but $\alpha_l \neq 0$. The details are left out.

It is hence a fact that

$$\#I_1, \#I_1 + \#I_2, \ldots, \sum_{i=1}^{m} \#I_i$$

in the limit have a joint normal distribution.

To complete the proof we need to calculate the correlations between the different $\#I_i$'s. If $j < i$ we have that $s_j - s_i \sim n^{-\gamma}$ where $\gamma = 1 - \max_{j \leqslant k < i} \theta_k$. Set

$$X_k = \sum_{m=1}^{k} \#I_m.$$

From a straightforward calculation (as above) we get that

$$\operatorname{Var}(X_i - X_j) = \operatorname{Var}\left(\sum_{k=j+1}^{i} \#I_k\right) = \operatorname{Var}\left(\# \bigcup_{k=j+1}^{i} I_k\right) = \frac{\gamma}{\pi^2} \log n + \mathcal{O}(\log\log n).$$

Since

$$\operatorname{Var}(X_k) = \frac{1}{2\pi^2} \log n + \mathcal{O}(\log\log n)$$

the correlation ρ is given by

$$\rho(X_i, X_j) = \frac{\frac{1}{2}(\operatorname{Var}(X_j) + \operatorname{Var}(X_i) - \operatorname{Var}(X_i - X_j))}{\sqrt{\operatorname{Var}(X_i)\operatorname{Var}(X_j)}} = \gamma + \mathrm{o}(1). \qquad \square$$

Proof of Theorem 1.4. This proof is of course very similar to the previous one so some details will be skipped.

With notation as in the formulation of Theorem 1.4 the intervals of interest (cf. previous proof) are in this case

$$I_1 = \left(\sqrt{2n}\left(1 - C_1\left(\frac{k_1}{n}\right)^{2/3}\right) + x_1 C_2\left(\frac{\log k_1}{n^{1/3}k_1^{2/3}}\right)^{1/2}, \infty\right),$$

$$I_i = \left(\sqrt{2n}\left(1 - C_1\left(\frac{k_i}{n}\right)^{2/3}\right) + x_i C_2\left(\frac{\log k_i}{n^{1/3}k_i^{2/3}}\right)^{1/2},\right.$$

$$\left.\sqrt{2n}\left(1 - C_1\left(\frac{k_{i-1}}{n}\right)^{2/3}\right) + x_{i-1} C_2\left(\frac{\log k_{i-1}}{n^{1/3}k_{i-1}^{2/3}}\right)^{1/2}\right],$$

where C_1, C_2 are known constants and $2 \leqslant i \leqslant m$. Given any $\{x_i\}$ it is straightforward to show that for n large enough $\{I_i\}$ really are intervals. As in the previous proof we want to show that

$$\#I_1, \#I_2, \ldots, \#I_m$$

are jointly normally distributed. The way to prove this is the same as before but some details are different. By Lemma 2.3 we need to show that

$$\log n = \mathcal{O}\left(\operatorname{Var}\left(\sum_{i=1}^{m} \alpha_i \#I_i\right)\right)$$

for any real α_i's such that for some i $\alpha_i \neq 0$.

Let $t = t(n)$ be such that $t \to 1^-$ as $n \to \infty$ and $n^{\varepsilon-2/3} \leqslant 1-t \leqslant n^{-\varepsilon}$ for some $0 < \varepsilon < 1/3$. From the proof of Lemma 2.3 we have that in the sets

$$\Omega_t = \left\{ t \leqslant x \leqslant t + \frac{1-t}{\log n}, t - \frac{1-t}{\log n} \leqslant y \leqslant t \right\}$$

it holds that

$$2nK_n^2(\sqrt{2n}x, \sqrt{2n}y) = \frac{1}{2\pi^2(x-y)^2} + \mathcal{O}\left(\frac{1}{\log n}\right).$$

Returning to the variance calculation we first assume that $\alpha_1 \neq 0$. We know from the previous proof that in this case it is sufficient to to show that

$$\iint\limits_{I_1 \times (I^c \setminus \bigcup_{i=2}^m I_i)} K_n^2(x, y)\,\mathrm{d}y\,\mathrm{d}x$$

is of order $\log n$. In fact since the integrand is non-negative it is enough if

$$\iint\limits_{I^* \times I_*} \frac{1}{(x-y)^2}\,\mathrm{d}y\,\mathrm{d}x$$

is of order $\log n$ where

$$I^* = \left(t_1 + r_1, t_1 + \frac{1-t_1}{\log n}\right),$$
$$I_* = \left(t_1 - \frac{1-t_1}{\log n}, t_m\right)$$

and

$$t_i = 1 - C_1\left(\frac{k_i}{n}\right)^{2/3},$$
$$r_i = x_i C_2\left(\frac{\log k_i}{n^{1/3}k_i^{2/3}}\right).$$

An elementary calculation shows that this integral is indeed of order $\log n$.

If $\alpha_1 = \cdots = \alpha_{k-1} = 0$ but $\alpha_k \neq 0$ it is sufficient that the integral

$$\iint\limits_{J^* \times J_*} \frac{1}{(x-y)^2}\,\mathrm{d}y\,\mathrm{d}x$$

is of order $\log n$ where

$$J^* = \left(t_{k-1} + r_{k-1}, t_{k-1} + \frac{1-t_{k-1}}{\log n}\right),$$
$$J_* = (t_k, t_{k-1}).$$

Again we get the size $\log n$. This proves that we get a Normal Distribution in the limit. The calculations of the correlations are very similar to the bulk case and the details are not presented here. □

4. The expected number of eigenvalues in I_n

In this section and the next we shall need asymptotics for the Airy function and the Hermite polynomials. In [4] the asymptotics for a class containing the Hermite case was studied. It is shown there that for fixed $\delta > 0$ the following holds:

1. $-1+\delta \leqslant x \leqslant 1-\delta$.

$$h_n(\sqrt{2n}x)\exp[-nx^2] = \left(\frac{2}{\pi\sqrt{2n}}\right)^{1/2}\frac{1}{(1-x^2)^{1/4}}\left(\cos\left[2nF(x)-\frac{1}{2}\arcsin(x)\right]+\mathcal{O}(n^{-1})\right).$$

2. $1-\delta \leqslant x < 1$.

$$\begin{aligned}h_n(\sqrt{2n}x)\mathrm{e}^{-nx^2} = (2n)^{-1/4}\biggl\{&\left(\frac{1+x}{1-x}\right)^{1/4}\bigl[3nF(x)\bigr]^{1/6}\operatorname{Ai}\bigl(-\bigl[3nF(x)\bigr]^{2/3}\bigr)\bigl(1+\mathcal{O}(n^{-1})\bigr)\\ &-\left(\frac{1-x}{1+x}\right)^{1/4}\bigl[3nF(x)\bigr]^{-1/6}\operatorname{Ai}'\bigl(-\bigl[3nF(x)\bigr]^{2/3}\bigr)\bigl(1+\mathcal{O}(n^{-1})\bigr)\biggr\}.\end{aligned}$$

3. $1 < x \leqslant 1+\delta$.

$$\begin{aligned}h_n(\sqrt{2n}x)\mathrm{e}^{-nx^2} = (2n)^{-1/4}\biggl\{&\left(\frac{x+1}{x-1}\right)^{1/4}\bigl[3nF(x)\bigr]^{1/6}\operatorname{Ai}\bigl(\bigl[3nF(x)\bigr]^{2/3}\bigr)\\ &-\left(\frac{x-1}{x+1}\right)^{1/4}\bigl[3nF(x)\bigr]^{-1/6}\operatorname{Ai}'\bigl(\bigl[3nF(x)\bigr]^{2/3}\bigr)\biggr\}\bigl(1+\mathcal{O}(n^{-1})\bigr).\end{aligned}$$

4. $x > 1+\delta$.

$$h_n(\sqrt{2n}x)\mathrm{e}^{-nx^2} = \mathcal{O}\bigl(n^{-1/4}\mathrm{e}^{-nF(x)}\bigr).$$

In these expressions Ai stands for the Airy function and

$$F(x) = \left|\int_x^1 \bigl|\sqrt{1-y^2}\bigr|\,\mathrm{d}y\right|. \tag{4.1}$$

There are of course also similar asymptotics for the Hermite polynomials near the point -1.

The Airy function is bounded on the real line. It is exponentially small in x on $\mathbb{R}_+$ and for $r > 0$ it holds that [8]

$$\begin{aligned}\operatorname{Ai}(-r) &= \pi^{-1/2}r^{-1/4}\left\{\cos\left[\frac{2}{3}r^{3/2}-\frac{\pi}{4}\right]+\mathcal{O}(r^{-3/2})\right\},\\ \operatorname{Ai}'(-r) &= \pi^{-1/2}r^{1/4}\left\{\sin\left[\frac{2}{3}r^{3/2}-\frac{\pi}{4}\right]+\mathcal{O}(r^{-3/2})\right\}.\end{aligned}$$

Proof of Lemma 2.1. Set

$$f_n(t) = t + x\frac{\sqrt{\log n}}{2n}.$$

We have that

$$\mathbb{E}[\#I_n] = \int_{f_n(t)}^{\infty} n\rho_n(x)\,\mathrm{d}x,$$

where ρ_n is the scaled density for the eigenvalues (the limiting density has support in $[-1,1]$). From symmetry one gets

$$\int_{f_n(t)}^{\infty} n\rho_n(x)\,\mathrm{d}x = \frac{n}{2} - \int_{0}^{f_n(t)} n\rho_n(x)\,\mathrm{d}x.$$

Formula (4.2) in [6] applied to the hermitian case says that

$$n\rho_n(x) = n\cdot\frac{2}{\pi}\sqrt{1-x^2} + \frac{1}{4\pi}\left(\frac{1}{x-1}-\frac{1}{x+1}\right)\cos\left[n\frac{2}{\pi}\int_x^1\sqrt{1-y^2}\,\mathrm{d}y\right] + \mathcal{O}(n^{-1}).$$

This formula is valid in the interval $[-1+\delta, 1-\delta]$ for any (fixed) $\delta>0$. We now get

$$\begin{aligned}
\mathbb{E}[\#I_n] &= \frac{n}{2} - n\frac{2}{\pi}\int_0^{f_n(t)}\sqrt{1-x^2}\,\mathrm{d}x + \mathcal{O}(n^{-1}) = n - n\frac{2}{\pi}\int_{-1}^{f_n(t)}\sqrt{1-x^2}\,\mathrm{d}x + \mathcal{O}(n^{-1})\\
&= n - n\frac{2}{\pi}\left(\int_{-1}^{t}\sqrt{1-x^2}\,\mathrm{d}x + \sqrt{1-t^2}\,x\frac{\sqrt{\log n}}{2n} + \mathcal{O}\left(\frac{\log n}{n^2}\right)\right) + \mathcal{O}(n^{-1})\\
&= n - k - \frac{x}{\pi}\sqrt{(1-t^2)\log n} + \mathcal{O}\left(\frac{\log n}{n}\right). \qquad \square
\end{aligned}$$

Proof of Lemma 2.2. From formula (4.4) and (4.21) in the paper [6] one gets after some minor calculations that

$$\begin{aligned}
n\rho_n(x) = {} & \left(\frac{\Phi'(x)}{4\Phi(x)} - \frac{\gamma'(x)}{\gamma(x)}\right)\bigl[2\,\mathrm{Ai}\bigl(\Phi(x)\bigr)\,\mathrm{Ai}'\bigl(\Phi(x)\bigr)\bigr] + \Phi'(x)\bigl[\bigl(\mathrm{Ai}'\bigl(\Phi(x)\bigr)\bigr)^2 - \Phi(x)\bigl(\mathrm{Ai}\bigl(\Phi(x)\bigr)\bigr)^2\bigr]\\
& + \mathcal{O}\left(\frac{1}{n(\sqrt{1-x})}\right)
\end{aligned}$$

in a fixed neighborhood of $[0,1]$. Here ρ_n is the scaled density for the eigenvalues so that

$$g(t) = \int_t^{\infty} n\rho_n(x)\,\mathrm{d}x.$$

The functions γ and Φ are given by

$$\begin{aligned}
\gamma(x) &= \left(\frac{x-1}{x+1}\right)^{1/4},\\
\Phi(x) &= \begin{cases} -\bigl(3n\int_x^1\sqrt{1-y^2}\,\mathrm{d}y\bigr)^{2/3} & \text{if } x\leqslant 1,\\ \bigl(3n\int_1^x\sqrt{y^2-1}\,\mathrm{d}y\bigr)^{2/3} & \text{if } x>1.\end{cases}
\end{aligned}$$

The function γ is evaluated taking the limit from the upper half plane using the principal branch.

The fact that the asymptotics only holds for, $[0,1+\delta]$, for some $\delta>0$ (independent of n) is not a problem. It is not difficult to show that for $x\geqslant 1+\delta$ $\rho_n(x)$ is exponentially small in n and exponentially decaying in x.

We now look at the different terms in the asymptotical expression for ρ_n above. When looking at the asymptotics for Ai and Ai$'$ it easy to see that

$$\bigl|\mathrm{Ai}(x)\,\mathrm{Ai}'(x)\bigr| = \mathcal{O}(1).$$

This together with the fact that

$$\left(\frac{\Phi'(x)}{4\Phi(x)}-\frac{\gamma'(x)}{\gamma(x)}\right)=\mathcal{O}(1)$$

gives

$$\int_t^{1+\delta}\left(\frac{\Phi'(x)}{4\Phi(x)}-\frac{\gamma'(x)}{\gamma(x)}\right)\bigl[2\,\mathrm{Ai}\bigl(\Phi(x)\bigr)\,\mathrm{Ai}'\bigl(\Phi(x)\bigr)\bigr]\,\mathrm{d}x=\mathcal{O}(1).$$

The main contribution comes from the second term. In fact a primitive function can be found for this expression:

$$\begin{aligned}
&\int_t^{1+\delta}\Phi'(x)\bigl[\bigl(\mathrm{Ai}'\bigl(\Phi(x)\bigr)\bigr)^2-\Phi(x)\bigl(\mathrm{Ai}\bigl(\Phi(x)\bigr)\bigr)^2\bigr]\,\mathrm{d}x=\bigl[y=\Phi(x)\bigr]=\int_{\Phi(t)}^{\Phi(1+\delta)}\bigl(\mathrm{Ai}'(y)\bigr)^2-y\bigl(\mathrm{Ai}(y)\bigr)^2\,\mathrm{d}y\\
&\quad=-\left[\frac{2}{3}\bigl(y^2\bigl(\mathrm{Ai}(y)\bigr)^2-y\bigl(\mathrm{Ai}'(y)\bigr)^2\bigr)-\frac{1}{3}\mathrm{Ai}(y)\,\mathrm{Ai}'(y)\right]_{\Phi(t)}^{\Phi(1+\delta)}\\
&\quad=\frac{2}{3}\bigl(\Phi(t)^2\bigl(\mathrm{Ai}\bigl(\Phi(t)\bigr)\bigr)^2-\Phi(t)\bigl(\mathrm{Ai}'\bigl(\Phi(t)'\bigr)\bigr)^2\bigr)-\frac{1}{3}\mathrm{Ai}\bigl(\Phi(t)\bigr)\,\mathrm{Ai}'\bigl(\Phi(t)\bigr)+\mathcal{O}\bigl(\exp-[cn]\bigr).
\end{aligned}$$

Here c is a positive constant. Integrating the third term only gives a contribution of order n^{-1}. One can now use the asymptotics for the Airy function and it's derivative to get the stated result. □

5. The variance of the number of eigenvalues in I_n

Proof of Lemma 2.3. The proof will be divided into two basic cases. The first case is when $1-t>\delta$ for a fix $\delta>0$, i.e. in the bulk. The second case is when $t=t(n)\to 1^-$ as $n\to\infty$ i.e. near the spectrum edge (considering the right edge here).

First define $I_n=[t\sqrt{2n},\infty)$ and $\#I_n$ as the number of eigenvalues in I_n. It is a fact (see Appendix B) that

$$\mathrm{Var}(I_n)=\int_{I_n}\int_{\mathbb{R}}K_n^2(x,y)\,\mathrm{d}x\,\mathrm{d}y-\int_{I_n}\int_{I_n}K_n^2(x,y)\,\mathrm{d}x\,\mathrm{d}y=\int_{I_n}\int_{I_n^c}K_n^2(x,y)\,\mathrm{d}x\,\mathrm{d}y.$$

Here K_n is the usual determinant kernel for the Hermitian ensemble. The advantage with this representation is that there is only one singular point in the Christoffel–Darboux representation of $K_n(x,y)$:

$$K_n(x,y)=\sqrt{\frac{n}{2}}\frac{h_n(x)h_{n-1}(y)-h_{n-1}(x)h_n(y)}{x-y}\exp\left(-\frac{1}{2}(x^2+y^2)\right).$$

Case I (the bulk). After a change of variables ($x\to\sqrt{2n}x$) we get the integrand

$$\bigl[\sqrt{2n}K_n(\sqrt{2n}x,\sqrt{2n}y)\bigr]^2.$$

First consider the domain where both variables are in the bulk:

$$\Gamma=\bigl\{(x,y);\,t\leqslant x\leqslant 1-\delta,\,-1+\delta\leqslant y\leqslant t\bigr\}.\tag{5.1}$$

In Γ h_n has asymptotics as

$$h_n(\sqrt{2n}x)\exp[-nx^2]=\left(\frac{2}{\pi\sqrt{2n}}\right)^{1/2}\frac{1}{(1-x^2)^{1/4}}\left(\cos\left[2nF(x)-\frac{1}{2}\arcsin(x)\right]+\mathcal{O}(n^{-1})\right).$$

Here

$$F(x)=\int_x^1 \sqrt{1-z^2}\,\mathrm{d}z=\frac{1}{2}\bigl(\arccos x - x\sqrt{1-x^2}\bigr).$$

The asymptotics for h_{n-1} becomes

$$\begin{aligned}
h_{n-1}(\sqrt{2n}x)\exp[-nx^2] &= \left(\frac{2}{\pi\sqrt{2(n-1)}}\right)^{1/2}\frac{1}{(1-x_n^2)^{1/4}}\\
&\quad\times\left(\cos\left[2(n-1)F(x_n)-\frac{1}{2}\arcsin(x_n)\right]+\mathcal{O}(n^{-1})\right)\\
&= \left(\frac{2}{\pi\sqrt{2n}}\right)^{1/2}\frac{1}{(1-x^2)^{1/4}}\left(\cos\left[2(n-1)F(x_n)-\frac{1}{2}\arcsin(x_n)\right]+\mathcal{O}(n^{-1})\right),
\end{aligned}$$

where $x_n=\sqrt{\frac{n}{n-1}}x$. A Taylor expansion gives

$$F(x_n)=F(x)-\frac{x}{2(n-1)}\sqrt{1-x^2}+\mathcal{O}(n^{-2})$$

leading to

$$2(n-1)F(x_n)=2nF(x)-2F(x)-x\sqrt{1-x^2}+\mathcal{O}(n^{-1})=2nF(x)-\arccos x+\mathcal{O}(n^{-1}).$$

One can now write

$$\begin{aligned}
&h_n(x\sqrt{2n})h_{n-1}(\sqrt{2n}y)\exp\bigl[-n(x^2+y^2)\bigr]=\frac{2}{\pi\sqrt{2n}(1-x^2)^{1/4}(1-y^2)^{1/4}}\\
&\quad\times\cos\left[2nF(x)-\frac{1}{2}\arcsin x\right]\cos\left[2nF(y)-\frac{1}{2}\arcsin y-\arccos y\right]+\mathcal{O}(n^{-3/2}).
\end{aligned}$$

Set, for simplicity,

$$\begin{aligned}
\alpha_x &= 2nF(x)-\frac{1}{2}\arcsin x,\\
\theta_x &= \arccos x.
\end{aligned}$$

By the Christoffel–Darboux formula

$$\sqrt{2n}K_n(\sqrt{2n}x,\sqrt{2n}y)=\frac{1}{\pi(1-x^2)^{1/4}(1-y^2)^{1/4}}\,\frac{\cos\alpha_x\cos[\alpha_y-\theta_y]-\cos[\alpha_x-\theta_x]\cos\alpha_y+\mathcal{O}(n^{-1})}{x-y}.$$

To prepare for integration we now divide Γ into four disjoint sets. Set

$$\begin{aligned}
\Gamma_0 &= \left\{(x,y);\, t\leqslant x\leqslant t+\frac{1}{n},\, t-\frac{1}{n}\leqslant y\leqslant t\right\},\\
\Gamma_1 &= \Gamma_1^1\cup\Gamma_1^2=\left\{(x,y);\, t\leqslant x\leqslant t+\frac{1-t}{r(n)},\, t-\frac{t+1}{r(n)}\leqslant y\leqslant t-\frac{1}{n}\right\}\\
&\qquad\cup\left\{(x,y);\, t+\frac{1}{n}\leqslant x\leqslant t+\frac{1-t}{r(n)},\, t-\frac{1}{n}\leqslant y\leqslant t\right\},\\
\Gamma_2 &= \Gamma\setminus(\Gamma_0\cup\Gamma_1),
\end{aligned}$$

where $r(n)=\log n$ and Γ was defined in (5.1).

Γ_0: When integrating over Γ_0 one can use the fact that

$$\sqrt{2n}K_n(\sqrt{2n}x,\sqrt{2n}y) \leqslant Cn\frac{\sin(x-y)}{x-y}$$

where $C > 0$. Hence

$$\int_{\Gamma_0} \left[\sqrt{2n}K_n(\sqrt{2n}x,\sqrt{2n}y)\right]^2 \mathrm{d}x\,\mathrm{d}y = \mathcal{O}(1).$$

Γ_1: In Γ_1 we have

$$\theta_x = \arccos x = \arccos t + \mathcal{O}\left(\frac{1}{r(n)}\right)$$

and of course also the equivalent for θ_y. Defining $\theta = \arccos t$ we get by the use of some trigonometric identities that

$$\begin{aligned}\cos\alpha_x\cos[\alpha_y-\theta_y]-\cos[\alpha_x-\theta_x]\cos\alpha_y &= \cos\alpha_x\cos[\alpha_y-\theta]-\cos[\alpha_x-\theta]\cos\alpha_y+\mathcal{O}\left(\frac{1}{r(n)}\right)\\ &= \sqrt{1-t^2}\sin[\alpha_y-\alpha_x]+\mathcal{O}\left(\frac{1}{r(n)}\right).\end{aligned}$$

Since

$$\frac{\sqrt{1-t^2}}{(1-x^2)^{1/4}(1-y^2)^{1/4}} = 1+\mathcal{O}\left(\frac{1}{r(n)}\right)$$

and

$$\alpha_y-\alpha_x = 2n\big(F(y)-F(x)\big)+\mathcal{O}\left(\frac{1}{r(n)}\right)$$

we now have

$$\begin{aligned}\iint_{\Gamma_1}\left[\sqrt{2n}K_n(\sqrt{2n}x,\sqrt{2n}y)\right]^2 \mathrm{d}x\,\mathrm{d}y &= \iint_{\Gamma_1^1}\frac{1}{\pi^2}\frac{\sin^2[2n(F(y)-F(x))]+\mathcal{O}(\frac{1}{r(n)})}{(x-y)^2}\,\mathrm{d}x\,\mathrm{d}y+\iint_{\Gamma_1^2}\frac{\mathcal{O}(1)}{(x-y)^2}\,\mathrm{d}x\,\mathrm{d}y\\ &= \frac{1}{2\pi^2}\iint_{\Gamma_1^1}\frac{1-\cos[4n(F(y)-F(x))]}{(x-y)^2}\,\mathrm{d}x\,\mathrm{d}y+\mathcal{O}\big(\log r(n)\big)\\ &= \frac{1}{2\pi^2}\log n-\frac{1}{2\pi^2}\iint_{\Gamma_1^1}\frac{\cos[4n(F(y)-F(x))]}{(x-y)^2}\,\mathrm{d}x\,\mathrm{d}y+\mathcal{O}\big(\log r(n)\big).\end{aligned}$$

The remaining integral is not bigger than a constant as will now be shown. a partial integration in the y-variable gives

$$\begin{aligned}\iint_{\Gamma_1^1}\frac{\cos[4n(F(y)-F(x))]}{(x-y)^2}\,\mathrm{d}x\,\mathrm{d}y = &\int_t^{t+\frac{(1-t)}{r(n)}}\left(\left[\frac{\sin[4n(F(x)-F(y))]}{4nF'(y)(x-y)^2}\right]_{t-\frac{t+1}{r(n)}}^{t-1/n}\right.\\ &\left.-\int_{t-\frac{t+1}{r(n)}}^{t-\frac{1}{n}}\sin\big[4n\big(F(x)-F(y)\big)\big]\left(\frac{1}{4n[F'(y)(x-y)^2]}\right)_y'\mathrm{d}y\right)\mathrm{d}x\\ = I_1-I_2.&\end{aligned}$$

Both the integrals are easy to estimate:

$$|I_1| \leqslant C \int_t^{t+\frac{1-t}{r(n)}} \frac{1}{n(x-y)^2}\,\mathrm{d}x = \mathcal{O}\left(\frac{1}{n\min(x-y)}\right) = \mathcal{O}(1).$$

We have

$$\left(\left[F'(y)(x-y)^2\right]^{-1}\right)'_y = -\frac{y}{(1-y^2)^{3/2}(x-y)^2} - \frac{2}{\sqrt{1-y^2}(x-y)^3}$$

which gives

$$|I_2| \leqslant C \iint_{\Gamma_1^1} \frac{1}{n(x-y)^3} = \mathcal{O}(1).$$

Above $C > 0$.

Γ_2: In Γ_2 it holds that

$$\left[\sqrt{2n}K_n(\sqrt{2n}x, \sqrt{2n}y)\right]^2 = \mathcal{O}\left(\frac{1}{(x-y)^2}\right)$$

and trivial calculations give

$$\iint_{\Gamma_2} \frac{1}{(x-y)^2}\,\mathrm{d}x\,\mathrm{d}y = \mathcal{O}\left(\log r(n)\right).$$

To complete case I we must also integrate over $I_n \times I_n^c \setminus \Gamma$. The asymptotical expression for h_n is different but there are no difficulties. One can just take absolute values in the integral and the result is $\mathcal{O}(1)$.

Case II (the spectrum edge). First consider the subdomain

$$\Omega = \left\{(x, y); t \leqslant x \leqslant 1 - Cn^{-1}, 1-\delta \leqslant y \leqslant t\right\},$$

where C is a large positive constant. After a change of variables the contribution J_Ω from $\sqrt{2n}\cdot\Omega$ to the variance can be written as

$$J_\Omega = \iint_\Omega \left[\sqrt{2n}K_n(\sqrt{2n}x, \sqrt{2n}y)\right]^2\,\mathrm{d}x\,\mathrm{d}y.$$

In order to deal with this integral we must first study the integrand and, via Christoffel–Darboux, especially the difference

$$D = h_n \exp\left(-n(x^2+y^2)\right)(\sqrt{2n}x)h_{n-1}(\sqrt{2n}y) - h_{n-1}(\sqrt{2n}x)h_n(\sqrt{2n}y). \tag{5.2}$$

We will show that in Ω it holds that

$$\begin{aligned} D = {} & \frac{\text{const}}{(4n(n-1))^{1/4}}\left[\mathrm{Ai}\left(-\left[3nF(x)\right]^{2/3}\right)\mathrm{Ai}'\left(-\left[3nF(y)\right]^{2/3}\right) - \mathrm{Ai}'\left(-\left[3nF(x)\right]^{2/3}\right)\mathrm{Ai}\left(-\left[3nF(y)\right]^{2/3}\right)\right] \\ & + \mathcal{O}\left(\frac{1}{n(1-x)}\right) + \mathcal{O}\left(\frac{(1-y)^{3/4}}{(1-x)^{1/4}}\right). \end{aligned}$$

Here Ai stands for the Airy function and

$$F(x) = \int_x^1 \sqrt{1-t^2}\,\mathrm{d}t.$$

In Ω h_n has the following asymptotics:

$$h_n(\sqrt{2n}x)\exp(-nx^2) = (2n)^{-1/4}\left\{\left(\frac{1+x}{1-x}\right)^{1/4}\left[3nF(x)\right]^{1/6}\mathrm{Ai}\left(-\left[3nF(x)\right]^{2/3}\right)\left(1+\mathcal{O}(n^{-1})\right)\right.$$
$$\left. -\left(\frac{1-x}{1+x}\right)^{1/4}\left[3nF(x)\right]^{-1/6}\mathrm{Ai}'\left(-\left[3nF(x)\right]^{2/3}\right)\left(1+\mathcal{O}(n^{-1})\right)\right\}.$$

If, for the moment, disregarding the $\mathcal{O}(n^{-1})$ terms in the h_n-asymtotics (5.2) can be written as a sum of four differences D_1–D_4:

$$\begin{aligned}
\left(4n(n-1)\right)^{1/4}D_1 = {} & \left(\frac{1+x}{1-x}\right)^{1/4}\left(\frac{1+y_n}{1-y_n}\right)^{1/4}\left[3nF(x)\right]^{1/6}\left[3n'F(y_n)\right]^{1/6}\\
& \times\mathrm{Ai}\left(-\left[3nF(x)\right]^{2/3}\right)\mathrm{Ai}\left(-\left[3n'F(y_n)\right]^{2/3}\right)\\
& -\left(\frac{1+x_n}{1-x_n}\right)^{1/4}\left(\frac{1+y}{1-y}\right)^{1/4}\left[3n'F(x_n)\right]^{1/6}\left[3nF(y)\right]^{1/6}\\
& \times\mathrm{Ai}\left(-\left[3n'F(x_n)\right]^{2/3}\right)\mathrm{Ai}\left(-\left[3nF(y)\right]^{2/3}\right),\\
\left(4n(n-1)\right)^{1/4}D_2 = {} & \left(\frac{1+x}{1-x}\right)^{1/4}\left(\frac{1-y_n}{1+y_n}\right)^{1/4}\left[3nF(x)\right]^{1/6}\left[3n'F(y_n)\right]^{-1/6}\\
& \times\mathrm{Ai}\left(-\left[3nF(x)\right]^{2/3}\right)\mathrm{Ai}'\left(-\left[3n'F(y_n)\right]^{2/3}\right)\\
& -\left(\frac{1+x_n}{1-x_n}\right)^{1/4}\left(\frac{1-y}{1+y}\right)^{1/4}\left[3n'F(x_n)\right]^{1/6}\left[3nF(y)\right]^{-1/6}\\
& \times\mathrm{Ai}\left(-\left[3n'F(x_n)\right]^{2/3}\right)\mathrm{Ai}'\left(-\left[3nF(y)\right]^{2/3}\right),\\
\left(4n(n-1)\right)^{1/4}D_3 = {} & \left(\frac{1-x_n}{1+x_n}\right)^{1/4}\left(\frac{1+y}{1-y}\right)^{1/4}\left[3n'F(x_n)\right]^{-1/6}\left[3nF(y)\right]^{1/6}\\
& \times\mathrm{Ai}'\left(-\left[3n'F(x_n)\right]^{2/3}\right)\mathrm{Ai}\left(-\left[3nF(y)\right]^{2/3}\right)\\
& -\left(\frac{1-x}{1+x}\right)^{1/4}\left(\frac{1+y_n}{1-y_n}\right)^{1/4}\left[3nF(x)\right]^{-1/6}\left[3n'F(y_n)\right]^{1/6}\\
& \times\mathrm{Ai}'\left(-\left[3nF(x)\right]^{2/3}\right)\mathrm{Ai}\left(-\left[3n'F(y_n)\right]^{2/3}\right),\\
\left(4n(n-1)\right)^{1/4}D_4 = {} & \left(\frac{1-x}{1+x}\right)^{1/4}\left(\frac{1-y_n}{1+y_n}\right)^{1/4}\left[3nF(x)\right]^{-1/6}\left[3n'F(y_n)\right]^{-1/6}\\
& \times\mathrm{Ai}'\left(-\left[3nF(x)\right]^{2/3}\right)\mathrm{Ai}'\left(-\left[3n'F(y_n)\right]^{2/3}\right)\\
& -\left(\frac{1-x_n}{1+x_n}\right)^{1/4}\left(\frac{1-y}{1+y}\right)^{1/4}\left[3n'F(x_n)\right]^{-1/6}\left[3nF(y)\right]^{-1/6}\\
& \times\mathrm{Ai}'\left(-\left[3n'F(x_n)\right]^{2/3}\right)\mathrm{Ai}'\left(-\left[3nF(y)\right]^{2/3}\right).
\end{aligned}$$

In the above $n' = n-1$ and $x_n = \sqrt{\frac{n}{n-1}}x$. Note that $x_n < 1$ in Ω.

D_1: A calculation using the series expansion

$$\frac{F^{1/6}(x)}{(1-x)^{1/4}} = c_0 + c_1(1-x) + \cdots$$

gives

$$\begin{aligned}
&\left(\frac{1+x_n}{1-x_n}\right)^{1/4}\left(\frac{1+y}{1-y}\right)^{1/4}\bigl[3(n-1)F(x_n)\bigr]^{1/6}\bigl[3nF(y)\bigr]^{1/6} \\
&\quad=\left(\frac{1+x}{1-x}\right)^{1/4}\left(\frac{1+y}{1-y}\right)^{1/4}\bigl[3nF(x)\bigr]^{1/6}\bigl[3nF(y)\bigr]^{1/6}+\mathcal{O}\bigl(n^{1/3}(1-x)\bigr) \\
&\quad=a_1n^{1/3}+\mathcal{O}\bigl(n^{1/3}(1-y)\bigr),
\end{aligned}$$

where

$$a_1=\lim_{x\to 1^-}\sqrt{1+x}\,\frac{(3F(x))^{1/3}}{\sqrt{1-x}}.$$

Since

$$Ai\bigl(-\bigl[3nF(x)\bigr]^{2/3}\bigr)=\mathcal{O}\left(\frac{1}{n^{1/6}(1-x)^{1/4}}\right)$$

it holds that

$$\begin{aligned}
\bigl(4n(n-1)\bigr)^{1/4}D_1=a_1n^{1/3}\bigl[&\mathrm{Ai}\bigl(-\bigl[3nF(x)\bigr]^{2/3}\bigr)\,\mathrm{Ai}\bigl(-\bigl[3n'F(y_n)\bigr]^{2/3}\bigr) \\
&-\mathrm{Ai}\bigl(-\bigl[3n'F(x_n)\bigr]^{2/3}\bigr)\,\mathrm{Ai}\bigl(-\bigl[3nF(y)\bigr]^{2/3}\bigr)\bigr]+\mathcal{O}\left(\frac{(1-y)^{3/4}}{(1-x)^{1/4}}\right).
\end{aligned}$$

D_2–D_4: The same procedure as in the previous case gives

$$\begin{aligned}
\bigl(4n(n-1)\bigr)^{1/4}D_2=\mathcal{O}(1)\bigl[&\mathrm{Ai}\bigl(-\bigl[3nF(x)\bigr]^{2/3}\bigr)\,\mathrm{Ai}'\bigl(-\bigl[3n'F(y_n)\bigr]^{2/3}\bigr) \\
&-\mathrm{Ai}\bigl(-\bigl[3n'F(x_n)\bigr]^{2/3}\bigr)\,\mathrm{Ai}'\bigl(-\bigl[3nF(y)\bigr]^{2/3}\bigr)\bigr]+\mathcal{O}\left(\frac{(1-y)^{5/4}}{(1-x)^{1/4}}\right), \\
\bigl(4n(n-1)\bigr)^{1/4}D_3=\mathcal{O}(1)\bigl[&\mathrm{Ai}'\bigl(-\bigl[3n'F(x_n)\bigr]^{2/3}\bigr)\,\mathrm{Ai}\bigl(-\bigl[3nF(y)\bigr]^{2/3}\bigr) \\
&-\mathrm{Ai}'\bigl(-\bigl[3nF(x)\bigr]^{2/3}\bigr)\,\mathrm{Ai}\bigl(-\bigl[3n'F(y_n)\bigr]^{2/3}\bigr)\bigr]+\mathcal{O}\left(\frac{(1-y)^{5/4}}{(1-x)^{1/4}}\right), \\
\bigl(4n(n-1)\bigr)^{1/4}D_4=\mathcal{O}(n^{-1/3})\bigl[&\mathrm{Ai}'\bigl(-\bigl[3nF(x)\bigr]^{2/3}\bigr)\,\mathrm{Ai}'\bigl(-\bigl[3n'F(y_n)\bigr]^{2/3}\bigr) \\
&-\mathrm{Ai}'\bigl(-\bigl[3n'F(x_n)\bigr]^{2/3}\bigr)\,\mathrm{Ai}'\bigl(-\bigl[3nF(y)\bigr]^{2/3}\bigr)\bigr]+\mathcal{O}\bigl((1-y)^{3/2}\bigr).
\end{aligned}$$

Now consider the difference still left in D_1:

$$\mathrm{Ai}\bigl(-\bigl[3nF(x)\bigr]^{2/3}\bigr)\,\mathrm{Ai}\bigl(-\bigl[3n'F(y_n)\bigr]^{2/3}\bigr)-\mathrm{Ai}\bigl(-\bigl[3n'F(x_n)\bigr]^{2/3}\bigr)\,\mathrm{Ai}\bigl(-\bigl[3nF(y)\bigr]^{2/3}\bigr).$$

To deal with this expression we shall first investigate the argument

$$\bigl[3n'F(x_n)\bigr]^{2/3}=\left[3(n-1)F\left(\sqrt{\frac{n}{n-1}}x\right)\right]^{2/3}.$$

A simple integration shows that

$$F(x)=\int_x^1\sqrt{1-t^2}\,\mathrm{d}t=\frac{1}{2}\bigl(\arccos x-x\sqrt{1-x^2}\bigr)$$

and since

$$x_n=\sqrt{\frac{n}{n-1}}x=x+\frac{x}{2(n-1)}+\mathcal{O}(n^{-2})$$

we have

$$F(x_n) = F(x) + F'(x)\left(\frac{x}{2(n-1)} + \mathcal{O}(n^{-2})\right) + \mathcal{O}(F''(x)n^{-2}) = F(x) - \frac{x\sqrt{1-x^2}}{2(n-1)} + \mathcal{O}\left(\frac{1}{n^2\sqrt{1-x}}\right)$$

and hence

$$3n'F(x_n) = 3(n-1)F(x) - \frac{3}{2}x\sqrt{1-x^2} + \mathcal{O}\left(\frac{1}{n\sqrt{1-x}}\right) = 3nF(x) - \frac{3}{2}\arccos x + \mathcal{O}\left(\frac{1}{n\sqrt{1-x}}\right).$$

The argument can now finally be written as

$$-\left[3n'F(x_n)\right]^{2/3} = -\left[3nF(x)\right]^{2/3} + \frac{\arccos x}{(3nF(x))^{1/3}} + \mathcal{O}\left(\frac{1}{n^{4/3}(1-x)}\right). \tag{5.3}$$

Note in the last expression that

$$\frac{\arccos x}{(3nF(x))^{1/3}} \sim n^{-1/3}.$$

It is now possible to expand the difference in a Taylor series around the point $-[3nF(x)]^{2/3}$ and the result is

$$\begin{aligned}&\frac{a_2}{n^{1/3}}\left[\mathrm{Ai}\left(-\left[3nF(x)\right]^{2/3}\right)\mathrm{Ai}'\left(-\left[3nF(y)\right]^{2/3}\right) - \mathrm{Ai}'\left(-\left[3nF(x)\right]^{2/3}\right)\mathrm{Ai}\left(-\left[3nF(y)\right]^{2/3}\right)\right]\\ &\quad + \mathcal{O}\left(\frac{1}{n^{4/3}(1-x)}\right) + \mathcal{O}\left(\frac{(1-y)^{3/4}}{n^{1/3}(1-x)^{1/4}}\right)\end{aligned}$$

where a_2 is defined by

$$a_2 = \lim_{x\to 1^-}\frac{\arccos x}{(3F(x))^{1/3}}.$$

Similar computations can be done in D_2–D_4 and one then ends up with

$$\left(4n(n-1)\right)^{1/4}(D_2+D_3+D_4) = \mathcal{O}\left((1-y)^{1/2}\right).$$

Adding everything up we now have

$$\begin{aligned}D = {}&\frac{a_1a_2}{(4n(n-1))^{1/4}}\left[\mathrm{Ai}\left(-\left[3nF(x)\right]^{2/3}\right)\mathrm{Ai}'\left(-\left[3nF(y)\right]^{2/3}\right) - \mathrm{Ai}'\left(-\left[3nF(x)\right]^{2/3}\right)\mathrm{Ai}\left(-\left[3nF(y)\right]^{2/3}\right)\right]\\ &+ \mathcal{O}\left(\frac{1}{n(1-x)}\right) + \mathcal{O}\left(\frac{(1-y)^{3/4}}{(1-x)^{1/4}}\right).\end{aligned} \tag{5.4}$$

As we shall see the main contribution will come from the domain

$$\Omega_1 = \left\{(x,y);\, t \leqslant x \leqslant t + \frac{1-t}{r(n)},\, t - \frac{1-t}{r(n)} \leqslant y \leqslant t - \varepsilon\right\}.$$

Here $r(n)$ is a function tending slowly to infinity as n tends to infinity and $\varepsilon = \frac{1}{n(1-t)^{1/2}}$. The size of the expected distance between two eigenvalues at t is ε. The reason why this ε is necessary lies in the asymptotics for the Hermite polynomials. The error term given there however small will cause problems since the integral

$$\int_t^{t+\varepsilon}\int_{t-\varepsilon}^{t}\frac{1}{(x-y)^2}\,\mathrm{d}y\,\mathrm{d}x$$

is divergent.

From the asymptotics of the Airy function and it's derivative we have that in Ω_1

$$\begin{aligned}\mathrm{Ai}\big(-[3nF(x)]^{2/3}\big)\,\mathrm{Ai}'\big(-[3nF(y)]^{2/3}\big) &= \left(\frac{1}{\sqrt{\pi}}\big(nF(x)\big)^{-1/6}\sin\left[2nF(x)+\frac{\pi}{4}\right]+\mathcal{O}\big(\big(nF(x)\big)^{-7/6}\big)\right)\\ &\quad\times\left(\frac{1}{\sqrt{\pi}}\big(nF(y)\big)^{1/6}\sin\left[2nF(y)-\frac{\pi}{4}\right]+\mathcal{O}\big(\big(nF(y)\big)^{-5/6}\big)\right)\\ &= \frac{1}{\pi}\left(\frac{F(y)}{F(x)}\right)^{1/6}\sin\left[2nF(x)+\frac{\pi}{4}\right]\sin\left[2nF(y)-\frac{\pi}{4}\right]\\ &\quad+\mathcal{O}\big(\big(nF(x)\big)^{-1}\big).\end{aligned}$$

If we define r by

$$\frac{1}{r(n)}=\max\left(\sqrt{1-t},\,\frac{1}{\log[n(1-t)^{3/2}]}\right)$$

then in Ω_1 it holds that

$$\begin{aligned}&\left(\frac{F(y)}{F(x)}\right)^{1/6}=1+\mathcal{O}\big(\big(r(n)\big)^{-1}\big),\\ &\left(\frac{F(x)}{F(y)}\right)^{1/6}=1+\mathcal{O}\big(\big(r(n)\big)^{-1}\big),\\ &\mathcal{O}\left(\frac{1}{n(1-x)}\right)=\mathcal{O}\big(\big(nF(x)\big)^{-1}\big)=\mathcal{O}\big(\big(r(n)\big)^{-1}\big),\\ &\mathcal{O}\left(\frac{(1-y)^{3/4}}{(1-x)^{1/4}}\right)=\mathcal{O}\big(\big(r(n)\big)^{-1}\big).\end{aligned}$$

From this it follows that in Ω_1 D can be written as

$$\begin{aligned}\frac{(4n(n-1))^{1/4}}{a_1a_2}D &= \frac{1}{\pi}\left(\sin\left[2nF(x)+\frac{\pi}{4}\right]\sin\left[2nF(y)-\frac{\pi}{4}\right]\right.\\ &\quad\left.-\sin\left[2nF(x)-\frac{\pi}{4}\right]\sin\left[2nF(y)+\frac{\pi}{4}\right]\right)+\mathcal{O}\big(\big(r(n)\big)^{-1}\big)\\ &=\frac{1}{\pi}\sin\big[2n\big(F(x)-F(y)\big)\big]+\mathcal{O}\big(\big(r(n)\big)^{-1}\big).\end{aligned}$$

The nominator in the integral of interest is

$$\begin{aligned}\frac{n}{2\sqrt{n(n-1)}}D^2 &= \frac{(a_1a_2)^2}{4\pi^2}\sin^2\big[2n\big(F(x)-F(y)\big)\big]+\mathcal{O}\big(\big(r(n)\big)^{-1}\big)\\ &=\frac{1}{2\pi^2}\big(1-\cos\big[4n\big(F(x)-F(y)\big)\big]\big)+\mathcal{O}\big(\big(r(n)\big)^{-1}\big).\end{aligned}$$

It has here been used that $a_1a_2=2$. A simple integration gives

$$\iint\limits_{\Omega_1'}\frac{1}{(x-y)^2}\,\mathrm{d}x\,\mathrm{d}y=\log\big[n(1-t)^{3/2}\big]+\mathcal{O}\big(\log r(n)\big).$$

The integral

$$I=\iint\limits_{\Omega_1}\frac{\cos[4n(F(x)-F(y))]}{(x-y)^2}\,\mathrm{d}x\,\mathrm{d}y$$

is $\mathcal{O}(1)$: by doing a partial integration I can be split into two integrals:

$$I = \int_{t}^{t+\frac{1-t}{r(n)}} \left(\left[\frac{\sin[4n(F(x)-F(y))]}{-4nF'(y)(x-y)^2} \right]_{t-\frac{1-t}{r(n)}}^{t-\varepsilon} + \int_{t-\frac{1-t}{r(n)}}^{t-\varepsilon} \sin\bigl[4n\bigl(F(x)-F(y)\bigr)\bigr] \left(\frac{1}{4nF'(y)(x-y)^2} \right)'_y \mathrm{d}y \right) \mathrm{d}x$$

$$= I_1 + I_2,$$

$$|I_1| \leqslant 2 \int_{t}^{t+\frac{1-t}{r(n)}} \frac{1}{4n\sqrt{1-t}(x-(t-\varepsilon))^2} \,\mathrm{d}x = \frac{\varepsilon}{2} \left[\frac{-1}{x-t+\varepsilon} \right]_t^{t+\frac{1-t}{r(n)}} \leqslant \frac{\varepsilon}{2} \cdot \frac{2}{\varepsilon} = 1.$$

Since

$$\left(\left[F'(y)(x-y)^2\right]^{-1}\right)'_y = -\frac{y}{(1-y^2)^{3/2}(x-y)^2} - \frac{2}{\sqrt{1-y^2}(x-y)^3}$$

we get

$$|I_2| \leqslant C\left(\iint_{\Omega_1} \frac{1}{n(1-y)^{3/2}(x-y)^2} \,\mathrm{d}x\,\mathrm{d}y + \iint_{\Omega_1} \frac{1}{n\sqrt{1-y}(x-y)^3} \,\mathrm{d}x\,\mathrm{d}y \right).$$

The first part is small:

$$\iint_{\Omega_1} \frac{1}{n(1-y)^{3/2}(x-y)^2} \,\mathrm{d}x\,\mathrm{d}y \leqslant \frac{1}{n(1-t)^{3/2}} \iint_{\Omega_1} \frac{1}{(x-y)^2} \,\mathrm{d}x\,\mathrm{d}y = \mathcal{O}\left(\frac{\log[n(1-t)^{3/2}]}{n(1-t)^{3/2}} \right).$$

The second part is also easily estimated:

$$\iint_{\Omega_1} \frac{1}{n\sqrt{1-y}(x-y)^3} \,\mathrm{d}x\,\mathrm{d}y \leqslant \varepsilon \iint_{\Omega_1} \frac{1}{(x-y)^3} \,\mathrm{d}x\,\mathrm{d}y = \mathcal{O}(1).$$

This concludes the calculations in Ω_1.

The calculations made above can also be applied to the small slice

$$\left\{ (x,y);\, t+\varepsilon \leqslant x \leqslant t + \frac{1-t}{r(n)},\, t-\varepsilon \leqslant y \leqslant t \right\}$$

and the result is $\mathcal{O}(\log[r(n)])$.

The corner

$$\Omega_0 = \left\{ (x,y);\, t \leqslant x \leqslant t+\varepsilon,\, t-\varepsilon \leqslant y \leqslant t \right\}$$

requires a special technique. In this domain a different representation of K_n will be used, namely

$$K_n(x,y) = \sum_{i=0}^{n-1} p_i(x) p_i(y) \exp\left(-\frac{1}{2}[x^2+y^2] \right).$$

By use of the Cauchy–Schwartz inequality we have

$$K_n^2(x,y) \leqslant K_n(x,x) K_n(y,y).$$

Having separated the variables one can now use the calculations of the expected value giving

$$\int_{t-\varepsilon'}^{t} \int_{t}^{t+\varepsilon'} \left(\sqrt{2n} K_n(\sqrt{2n}x, \sqrt{2n}y) \right)^2 \mathrm{d}x\,\mathrm{d}y = \mathcal{O}(1).$$

Note that

$$\int_t^{t+\varepsilon'} K_n(\sqrt{2n}x, \sqrt{2n}x)\,\mathrm{d}x = g(t) - g(t+\varepsilon')$$

where, as usual, $g(t)$ is the expected value.

Now we shall look at the other part still left of Ω. This domain can conveniently be written as $\Omega_2 \cup \Omega_3$ where

$$\Omega_2 = \left\{(x,y); t \leqslant x \leqslant 1 - Cn^{-1}, 1-\delta \leqslant y \leqslant t - \frac{1-t}{r(n)}\right\}$$

and

$$\Omega_3 = \left\{(x,y); t + \frac{1-t}{r(n)} \leqslant x \leqslant 1 - Cn^{-1}, t - \frac{1-t}{r(n)} \leqslant y \leqslant t\right\}.$$

When looking at the expression for D in (5.4) above it is clear that every term is smaller than

$$n^{-1/2}\,\mathrm{Ai}\big(-\big[3nF(x)\big]^{2/3}\big)\,\mathrm{Ai}'\big(-\big[3nF(y)\big]^{2/3}\big) = \mathcal{O}\left(n^{-1/2}\left(\frac{1-y}{1-x}\right)^{1/4}\right).$$

This means that it is sufficient to calculate the integrals

$$\iint_{\Omega_i} \frac{\sqrt{1-y}}{\sqrt{1-x}(x-y)^2}\,\mathrm{d}x\,\mathrm{d}y, \quad i = 2, 3.$$

The calculations are straightforward so some details will be skipped. When first integrating with respect to the x-variable one gets

$$\begin{aligned}\int_{L_1}^{H_1} \frac{\sqrt{1-y}}{\sqrt{1-x}(x-y)^2}\,\mathrm{d}x &= \frac{1}{2(1-y)}\log\left[\frac{(\sqrt{1-y}+\sqrt{1-L_1})(\sqrt{1-y}-\sqrt{1-H_1})}{(\sqrt{1-y}+\sqrt{1-H_1})(\sqrt{1-y}-\sqrt{1-L_1})}\right] \\ &\quad + \frac{1}{\sqrt{1-y}}\left(\frac{1}{\sqrt{1-y}-\sqrt{1-L_1}} - \frac{1}{\sqrt{1-y}-\sqrt{1-H_1}}\right. \\ &\quad \left. + \frac{1}{\sqrt{1-y}+\sqrt{1-H_1}} - \frac{1}{\sqrt{1-y}+\sqrt{1-L_1}}\right).\end{aligned}$$

Ω_2: Letting $H_1 = 1$ instead of $1 - Cn^{-1}$ we get nicer expressions. This is allowed since the domain of integration becomes larger. The task is to get an upper bound for the integrals

$$A = \int_{L_2}^{H_2} \frac{1}{2(1-y)}\log\left[\frac{\sqrt{1-y}+\sqrt{1-L_1}}{\sqrt{1-y}-\sqrt{1-L_1}}\right]\mathrm{d}y = \int_{\sqrt{1-H_2}}^{\sqrt{1-L_2}} \frac{1}{z}\log\left[\frac{z+\sqrt{1-L_1}}{z-\sqrt{1-L_1}}\right]\mathrm{d}z$$

and

$$\begin{aligned}B &= \int_{L_2}^{H_2} \frac{1}{\sqrt{1-y}}\left(\frac{1}{\sqrt{1-y}-\sqrt{1-L_1}} - \frac{1}{\sqrt{1-y}+\sqrt{1-L_1}}\right)\mathrm{d}y \\ &= 2\int_{\sqrt{1-H_2}}^{\sqrt{1-L_2}} \left(\frac{1}{z-\sqrt{1-L_1}} - \frac{1}{z+\sqrt{1-L_1}}\right)\mathrm{d}z,\end{aligned}$$

where

$$L_2 = 1 - \delta, \quad H_2 = t - \frac{1-t}{r(n)} \quad \text{and} \quad L_1 = t.$$

When manipulating the integrand in A one gets

$$\frac{1}{z}\log\left[1 + 2\frac{\sqrt{1-L_1}}{z-\sqrt{1-L_1}}\right] = \frac{1}{z}\mathcal{O}\left(\frac{\sqrt{1-L_1}}{z-\sqrt{1-L_1}}\right).$$

Some algebra shows that

$$\frac{\sqrt{1-L_1}}{z(z-\sqrt{1-L_1})} = \frac{1}{z-\sqrt{1-L_1}} - \frac{1}{z}$$

which can easily be integrated:

$$A \leqslant C\left[\log\left[\frac{z-\sqrt{1-L_1}}{z}\right]\right]_{\sqrt{1-H_2}}^{\sqrt{1-L_2}} = \mathcal{O}\big(\log r(n)\big).$$

The integral B is even easier and one gets

$$B = 2\left[\log\left[\frac{z-\sqrt{1-L_1}}{z+\sqrt{1-L_1}}\right]\right]_{\sqrt{1-H_2}}^{\sqrt{1-L_2}} = \mathcal{O}\big(\log r(n)\big).$$

Ω_3: The same procedure as in Ω_2 gives that the contribution to the variance from this domain is $o(1)$.

We shall now consider the thin strip

$$\Omega_4 = \{x, y; 1 - Cn^{-1} \leqslant x \leqslant 1 + Cn^{-1}, 1 - \delta \leqslant y \leqslant t\}.$$

The asymptotics here are similar to those in Ω and hence many of the calculations already done can be applied here as well. As before D can be split up in $D_1 - D_4$ which can all be treated similarly. Therefore we only look at D_1 here. We have that

$$\begin{aligned}\big(4n(n-1)\big)^{1/4}D_1 = a_1 n^{1/3}\big[&\mathrm{Ai}\big(\mp\big[3nF(x)\big]^{2/3}\big)\,\mathrm{Ai}\big(-\big[3n'F(y_n)\big]^{2/3}\big)\\ &- \mathrm{Ai}\big(\mp\big[3n'F(x_n)\big]^{2/3}\big)\,\mathrm{Ai}\big(-\big[3nF(y)\big]^{2/3}\big)\big] + \mathcal{O}\left(\frac{(1-y)^{3/4}}{(1-x)^{1/4}}\right),\end{aligned}$$

where $\mp[3n'F(x_n)]^{2/3}$ means minus when $x_n < 1$ and plus otherwise (the equivalent for $\mp[3nF(x)]^{2/3}$). This follows from calculations done above and the asymptotics for the Hermite Polynomials when $x > 1$. In Ω_4 we have

$$\begin{aligned}&\mathrm{Ai}\big(\mp\big[3nF(x)\big]^{2/3}\big) = \mathrm{Ai}(0) + \mathcal{O}(n^{-1/3}),\\ &\mathrm{Ai}\big(\mp\big[3n'F(x_n)\big]^{2/3}\big) = \mathrm{Ai}(0) + \mathcal{O}(n^{-1/3})\end{aligned}$$

and by using Eq. (5.3) (for the y-variable) one gets

$$\big(4n(n-1)\big)^{1/4}D_1 = \mathcal{O}\left(\frac{(1-y)^{1/4}}{|1-x|^{1/4}}\right).$$

The error term here has actually already been dealt with in the estimations of the contribution coming from Ω_2.

Rather than to repeat a lot of calculations we now just give ideas of how to treat what's left of $[t,\infty) \times (-\infty, t]$.

In the domain

$$\{x, y; 1 + Cn^{-1} \leqslant x \leqslant 1 + \delta, 1 - \delta \leqslant y \leqslant t\}$$

one can perform much the same calculations as in Ω and the contribution is $\mathcal{O}(1)$. In

$$\{x, y; t \leqslant x \leqslant 1+\delta, -1-\delta \leqslant y \leqslant 1-\delta\}$$

one can use the fact that $x - y \geqslant \delta$ to show that the contribution from this domain is $\mathcal{O}(1)$. If $x \geqslant 1+\delta$ or $y \leqslant -1-\delta$ t one easily gets from the asymptotics for the Hermite Polynomials that $K_n(\sqrt{2n}x, \sqrt{2n}y)$ is exponentially small in n and exponentially decaying in x^2 (or y^2). Thus the contribution from this domain is $\mathrm{o}(1)$. $\square$

Acknowledgements

I would like to thank my advisor Kurt Johansson for presenting this problem to me and for many useful discussions. I also thank Kenneth McLaughlin for drawing my attention to his and N.M. Ercolani's work on the asymptotics of the Partition Function for random matrices.

Appendix A. Some integrals

The following equalities hold:

$$\int_x^\infty \mathrm{Ai}^2(y)\,\mathrm{d}y = \mathrm{Ai}'^2(x) - x\,\mathrm{Ai}^2(x),$$

$$\int_x^\infty y\,\mathrm{Ai}^2(y)\,\mathrm{d}y = \frac{1}{3}\big(x\,\mathrm{Ai}'^2(x) - x^2\,\mathrm{Ai}^2(x) - \mathrm{Ai}(x)\,\mathrm{Ai}'(x)\big),$$

$$\int_x^\infty \mathrm{Ai}'^2(y)\,\mathrm{d}y = \frac{1}{3}\big(x^2\,\mathrm{Ai}^2(x) - x\,\mathrm{Ai}'^2(x) - 2\,\mathrm{Ai}(x)\,\mathrm{Ai}'(x)\big),$$

$$\int_x^\infty y^2\,\mathrm{Ai}^2(y)\,\mathrm{d}y = \frac{1}{5}\big(x^2\,\mathrm{Ai}'^2(x) - x^3\,\mathrm{Ai}^2(x) - 2x\,\mathrm{Ai}(x)\,\mathrm{Ai}'(x) + \mathrm{Ai}^2(x)\big),$$

$$\int_x^\infty y\,\mathrm{Ai}'^2(y)\,\mathrm{d}y = \frac{1}{5}\left(x^3\,\mathrm{Ai}^2(x) - x^2\,\mathrm{Ai}'^2(x) - 3x\,\mathrm{Ai}(x)\,\mathrm{Ai}'(x) + \frac{3}{2}\,\mathrm{Ai}^2(x)\right).$$

The first integral is obtained from one partial integration while remembering that

$$\mathrm{Ai}''(x) = x\,\mathrm{Ai}(x).$$

The integrals 3–5 can be obtained rather easily from the second which can be treated as follows:

Set

$$u_\alpha(x) = \mathrm{Ai}(\alpha x), \quad \alpha > 0.$$

The relationship

$$[u'_\alpha u_\beta - u_\alpha u'_\beta]' = u''_\alpha u_\beta - u_\alpha u''_\beta = x(\alpha^3 - \beta^3)u_\alpha u_\beta$$

holds since

$$u''_\alpha(x) = \alpha^2\,\mathrm{Ai}''(\alpha x) = \alpha^3 x\,\mathrm{Ai}(\alpha x) = \alpha^3 x u_\alpha(x).$$

Hence

$$\int_a^\infty x u_\alpha(x) u_\beta(x)\,\mathrm{d}x = \frac{1}{\alpha^3-\beta^3}[u'_\alpha u_\beta - u_\alpha u'_\beta]_a^\infty = \frac{u_\alpha(a)u'_\beta(a) - u'_\alpha(a)u_\beta(a)}{\alpha^3-\beta^3}.$$

The idea now is to let α, β tend to one. Set $\alpha = 1+h$ and $\beta = 1-h$ where $h > 0$ and small. The left hand side tends to

$$\int_a^\infty x\,\mathrm{Ai}^2(x)\,\mathrm{d}x$$

as $h \to 0^+$. Standard calculations show that at the same time the right hand side tends to

$$\frac{1}{3}\bigl(-a^2\,\mathrm{Ai}^2(a) - \mathrm{Ai}(a)\,\mathrm{Ai}'(a) + a\,\mathrm{Ai}'^2(a)\bigr).$$

Appendix B. Variance calculations

Let $I_1, \ldots, I_m$ be a set of disjoint intervals and $\#I_i$ be the number of eigenvalues of the GUE_n in the interval I_i. We shall give a formula for $\mathrm{Var}(\alpha_1\#I_1 + \cdots + \alpha_m\#I_m)$. We have

$$\#I_i = \sum_{k=1}^n \chi_{I_i}(x_k), \quad 1 \leqslant i \leqslant n,$$

where χ_B is the characteristic function for the set B and $\{x_k\}_1^n$ are the (not ordered) eigenvalues. The expected value is easy to compute:

$$E[\#I_i] = \int_{I_i} \rho_{n,1}(x)\,\mathrm{d}x = \int_{I_i} K_n(x,x)\,\mathrm{d}x.$$

The correlation functions $\rho_{n,k}$ were defined in the introduction. We also need to calculate $E[\#I_i^2]$:

$$\begin{aligned}
E[\#I_i^2] &= E\left[\sum_{j,k=1}^n \chi_{I_i}(x_k)\chi_{I_i}(x_j)\right] = \sum_{k=1}^n E\bigl[\chi_{I_i}(x_k)\bigr] + \sum_{j\neq k} E\bigl[\chi_{I_i}(x_k)\chi_{I_i}(x_j)\bigr] \\
&= \int_{I_i} K_n(x,x)\,\mathrm{d}x + \iint_{I_i\times I_i} \rho_{n,2}(x,y)\,\mathrm{d}x\,\mathrm{d}y \\
&= \int_{I_i} K_n(x,x)\,\mathrm{d}x + \left(\int_{I_i} K_n(x,x)\,\mathrm{d}x\right)^2 - \iint_{I_i\times I_i} K_n^2(x,y)\,\mathrm{d}x\,\mathrm{d}y.
\end{aligned}$$

We now have that

$$\mathrm{Var}(\#I_i) = \int_{I_i} K_n(x,x)\,\mathrm{d}x - \iint_{I_i\times I_i} K_n^2(x,y)\,\mathrm{d}x\,\mathrm{d}y.$$

To get a more convenient formula to work with one can now use the identities [7] $K_n(x,y) = K_n(y,x)$ and

$$\int_{\mathbb{R}} K_n(x,y)K_n(y,z)\,\mathrm{d}y = K(x,z)$$

to get

$$\mathrm{Var}(\#I_i) = \int_{I_i} \left(\int_{\mathbb{R}} K_n^2(x, y)\,\mathrm{d}y \right) \mathrm{d}x - \iint_{I_i \times I_i} K_n^2(x, y)\,\mathrm{d}x\,\mathrm{d}y = \iint_{I_i \times I_i^c} K_n^2(x, y)\,\mathrm{d}x\,\mathrm{d}y.$$

In more generality one gets

$$E[\alpha_1 \# I_1 + \cdots + \alpha_m \# I_m] = \sum_{i=1}^m \alpha_i \int_{I_i} K_n(x, x)\,\mathrm{d}x$$

and

$$\begin{aligned}(\alpha_1 \# I_1 + \cdots + \alpha_m \# I_m)^2 &= \sum_{i=1}^m \alpha_i^2 \left(\sum_{k=1}^n \chi_{I_i}(x_k) \right)^2 + \sum_{i \neq j}^m \alpha_i \alpha_j \left(\sum_{k=1}^n \chi_{I_i}(x_k) \right) \left(\sum_{k=1}^n \chi_{I_j}(x_k) \right) \\ &= S_1 + S_2.\end{aligned}$$

From the calculations above we know that

$$E[S_1] = \sum_{i=1}^m \alpha_i^2 \left(\int_{I_i} K_n(x, x)\,\mathrm{d}x + \left(\int_{I_i} K_n(x, x)\,\mathrm{d}x \right)^2 - \iint_{I_i \times I_i} K_n^2(x, y)\,\mathrm{d}x\,\mathrm{d}y \right)$$

so it remains to calculate $E[S_2]$. We have

$$\left(\sum_{k=1}^n \chi_{I_i}(x_k) \right) \left(\sum_{k=1}^n \chi_{I_j}(x_k) \right) = \sum_{k \neq l}^n \chi_{I_i}(x_k) \chi_{I_i}(x_l)$$

and hence

$$E[S_2] = \sum_{i \neq j}^m \alpha_i \alpha_j \iint_{I_i \times I_i} \rho_{n,2}(x, y)\,\mathrm{d}x\,\mathrm{d}y = \sum_{i \neq j}^m \alpha_i \alpha_j \left(\int_{I_i} K_n(x, x)\,\mathrm{d}x \int_{I_j} K_n(x, x)\,\mathrm{d}x - \iint_{I_i \times I_j} K_n^2(x, y)\,\mathrm{d}x\,\mathrm{d}y \right).$$

Since

$$(E[\alpha_1 \# I_1 + \cdots + \alpha_m \# I_m])^2 = \sum_{i=1}^m \alpha_i^2 \left(\int_{I_i} K_n(x, x)\,\mathrm{d}x \right)^2 + \sum_{i \neq j}^m \alpha_i \alpha_j \int_{I_i} K_n(x, x)\,\mathrm{d}x \int_{I_j} K_n(x, x)\,\mathrm{d}x$$

we finally get (with manipulations as before)

$$\mathrm{Var}(\alpha_1 \# I_1 + \cdots + \alpha_m \# I_m) = \sum_{i=1}^m \alpha_i^2 \iint_{I_i \times I_i^c} K_n^2(x, y)\,\mathrm{d}x\,\mathrm{d}y - \sum_{i \neq j}^m \alpha_i \alpha_j \iint_{I_i \times I_j} K_n^2(x, y)\,\mathrm{d}x\,\mathrm{d}y.$$

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