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Coboundaries in L_0^∞

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Abstract

Let T be an ergodic automorphism of a probability space, f a bounded measurable function, $S_n(f) = \sum_{k=0}^{n-1} f \circ T^k$. It is shown that the property that the probabilities $\mu(|S_n(f)| > n)$ are of order n^{-p} roughly corresponds to the existence of an approximation in L^∞ of f by functions (coboundaries) $g - g \circ T$, $g \in L^p$. Similarly, the probabilities $\mu(|S_n(f)| > n)$ are exponentially small iff f can be approximated by coboundaries $g - g \circ T$ where g have finite exponential moments.

Résumé

Soit T un automorphisme ergodique d'un espace probabilisé, f une fonction bornée mesurable et $S_n(f) = \sum_{k=0}^{n-1} f \circ T^k$. Une correspondance est établie entre l'existence de l'estimation des probabilités $\mu(|S_n(f)| > n)$ d'ordre n^{-p} et l'existence de l'approximation dans L^∞ de la fonction f par des cobords $g - g \circ T$ où g est "presque" dans L^p. De manière similaire, les probabilités $\mu(|S_n(f)| > n)$ sont d'ordre e^{-cn}, pour un certain $c > 0$, $n = 1, 2 \ldots$, si et seulement si f admet une approximation dans L^∞ par des cobords $g - g \circ T$ avec g ayant des moments exponentiels.

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1. Introduction and results

Let $(\Omega, \mathcal{A}, \mu)$ be a probability space and $T : \Omega \to \Omega$ a bijective, bimeasurable and measure preserving mapping. Throughout the paper we shall suppose that T is ergodic and aperiodic. For a measurable function f on Ω we denote

$$S_n(f) = \sum_{i=0}^{n-1} f \circ T^i.$$

L_0^p denotes the space of all $f \in L^p$ with zero mean, $1 \leqslant p \leqslant \infty$. If $f = g - g \circ T$ with g a measurable function, then we say that f is a coboundary. The function g is then called the cobounding function. We shall study the approximation of functions from L_0^∞ by coboundaries. The main results are presented in Theorems 1, 2, and 3; they show a relationship between the moments of the cobounding function g of the approximating coboundary and probabilities of large deviations of the stochastic process $(f \circ T^i)$.

It is well known that for $1 \leqslant p < \infty$, the coboundaries with a cobounding function in L^p are a dense subset of L_0^p (because L^∞ is a dense subset of L^p, the coboundaries $g - g \circ T$ with g bounded thus form a dense subset of all L_0^p, $1 \leqslant p < \infty$). As an immediate consequence of the density of the sets of coboundaries in L^p spaces we get the von Neumann's ergodic theorem in these spaces:

$$\left\| \frac{1}{n} S_n(f) - Ef \right\|_p \to 0$$

for any $f \in L^p$, $1 \leqslant p < \infty$ (cf. e.g. [10, p. 21]).

For $p = \infty$ the things are more complicated:

Theorem A. *Let $f \in L_0^1$ and $\varepsilon > 0$. Then there exists a measurable function g such that*

$$\left\| f - (g - g \circ T) \right\|_\infty < \varepsilon.$$

The theorem follows from [7, Corollary 3] ([8] in English). For completeness, we shall show a proof the idea of which is due to Michael Keane.

There exist, however, bounded functions which cannot be (in L^∞) approximated by any coboundary with an integrable cobounding function g:

Theorem B. *Let φ be a positive real function, $\lim_{t \to -\infty} \varphi(t) = \lim_{t \to \infty} \varphi(t) = \infty$. Then there exists a function $f \in L_0^\infty$ with $\|f\|_\infty = 1$ such that for each measurable function g with $\int \varphi \circ g \, d\mu < \infty$,*

$$\left\| f - (g - g \circ T) \right\|_\infty \geqslant 1/2.$$

In particular, for any $p > 0$ there exists f with $\|f\|_\infty = 1$ s.t. $\|f - (g - g \circ T)\|_\infty \geqslant 1/2$ for each $g \in L^p$.

This result is not completely new either; a version of it can be found in the work of A. Katok [6].

The main aim of this paper is to show that the set of $f \in L_0^\infty$ which can be approximated by coboundaries whose cobounding functions have finite moments can be characterized by the probabilities of large deviations. The first two theorems show that the integrability of $|f|^p$ is "almost equivalent" to the property that the probabilities $\mu(|S_k(f)| > xk)$ are of order $1/k^p$. The third proposition extends the result to functions with exponential moments.

Theorem 1. *Let $f \in L_0^\infty$, $p \geqslant 1$. If for every $\delta > 0$ there exists a $g \in L^p$ with $\|f - (g - g \circ T)\|_\infty < \delta$ then*

(i) *for each* $\varepsilon > 0$

$$\sum_{k=1}^{\infty} k^{p-1}\mu\bigl(\bigl|S_k(f)\bigr| > \varepsilon k\bigr) < \infty,$$

(ii) *for each* $\varepsilon > 0$ *there exists a* $c_\varepsilon > 0$ *such that*

$$\mu\bigl(\bigl|S_k(f)\bigr| > \varepsilon k\bigr) < c_\varepsilon \cdot \frac{1}{k^p} \quad \text{for all } k = 1, 2, \ldots.$$

Theorem 2. *Let* $f \in L_0^\infty$, $p \geqslant 1$. *If for every* $x > 0$ *there exists a* $0 < c_x < \infty$ *such that for all* k,

$$\mu\bigl(\bigl|S_k(f)\bigr| > xk\bigr) < c_x k^{-p}$$

then for all $\varepsilon > 0$ *and every measurable function* $v : \mathbb{R}^+ \to \mathbb{R}^+$ *such that*

$$\sum_{j=1}^{\infty} \frac{1}{j v(aj)} < \infty \quad \text{for all } a > 0 \quad \text{and} \quad v(x), \quad \frac{x^p}{v(x)} \quad \text{are increasing}$$

there exists a measurable function g *such that*

$$E\left(\frac{|g|^p}{v(|g|)}\right) < \infty \quad \text{and} \quad \bigl\|f - (g - g \circ T)\bigr\|_\infty < \varepsilon.$$

In particular, for any $\delta > 0$ *we can find* $g \in L^{p-\delta}$.

Theorem 3. *Let* $f \in L_0^\infty$. *It is equivalent*

(i) *For every* $\varepsilon > 0$ *there exists a* $c_\varepsilon > 0$ *and* $n_\varepsilon \in \mathbb{N}$ *such that* $\mu(|S_n(f)| > \varepsilon n) < \mathrm{e}^{-c_\varepsilon n}$ *for all* $n \geqslant n_\varepsilon$.
(ii) *For every* $\varepsilon > 0$ *there exists a measurable function* g *and* $c > 0$ *such that* $E\mathrm{e}^{c|g|} < \infty$ *and*

$$\bigl\|f - (g - g \circ T)\bigr\|_\infty < \varepsilon.$$

For sequences of independent and weakly dependent random variables X_i, the probabilities of $\{\sum_{i=1}^n X_i > xn\}$ have been analyzed in detail before. For example, by Azuma's inequality [4] an exponential bound like in (i) of Theorem 3 exists whenever (X_i) is a uniformly bounded sequence of martingale differences. Therefore, if $f \in L_0^\infty$ and $(f \circ T^i)$ is a martingale difference sequence (or even a sequence of mutually independent random variables) then f can be approximated by coboundaries whose transfer functions have finite exponential moments.

Generic properties of sets of functions f for which the probabilities $\mu(S_n(f) > xn)$ have a particular asymptotic behaviour are studied in the paper [9].

2. Proofs

Proof of Theorem A. By the Birkhoff (almost sure) Ergodic Theorem for any $\eta > 0$ there exists $A \in \mathcal{A}$ with $\mu(A) > 0$ and $n_0 \in \mathbb{N}$ such that for each $\omega \in A$ we have

$$\bigl|(1/n)S_n(f)(\omega)\bigr| < \eta \quad \text{for all } n \geqslant n_0.$$

Let A^k be the set of points whose return time to A is k: $A^k = \{\omega \in A\colon\ T^k\omega \in A,\ T^i\omega \notin A \text{ for } 0 < i < k\}$, $k = 1, 2, \ldots$, let $A = \bigcup_{k=1}^\infty A^k$. Because T is ergodic, the set $\bigcup_{k=1}^\infty \bigcup_{i=0}^{k-1} T^i A^k$ has measure 1; without loss of generality we can suppose that it equals Ω.

It is known that we can also suppose that $A^k = \emptyset$ for $1 \leqslant k \leqslant n_0 - 1$. This is the case if $A, TA, \ldots, T^{n_0-1}A$ is a Rokhlin tower. If not, we can recursively find an adequate subset of A of strictly positive measure because for any A of positive measure and $k \geqslant 0$, $B \subset A$, $0 < \mu(B) < \mu(A)$, we by ergodicity have $\mu(B \setminus T^{k+1}B) > 0$, hence $B, \ldots, T^{k+1}B$ are disjoint.

For each $\omega \in \Omega$ there thus exist unique $k \geqslant n_0$ and $0 \leqslant i \leqslant k-1$ such that $\omega \in T^i A^k$.
Define

$$\xi(\omega) = T^{-i}\omega,$$
$$g(\omega) = f(\omega) - \frac{1}{k}\sum_{j=0}^{k-1} f\big(T^j \xi(\omega)\big),$$
$$h(\omega) = \sum_{j=0}^{i} g(T^{j-i}\omega) = \sum_{j=0}^{i} g\big(T^j \xi(\omega)\big).$$

For $\omega \in T^{k-1}A^k$, $h(\omega) = 0$. We thus have

$$g = h - h \circ T^{-1}.$$

From the definition of the set A it follows that

$$\|f - g\|_\infty < \eta. \qquad \square$$

In the proof of Theorem B we shall use the result by A. Alpern (cf. [1–3,5]).

Theorem C (A. Alpern). *Let $1 \leqslant n_k$, $k = 1, 2, \ldots,$ be positive integers whose least common divisor is 1, p_k positive reals, $\sum_{k=1}^\infty p_k n_k = 1$. Then there exist measurable sets F_k, such that $\mu(F_k) = p_k$, $T^i F_k$, $0 \leqslant i \leqslant n_k - 1$, $k = 1, 2, \ldots,$ are pairwise disjoint and $\mu(\bigcup_{k=1}^\infty \bigcup_{i=0}^{n_k-1} T^i F_k) = 1$.*

Proof of Theorem B. Without loss of generality we can suppose that φ is an even function. For $k = 1, 2, \ldots$ let r_k be a positive integer such that

$$\sum_{j=1}^{[(k-1)/2]} \varphi(j/2) \geqslant k r_k;$$

by the assumptions, $r_k \to \infty$. There thus exist (strictly) positive numbers p_k such that

$$\sum_{k=1}^\infty k p_k = 1,$$
$$\sum_{k=1}^\infty k r_k p_k = \infty.$$

By Theorem C there exist measurable sets A_k, $k = 1, 2, \ldots,$ such that $\mu(A_k) = p_k$ and $\{A_k, T^{-1}A_k, \ldots, T^{-k+1}A_k\}$, $k = 1, 2, \ldots$ are mutually disjoint Rokhlin towers. Let $A_k = B_k \cup C_k$ where $B_k \cap C_k = \emptyset$ and $\mu(B_k) = \mu(A_k)/2 = \mu(C_k)$, $k = 1, 2, \ldots.$ We define

$$f(\omega) = \begin{cases} 1 & \text{for } \omega \in \bigcup_{k=1}^\infty \bigcup_{i=0}^{k-1} T^{-i} B_k \\ -1 & \text{for } \omega \in \bigcup_{k=1}^\infty \bigcup_{i=0}^{k-1} T^{-i} C_k \\ 0 & \text{for } \omega \notin \bigcup_{k=1}^\infty \bigcup_{i=0}^{k-1} T^{-i} A_k. \end{cases}$$

Suppose that g, h are measurable functions, $\|h\|_\infty < 1/2$ and

$$f = g \circ T - g - h.$$

Then

$$g \circ T = f + g + h,$$
$$g \circ T^2 = f \circ T + g \circ T + h \circ T = f + f \circ T + h + h \circ T + g,$$
$$\dots$$
$$g \circ T^n = \sum_{i=0}^{n-1} f \circ T^i + \sum_{i=0}^{n-1} h \circ T^i + g.$$

We have

$$E\varphi \circ g \geqslant \sum_{k=1}^{\infty} \int_{A_k} \sum_{i=0}^{k-1} \varphi(g \circ T^i)\, d\mu.$$

Let us denote $\psi_j = \sum_{i=0}^{j-1}(f \circ T^i + h \circ T^i)$, $j = 1, 2, \dots$. For $\omega \in A_k$ we get $\sum_{j=0}^{k-1} \varphi(g(T^j\omega)) = \sum_{j=0}^{k-1} \varphi(\psi_j + g)$. We distinguish two possibilities:

1. The numbers $\psi_j(\omega) + g(\omega)$, $j = 0, \dots, k-1$, are all of the same sign. Then $\sum_{j=1}^{k-1} \varphi(\psi_j + g) \geqslant \sum_{j=1}^{k-1} \varphi(j/2)$.

2. The numbers $\psi_j(\omega) + g(\omega)$, $j = 1, \dots, k-1$, are not all of the same sign. Because $f(T^j\omega)$ are all 1 or all -1 while $|h| \leqslant 1/2$, the sequence $\psi_1(\omega), \dots, \psi_{k-1}(\omega)$ is monotone. Hence, there exists $1 \leqslant n \leqslant k-1$ such that $\sum_{j=1}^{k-1} \varphi(\psi_j + g) = \sum_{j=1}^{n} \varphi(\psi_j + g) + \sum_{j=n+1}^{k-1} \varphi(\psi_j + g) \geqslant \sum_{j=1}^{[(k-1)/2]} \varphi(j/2)$ where $[x]$ denotes the integer value of x.

We thus get $E\varphi(g) \geqslant \sum_{k=1}^{\infty} p_k \sum_{j=1}^{[(k-1)/2]} \varphi(j/2) \geqslant \sum_{k=1}^{\infty} k r_k p_k = \infty$. This finishes the proof. □

Proof of Theorem 1. Let $\varepsilon > 0$ be fixed. We put $0 < \delta < \varepsilon/2$, g is a function from L^p with $\|f - (g - g \circ T)\|_\infty < \delta$. Then

$$\big|S_n\big(f - (g - g \circ T)\big)\big| < \delta n < n\varepsilon/2$$

hence

$$\mu\big(\big|S_k(f)\big| > \varepsilon k\big) \leqslant \mu\big(\big|S_k(g - g \circ T)\big| > k\varepsilon/2\big) \leqslant 2\mu\big(|g| > k\varepsilon/4\big).$$

Because $g \in L^p$,

$$\sum_{l=1}^{\infty} \sum_{j=l^p}^{(l+1)^p-1} \mu\big(|g| > l+1\big) \leqslant \sum_{l=1}^{\infty} \sum_{j=l^p}^{(l+1)^p-1} \mu\big(|g| > j\big) < \infty$$

hence

$$\sum_{l=1}^{\infty} l^{p-1} \mu\big(|g| > l\big) < \infty.$$

The statement (ii) follows from

$$\mu\big(\big|S_n(g - g \circ T)\big| > \varepsilon n\big) \leqslant 2\mu\left(|g| > \frac{\varepsilon}{2} n\right) \leqslant 2 \int \frac{|g|^p}{(\frac{\varepsilon}{2} n)^p}\, d\mu = \frac{c_\varepsilon}{n^p}$$

where $c_\varepsilon = 2^{p+1} \int |g/\varepsilon|^p\, d\mu$. □

For the proof of Theorem 2 we shall need the following statement:

Proposition. *Let $f \in L_0^\infty$, $\varepsilon > 0$. Then under the assumptions of Theorem 2 there exists an integer $n_0 \geqslant 1$ and a set F of positive measure with*

$$F_k = \{\omega \in F \mid T^k\omega \in F,\ \forall 1 \leqslant i \leqslant k-1,\ T^i\omega \notin F\}, \quad k = 1, 2, \ldots,$$
$$F_\infty = \{\omega \in F \mid \forall 1 \leqslant i,\ T^i\omega \notin F\},$$

such that

(a) *for $\omega \in F_k$, $1 \leqslant k < \infty$,*

$$\left|S_k(f)(\omega)\right| \leqslant k\varepsilon \quad \text{and} \quad \left|S_j(f)(\omega)\right| > j\varepsilon \quad \text{for all } 1 \leqslant j \leqslant k - n_0$$

and

(b) *there exists a $0 < c < \infty$ such that for all n,*

$$\sum_{k=n}^{\infty} \mu(F_k) < cn^{-p-1}.$$

Proof. As in the proof of Theorem A we can show that there exists an integer $n_0 \geqslant 1$ and a set A of positive measure such that for all $\omega \in A$

(i) if $n \geqslant n_0$ then $|\sum_{i=0}^{n-1} f(T^{-i}\omega)| \leqslant \varepsilon n$,
(ii) if $1 \leqslant k \leqslant n_0 - 1$ then $T^{-k}\omega \notin A$.

For $\omega \in \Omega$ we define

$$\psi(\omega) = \begin{cases} \min\{n \geqslant 2 \mid |S_n(f)(\omega)| \leqslant \varepsilon n\} & \text{if } \exists n \geqslant 2,\ |S_n(f)(\omega)| \leqslant \varepsilon n, \\ \infty & \text{otherwise.} \end{cases}$$

For $\omega \in A$ we recursively define $\tau_k(\omega)$, $k = 0, 1, \ldots$ by

$$\tau_0(\omega) = 0, \quad \tau_{k+1}(\omega) = \tau_k(\omega) + \psi(T^{\tau_k(\omega)}\omega)$$

and put

$$\varphi(\omega) = \begin{cases} \min\{k \geqslant 1 \mid T^k\omega \in A\} & \text{if } \exists n \geqslant 1,\ T^n\omega \in A, \\ \infty & \text{otherwise,} \end{cases}$$
$$t(\omega) = \begin{cases} \sup\{0 \leqslant k \mid \tau_k(\omega) \leqslant \varphi(\omega)\} & \text{if } \varphi(\omega) < \infty, \\ \infty & \text{otherwise,} \end{cases}$$

Observation. *Let $\omega \in A$. If $\varphi(\omega), t(\omega) < \infty$ then*

$$\tau_{t(\omega)}(\omega) \leqslant \varphi(\omega),$$
$$\varphi(\omega) - \tau_{t(\omega)}(\omega) \leqslant n_0 - 1,$$
$$t(\omega) \geqslant 1.$$

The first inequality follows immediately from the definition of t, the second follows from (i), the third from the preceding ones and (ii).

Let $\omega \in A$. Define

$$F = \bigcup_{\omega \in A} \bigcup_{k=0}^{t(\omega)-1} T^{\tau_k(\omega)}\omega.$$

By the construction, the set F is measurable and satisfies (a).

Let us prove (b). Let $\delta > 0$. If $0 \leqslant i \leqslant \delta n$, $k \geqslant n_0 + n(1+\delta)$, $\omega \in F_k$, then by (a)

$$\left|\sum_{j=0}^{n+i-1} f(T^j\omega)\right| > (n+i)\varepsilon \geqslant n\varepsilon \quad \text{and} \quad \left|\sum_{j=0}^{i-1} f(T^j\omega)\right| \leqslant \delta n\|f\|_\infty.$$

Therefore,

$$\left|\sum_{j=0}^{n-1} f(T^jT^i\omega)\right| \geqslant \left|\sum_{j=0}^{n+i-1} f(T^j\omega)\right| - \left|\sum_{j=0}^{i-1} f(T^j\omega)\right| \geqslant n\varepsilon - n\delta\|f\|_\infty = n\big(\varepsilon - \delta\|f\|_\infty\big).$$

For n_0 defined at the beginning of the proof, for each $\delta > 0$, $\omega \in \bigcup_{k\geqslant n_0+n(1+\delta)} F_k$ and $0 \leqslant i \leqslant \delta n$, we then have $|S_n(f)(T^i\omega)| \geqslant n(\varepsilon - \delta\|f\|_\infty)$. There thus exists a constant c depending only on $\varepsilon - \delta\|f\|_\infty$,

$$[\delta n] \sum_{k\geqslant n_0+n(1+\delta)} \mu(F_k) < cn^{-p}$$

and the inequality (b) follows. □

Proof of Theorem 2. Let $1 \leqslant k < \infty$, $\omega \in F_k$, $0 \leqslant i \leqslant k-1$. We define

$$h(T^i\omega) = f(T^i\omega) - \frac{1}{k}\sum_{j=0}^{k-1} f(T^j\omega),$$

$$g(T^i\omega) = \sum_{j=0}^{i-1} h(T^j\omega).$$

On the rest of Ω we define $g = 0$. We then have

$$h = g \circ T - g$$

and

$$\big\|f - (g\circ T - g)\big\|_\infty \leqslant \varepsilon.$$

Let us denote $a = \|h\|_\infty$. Using (b) we calculate

$$E\left(\frac{|g|^p}{v(|g|)}\right) \leqslant \sum_{k=1}^\infty \left(\sum_{j=1}^k \frac{(aj)^p}{v(aj)}\right)\mu(F_k) \leqslant \sum_{j=1}^\infty \frac{(aj)^p}{v(aj)}\left(\sum_{k=j}^\infty \mu(F_k)\right) \leqslant c\sum_{j=1}^\infty \frac{(aj)^p}{v(aj)} j^{-p-1}$$
$$= ca^p \sum_{j=1}^\infty \frac{1}{jv(aj)} < \infty. \qquad \square$$

The proof of Theorem 3 is left as an exercise for the reader. For (i) ⇒ (ii) we can use the same construction as in the proof of Theorem 2, (ii) ⇒ (i) follows from $\mu(|S_n(f)| > \varepsilon) \leqslant \mu(|S_n(f - (g - g\circ T))| > \varepsilon/2) + \mu(|S_n(g - g\circ T)| > \varepsilon/2)$.

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Further reading

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