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Upper bound of a volume exponent for directed polymers in a random environment

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Abstract

We consider the model of directed polymers in a random environment introduced by Petermann: the random walk is $\mathbb{R}^d$-valued and has independent $\mathcal{N}(0, I_d)$-increments, and the random media is a stationary centered Gaussian process $(g(k,x),\ k \geqslant 1,\ x \in \mathbb{R}^d)$ with covariance matrix $\operatorname{cov}(g(i,x), g(j,y)) = \delta_{ij}\Gamma(x-y)$, where Γ is a bounded integrable function on $\mathbb{R}^d$. For this model, we establish an upper bound of the volume exponent in all dimensions d.

Résumé

On considère le modèle de polymères dirigés en environnement aléatoire introduit par Petermann : la marche aléatoire sous-jacente est à valeurs dans $\mathbb{R}^d$, ses incréments sont des variables indépendantes de loi $\mathcal{N}(0, I_d)$, et le milieu aléatoire est un processus gaussien stationnaire centré $(g(k,x),\ k \geqslant 1,\ x \in \mathbb{R}^d)$ de matrice de covariance $\operatorname{cov}(g(i,x), g(j,y)) = \delta_{ij}\Gamma(x-y)$, où Γ est une fonction bornée intégrable sur $\mathbb{R}^d$. Pour ce modèle, nous établissons une majoration de l'exposant de volume, pour toute dimension *d*.

MSC: 60K37; 60G42; 82D30

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1. Introduction

The model of directed polymers in a random environment was introduced by Imbrie and Spencer [7]. We focus here on a particular model studied by Petermann [8] in his thesis: let $(S_n)_{n \geqslant 0}$ be a random walk in $\mathbb{R}^d$ starting from the origin, with independent $\mathcal{N}(0, I_d)$-increments, defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and let $g = (g(k,x),\ k \geqslant 1,\ x \in \mathbb{R}^d)$ be a stationary centered Gaussian process with covariance matrix

$$\operatorname{cov}\big(g(i,x), g(j,y)\big) = \delta_{ij}\Gamma(x-y),$$

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where Γ is a bounded integrable function on $\mathbb{R}^d$. We suppose that this random media g is defined on a probability space $(\Omega^g, \mathcal{G}, P)$, where $(\mathcal{G}_n)_{n\geqslant 0}$ is the natural filtration:

$$\mathcal{G}_n = \sigma\bigl(g(k,x),\ 1\leqslant k\leqslant n,\ x\in\mathbb{R}^d\bigr)$$

for $n\geqslant 1$ ($\mathcal{G}_0$ being the trivial σ-algebra). We denote by $\mathbb{E}$ (respectively E) the expectation with respect to $\mathbb{P}$ (respectively P). We define the Gibbs measure $\langle . \rangle^{(n)}$ by:

$$\langle f\rangle^{(n)} = \frac{1}{Z_n}\mathbb{E}\bigl(f(S_1,\dots,S_n)\mathrm{e}^{\beta\sum_{k=1}^n g(k,S_k)}\bigr)$$

for any bounded function f on $(\mathbb{R}^d)^n$, where $\beta>0$ is a fixed parameter and Z_n is the partition function:

$$Z_n = \mathbb{E}\bigl(\mathrm{e}^{\beta\sum_{k=1}^n g(k,S_k)}\bigr).$$

Following Piza [9] we define the volume exponent

$$\xi = \inf\bigl\{\alpha>0\colon\ \langle \mathbb{1}_{\{\max_{k\leqslant n}|S_k|\leqslant n^{\alpha}\}}\rangle^{(n)} \underset{n\to\infty}{\to} 1 \text{ in P-probability}\bigr\}.$$

Here and in the sequel, $|x| = \max_{1\leqslant i\leqslant d}|x_i|$ for any $x=(x_1,\dots,x_d)\in\mathbb{R}^d$. Petermann obtained a result of superdiffusivity in dimension one, in the particular case where $\Gamma(x) = \frac{1}{2\lambda}\mathrm{e}^{-\lambda|x|}$ for some $\lambda>0$: he proved that $\xi\geqslant 3/5$ for all $\beta>0$ (for another result of superdiffusivity, see [6]).

Our main result gives on the contrary an upper bound for the volume exponent, in all dimensions:

$$\forall d\geqslant 1,\ \forall\beta>0\quad \xi\leqslant\frac{3}{4}. \tag{1}$$

This paper is organized as follows:

- In Section 2, we first extend exponential inequalities concerning independent Gaussian variables, proved by Carmona and Hu [1], to the case of a stationary Gaussian process. Then, following Comets, Shiga and Yoshida [2], we combine these inequalities with martingale methods and obtain a concentration inequality.
- In Section 3, we obtain an upper bound for ξ when we consider only the value of the walk S at time n, and not the maximal one before n. In fact we prove a stronger result, namely a large deviation principle for $(\langle\mathbb{1}_{S_n/n^{\alpha}\in .}\rangle^{(n)},\ n\geqslant 1)$ when $\alpha>3/4$. This result and its proof are an adaptation of the works of Comets and Yoshida on a continuous model of directed polymers [3].
- In Section 4, we establish (1).
- Appendix A is devoted to the proof of Lemma 2.4, used in Section 2, which gives a large deviation estimate for a sum of martingale-differences. It is a slight extension of a result of Lesigne and Volný [5, Theorem 3.2].

2. Preliminary: a concentration inequality

2.1. Exponential inequalities

Lemma 2.1. *Let $(g(x), x\in\mathbb{R}^d)$ be a family of Gaussian centered random variables with common variance $\sigma^2>0$. We fix $q,\beta>0$, $(x_1,\dots,x_n)\in(\mathbb{R}^d)^n$ and $(\lambda_1,\dots,\lambda_n)$ in $\mathbb{R}^n$. Then for any probability measure μ on $\mathbb{R}^d$:*

$$\mathrm{e}^{-\frac{\beta^2\sigma^2}{2}q}\leqslant \boldsymbol{E}\biggl(\frac{\mathrm{e}^{\beta\sum_{i=1}^n\lambda_i g(x_i)}}{\bigl(\int_{\mathbb{R}}\mathrm{e}^{\beta g(x)}\mu(dx)\bigr)^q}\biggr)\leqslant \mathrm{e}^{\frac{\beta^2\sigma^2}{2}(q+\sum_{i=1}^n|\lambda_i|)^2}.$$

The proof is identical with the one made by Carmona and Hu in a discrete framework (μ is the sum of Dirac masses), and is therefore omitted.

Lemma 2.2. *Let* $(g(x), x \in \mathbb{R}^d)$ *be a centered Gaussian process with covariance matrix* $\operatorname{cov}(g(x), g(y)) = \Gamma(x - y)$. *Let* $\sigma^2 = \Gamma(0)$, *and let* μ *be a probability measure on* $\mathbb{R}^d$. *Then for all* $\beta > 0$, *there are constants* $c_1 = c_1(\beta, \sigma^2) > 0$ *and* $c_2 = c_2(\beta, \sigma^2) > 0$ *such that*:

$$-c_1 \iint_{\mathbb{R}^d} \Gamma(x-y)\,\mu(dx)\,\mu(dy) \leqslant \boldsymbol{E}\left(\log \int_{\mathbb{R}} \mathrm{e}^{\beta g(x) - \frac{\beta^2\sigma^2}{2}}\mu(dx)\right) \leqslant -c_2 \iint_{\mathbb{R}^d} \Gamma(x-y)\,\mu(dx)\,\mu(dy).$$

In particular,

$$-c_1\sigma^2 \leqslant \boldsymbol{E}\left(\log \int_{\mathbb{R}^d} \mathrm{e}^{\beta g(x) - \frac{\beta^2\sigma^2}{2}}\mu(dx)\right) \leqslant 0.$$

Proof. Let $\{B_x(t), t \geqslant 0\}_{x \in \mathbb{R}^d}$ be the family of centered Gaussian processes such that

$$\boldsymbol{E}\big(B_x(t)B_y(s)\big) = \inf(s,t)\Gamma(x-y),$$

with $B_x(0) = 0$ for all $x \in \mathbb{R}^d$. Define

$$X(t) = \int_{\mathbb{R}^d} M_x(t)\,\mu(dx), \quad t \geqslant 0,$$

where $M_x(t) = \mathrm{e}^{\beta B_x(t) - \beta^2\sigma^2 t/2}$. Since $dM_x(t) = \beta M_x(t)\,dB_x(t)$, one has

$$d\langle M_x, M_y\rangle_t = \beta^2 M_x(t) M_y(t)\,d\langle B_x, B_y\rangle_t = \beta^2 \mathrm{e}^{\beta(B_x(t)+B_y(t)) - \beta^2\sigma^2 t}\Gamma(x-y)\,dt,$$

and $d\langle X, X\rangle_t = \iint_{\mathbb{R}^d} \beta^2 \mathrm{e}^{\beta(B_x(t)+B_y(t)) - \beta^2\sigma^2 t}\Gamma(x-y)\,\mu(dx)\,\mu(dy)\,dt$. Thus, by Ito's formula,

$$\boldsymbol{E}(\log X_1) = -\frac{\beta^2}{2} \iint_{\mathbb{R}^d} \mu(dx)\,\mu(dy)\,\Gamma(x-y) \int_0^1 \boldsymbol{E}\left(\frac{\mathrm{e}^{\beta(B_x(t)+B_y(t)) - \beta^2\sigma^2 t}}{X_t^2}\right) dt.$$

By Lemma 2.1, we have for all t:

$$\mathrm{e}^{-\beta^2\sigma^2 t} \leqslant \boldsymbol{E}\left(\frac{\mathrm{e}^{\beta(B_x(t)+B_y(t)) - \beta^2\sigma^2 t}}{X_t^2}\right) = \boldsymbol{E}\left(\frac{\mathrm{e}^{\beta(B_x(t)+B_y(t))}}{(\int_{\mathbb{R}} \mathrm{e}^{\beta B_x(t)}\mu(dx))^2}\right) \leqslant \mathrm{e}^{8\beta^2\sigma^2 t}.$$

Hence:

$$-\frac{\mathrm{e}^{8\beta^2\sigma^2} - 1}{16\sigma^2} \iint_{\mathbb{R}^d} \Gamma(x-y)\,\mu(dx)\,\mu(dy) \leqslant \boldsymbol{E}(\log X_1) \leqslant -\frac{1 - \mathrm{e}^{-\beta^2\sigma^2}}{2\sigma^2} \iint_{\mathbb{R}^d} \Gamma(x-y)\,\mu(dx)\,\mu(dy),$$

which concludes the proof since $X_1 \overset{d}{=} \int_{\mathbb{R}^d} \mathrm{e}^{\beta g(x) - \frac{\beta^2\sigma^2}{2}}\,\mu(dx)$. □

2.2. A concentration result

Proposition 2.3. *Let* $\nu > 1/2$. *For* $n \in \mathbb{N}$, $j \leqslant n$ *and* f_n *a nonnegative bounded function, such that* $\mathbb{E}(f_n(S_j)) > 0$. *We note* $W_{n,j} = \mathbb{E}(f_n(S_j)\mathrm{e}^{\beta \sum_{k=1}^n g(k,S_k)})$. *Then for* $n \geqslant n_0(\beta, \nu)$,

$$P\big(\big|\log W_{n,j} - E(\log W_{n,j})\big| \geqslant n^\nu\big) \leqslant \exp\left(-\frac{1}{4} n^{(2\nu-1)/3}\right).$$

Proof. We use the following lemma, whose proof is postponed to Appendix A.

Lemma 2.4. *Let* $(X_n^i,\ 1 \leqslant i \leqslant n)$ *be a martingale difference sequence and let* $M_n = \sum_{i=1}^n X_n^i$. *Suppose that there exists* $K > 0$ *such that* $\boldsymbol{E}(\mathrm{e}^{|X_n^i|}) \leqslant K$ *for all* i *and* n. *Then for any* $\nu > 1/2$, *and for* $n \geqslant n_0(K, \nu)$,

$$\boldsymbol{P}\big(|M_n| > n^\nu\big) \leqslant \exp\left(-\frac{1}{4} n^{(2\nu-1)/3}\right).$$

- We first assume that $f_n > 0$.
  To apply the Lemma 2.4, we define $X_{n,j}^i = E(\log W_{n,j} \mid \mathcal{G}_i) - E(\log W_{n,j} \mid \mathcal{G}_{i-1})$ so that

$$\log(W_{n,j}) - E(\log W_{n,j}) = \sum_{i=1}^n X_{n,j}^i.$$

It is sufficient to prove that there exists $K > 0$ such that $E(\mathrm{e}^{|X_{n,j}^i|}) \leqslant K$ for all i and (n, j).
For this, we introduce:

$$e_{n,j}^i = f_n(S_j) \exp\left(\sum_{1 \leqslant k \leqslant n,\ k \neq i} \beta g(k, S_k)\right), \quad W_{n,j}^i = \mathbb{E}\big(e_{n,j}^i\big).$$

$W_{n,j}^i > 0$ since we assumed that $f_n > 0$. If E_i is the conditional expectation with respect to $\mathcal{G}_i$, then $E_i(\log W_{n,j}^i) = E_{i-1}(\log W_{n,j}^i)$, so that:

$$X_{n,j}^i = E_i\big(\log Y_{n,j}^i\big) - E_{i-1}\big(\log Y_{n,j}^i\big), \tag{2}$$

with

$$Y_{n,j}^i = \mathrm{e}^{-\beta^2/2} \frac{W_{n,j}}{W_{n,j}^i} = \int_{\mathbb{R}^d} \mathrm{e}^{\beta g(i,x) - \beta^2/2} \mu_{n,j}^i(dx), \tag{3}$$

$\mu_{n,j}^i$ being the random probability measure:

$$\mu_{n,j}^i(dx) = \frac{1}{W_{n,j}^i} \mathbb{E}\big(e_{n,j}^i \mid S_i = x\big) \mathbb{P}(S_i \in dx).$$

Since $\mu_{n,j}^i$ is measurable with respect to $\mathcal{G}_{n,i} = \sigma(g(k,x),\ 1 \leqslant k \leqslant n,\ k \neq i,\ x \in \mathbb{R}^d)$, we deduce from Lemma 2.2 that there exists a constant $c = c(\beta) > 0$, which does not depend on (n, j, i), such that:

$$-c \leqslant E\left(\log \int_{\mathbb{R}^d} \mathrm{e}^{\beta g(i,x) - \beta^2/2} \mu_{n,j}^i(dx) \mid \mathcal{G}_{n,i}\right) \leqslant 0,$$

and since $\mathcal{G}_{i-1} \subset \mathcal{G}_{n,i}$, we obtain:

$$0 \leqslant -E_{i-1}\big(\log Y_{n,j}^i\big) \leqslant c. \tag{4}$$

Thus we deduce from (2) and (4) that for all $\theta \in \mathbb{R}$

$$E\big[\mathrm{e}^{\theta X_{n,j}^i}\big] \leqslant \mathrm{e}^{c\theta^+} E\big[\mathrm{e}^{\theta E_i(\log Y_{n,j}^i)}\big]$$

with $\theta^+ := \max(\theta, 0)$. By Jensen's inequality,

$$\mathrm{e}^{\theta E_i(\log Y_{n,j}^i)} \leqslant E_i\big[\big(Y_{n,j}^i\big)^\theta\big]$$

so that

$$E\big[e^{\theta X^i_{n,j}}\big] \leqslant e^{c\theta^+} E\big[\big(Y^i_{n,j}\big)^\theta\big] = e^{c\theta^+} E\big[E\big[\big(Y^i_{n,j}\big)^\theta \mid \mathcal{G}_{n,i}\big]\big].$$

Assume now that $\theta \in \{-1, 1\}$, hence in both cases, the function $x \mapsto x^\theta$ is convex; using (3), we obtain $(Y^i_{n,j})^\theta \leqslant \int_{\mathbb{R}^d} e^{\theta(\beta g(i,x) - \beta^2/2)} \mu^i_{n,j}(dx)$, so that:

$$E\big[\big(Y^i_{n,j}\big)^\theta \mid \mathcal{G}_{n,i}\big] \leqslant \int_{\mathbb{R}^d} E\big(e^{\theta(\beta g(i,x)-\beta^2/2)} \mid \mathcal{G}_{n,i}\big)\, \mu^i_{n,j}(dx) = \int_{\mathbb{R}^d} E\big(e^{\theta(\beta g(i,x)-\beta^2/2)}\big)\, \mu^i_{n,j}(dx)$$
$$= E\big(e^{\theta(\beta g(1,0)-\beta^2/2)}\big),$$

using that $g(i,x)$ is independent from $\mathcal{G}_{n,i}$, and is distributed as $g(1,0)$ for all i and x. We conclude that for all n and $1 \leqslant i, j \leqslant n$,

$$E\big[e^{|X^i_{n,j}|}\big] \leqslant E\big[e^{X^i_{n,j}}\big] + E\big[e^{-X^i_{n,j}}\big] \leqslant K := e^c + e^{\beta^2}.$$

- In the general case where $f_n \geqslant 0$, we introduce $h_n = f_n + \delta$ for some $0 < \delta < 1$. The first part of the proof applies to h_n: noting $W^\delta_{n,j} = \mathbb{E}(h_n(S_j) e^{\beta \sum_{k=1}^n g(k,S_k)})$, it remains to show that $\log W^\delta_{n,j} - E(\log W^\delta_{n,j}) \overset{P\text{-a.s.}}{\underset{\delta\to 0}{\to}} \log W_{n,j} - E(\log W_{n,j})$. Since f_n is bounded by some constant $C_n > 0$, the following inequality holds for all $0 < \delta < 1$: $\log W_{n,j} \leqslant \log W^\delta_{n,j} \leqslant \log((C_n + 1) Z_n)$. Since $0 \leqslant E \log Z_n \leqslant \log E Z_n = n\beta^2 \Gamma(0)/2 < \infty$, the conclusion follows from dominated convergence. □

Corollary 2.5. *Let $\nu > 1/2$. Let us fix a sequence of Borel sets $(B(j,n), n \geqslant 1,\ j \leqslant n)$. Then P-almost surely, there exists n_0 such that for every $n \geqslant n_0$, every $j \leqslant n$,*

$$\big|\log \langle \mathbb{1}_{S_j \in B(j,n)} \rangle^{(n)} - E\big(\log \langle \mathbb{1}_{S_j \in B(j,n)} \rangle^{(n)}\big)\big| \leqslant 2n^\nu.$$

Proof. Let us write $A_{n,j} = \{|\log \mathbb{E}(f_n(S_j) e^{\beta \sum_{k=1}^n g(k,S_k)}) - E[\log \mathbb{E}(f_n(S_j) e^{\beta \sum_{k=1}^n g(k,S_k)})]| \geqslant n^\nu\}$. Proposition 2.3 implies that

$$P\bigg(\bigcup_{j \leqslant n} A_{n,j}\bigg) \leqslant n \exp\bigg(-\frac{1}{4} n^{(2\nu-1)/3}\bigg).$$

Hence, by Borel–Cantelli, P-almost-surely there exists n_0 such that for every $n \geqslant n_0$ and every $j \leqslant n$:

$$\big|\log \mathbb{E}\big(f_n(S_j) e^{\beta \sum_{k=1}^n g(k,S_k)}\big) - E\big[\log \mathbb{E}\big(f_n(S_j) e^{\beta \sum_{k=1}^n g(k,S_k)}\big)\big]\big| \leqslant n^\nu.$$

Then one applies this result to $f_n(x) = \mathbb{1}_{x \in B(j,n)}$ and to $f_n(x) = 1$. □

3. A first result

In this section, we prove that a large deviation principle holds P-almost surely for the sequence of measures $(\langle \mathbb{1}_{S_n/n^\alpha \in .} \rangle^{(n)}, n \geqslant 1)$ if $\alpha > 3/4$. This was first proved by Comets and Yoshida [3, Theorem 2.4.4], for a model of directed polymers in which the random walk is replaced by a Brownian motion and the environment is given by a Poisson random measure on $\mathbb{R}_+ \times \mathbb{R}^d$.

Theorem 3.1. *Let $\alpha > 3/4$. Then a large deviation principle for $(\langle \mathbb{1}_{S_n/n^\alpha \in .} \rangle^{(n)}, n \geqslant 1)$ holds P-a.s., with the rate function $I(\lambda) = \|\lambda\|^2/2$ and the speed $n^{2\alpha-1}$, $\|.\|$ denoting the Euclidean norm on $\mathbb{R}^d$. In particular, for all $\varepsilon > 0$,*

$$\lim_{n\to\infty} -\frac{1}{n^{2\alpha-1}} \log \langle \mathbb{1}_{\|S_n\| \geqslant \varepsilon n^\alpha} \rangle^{(n)} = \frac{\varepsilon^2}{2} \quad P\text{-a.s.}$$

Remark 3.2. In particular this result implies that for all $\alpha > 3/4$,

$$\langle \mathbb{1}_{|S_n| \geqslant n^\alpha} \rangle^{(n)} \underset{n\to\infty}{\overset{P\text{-a.s.}}{\to}} 0. \tag{5}$$

Proof. Let us fix $\lambda \in \mathbb{R}^d$, $n \geqslant 1$, and then introduce the following martingale:

$$M_p^{\lambda,n} = \begin{cases} e^{\lambda.S_p - p\|\lambda\|^2/2} & \text{if } p \leqslant n, \\ e^{\lambda.S_n - n\|\lambda\|^2/2} & \text{if } p > n, \end{cases}$$

with $x.y$ denoting the scalar product between two vectors x and y in $\mathbb{R}^d$. If $\mathbb{Q}^{\lambda,n}$ is the probability defined by Girsanov's change associated to this positive martingale, Girsanov's formula ensures that, under $\mathbb{Q}^{\lambda,n}$, the process $(\tilde{S}_p := S_p - \lambda(p \wedge n),\ p \geqslant 1)$ has the same distribution as S under $\mathbb{P}$. Therefore:

$$\begin{aligned} Z_n\langle e^{\lambda.S_n}\rangle^{(n)} &= e^{n\|\lambda\|^2/2}\mathbb{E}\big(M_n^\lambda e^{\beta\sum_{k=1}^n g(k,S_k)}\big) \\ &= e^{n\|\lambda\|^2/2}\mathbb{E}\big(e^{\beta\sum_{k=1}^n g^{\lambda,n}(k,S_k+k\lambda)}\big) \\ &= e^{n\|\lambda\|^2/2}\mathbb{E}\big(e^{\beta\sum_{k=1}^n g^{\lambda,n}(k,S_k)}\big), \end{aligned}$$

where we denote by $g^{\lambda,n}$ the translated environment

$$g^{\lambda,n}(k,x) := g\big(k, x + \lambda(k \wedge n)\big).$$

By stationarity, this environment has the same distribution as $(g(k,x),\ k \geqslant 1,\ x \in \mathbb{R}^d)$, hence

$$\mathbb{E}\big(e^{\beta\sum_{k=1}^n g^{\lambda,n}(k,S_k)}\big) \overset{d}{=} \mathbb{E}\big(e^{\beta\sum_{k=1}^n g(k,S_k)}\big),$$

thus

$$E\log\langle e^{\lambda.S_n}\rangle^{(n)} = n\|\lambda\|^2/2. \tag{6}$$

Now let us fix $\alpha > 3/4$. With $n^{\alpha-1}\lambda$ instead of λ, (6) gives

$$E\log\langle e^{n^{\alpha-1}\lambda.S_n}\rangle^{(n)} = n^{2\alpha-1}\|\lambda\|^2/2. \tag{7}$$

Let us define $f_n(x) = e^{n^{\alpha-1}\lambda.x}$. This function is positive and $\mathbb{E}(f_n(S_n)e^{\beta\sum_{k=1}^n g(k,S_k)}) < \infty$, so that the result of Corollary 2.5 is still true with $f_n(x)$ instead of $\mathbb{1}_{x\in B(j,n)}$. Since $2\alpha - 1 > 1/2$, this implies

$$\lim_{n\to\infty} \frac{1}{n^{2\alpha-1}}\big(\log\langle e^{n^{\alpha-1}\lambda.S_n}\rangle^{(n)} - E\log\langle e^{n^{\alpha-1}\lambda.S_n}\rangle^{(n)}\big) = 0 \quad P\text{-a.s.} \tag{8}$$

From (7) and (8), we get:

$$\lim_{n\to\infty} \frac{1}{n^{2\alpha-1}}\log\langle e^{n^{\alpha-1}\lambda.S_n}\rangle^{(n)} = \|\lambda\|^2/2 \quad P\text{-a.s.}$$

Let us define $h_n(\lambda) = \frac{1}{n^{2\alpha-1}}\log\langle e^{n^{\alpha-1}\lambda.S_n}\rangle^{(n)}$ and $h(\lambda) = \|\lambda\|^2/2$. From what we proved, we deduce the existence of $A \subset \Omega^g$, with $P(A) = 1$, on which $h_n(\lambda) \to h(\lambda)$ for all $\lambda \in \mathbb{Q}^d$. Now we show that on A the convergence actually holds for all $\lambda \in \mathbb{R}^d$, by using that the functions h_n are convex and h is continuous. To this goal, we can reduce the proof to the case $d = 1$. Indeed if $d \geqslant 2$, we fix $(d-1)$ coordinates in $\mathbb{Q}^{d-1}$ and use that h_n is still convex as a function of the last coordinate, and then repeat the process. So let us assume $d = 1$ and fix $\lambda \in \mathbb{R}^*$. There exist two sequences $(a_i,\ i \geqslant 0)$ and $(b_i,\ i \geqslant 0)$ in $\mathbb{Q}^{\mathbb{N}}$ that converge to λ, $(a_i,\ i \geqslant 0)$ being increasing and $(b_i,\ i \geqslant 0)$ decreasing. Let us fix $i \geqslant 1$ and $n \geqslant 1$. Since h_n is convex and satisfies $h_n(0) = 0$, the function $x \to h_n(x)/x$ is increasing. Hence the following inequalities hold:

$$\frac{h_n(a_i)}{a_i} \leqslant \frac{h_n(\lambda)}{\lambda} \leqslant \frac{h_n(b_i)}{b_i}.$$

Since h_n converges towards h on $\mathbb{Q}$, it follows that

$$\frac{h(a_i)}{a_i} \leqslant \liminf_{n\to\infty} \frac{h_n(\lambda)}{\lambda} \leqslant \limsup_{n\to\infty} \frac{h_n(\lambda)}{\lambda} \leqslant \frac{h(b_i)}{b_i}.$$

The limit function h being continuous, we obtain by letting $i \to +\infty$ that the limit of $h_n(\lambda)/\lambda$ exists and is equal to $h(\lambda)/\lambda$, which proves that $h_n(\lambda) \to h(\lambda)$ for all $\lambda \in \mathbb{R}$.

One then concludes by the Gärtner–Ellis–Baldi theorem (see [4]). □

4. Upper bound of the volume exponent

We now extend the result (5) to the maximal deviation from the origin:

Theorem 4.1. *For all $d \geqslant 1$ and $\alpha > 3/4$,*

$$\langle \mathbb{1}_{\{\max_{k\leqslant n} |S_k| \geqslant n^\alpha\}} \rangle^{(n)} \underset{n\to\infty}{\overset{P\text{-a.s.}}{\to}} 0.$$

Proof. We will use the following notations: for $x \in \mathbb{R}^d$ and $r \geqslant 0$, $B(x,r) = \{y \in \mathbb{R}^d,\ |y-x| \leqslant r\}$. For $\alpha \geqslant 0$ and $j = (j_1, \ldots, j_d) \in \mathbb{Z}^d$, $B_j^\alpha = B(jn^\alpha, n^\alpha)$. We will use the fact that the union of the balls $(B_j^\alpha,\ j \in (2\mathbb{Z})^d \backslash \{0\})$ form a partition of $\mathbb{R}^d \backslash B(0, n^\alpha)$.

We first prove the following upper bound:

Proposition 4.2. *Let $n \geqslant 0$ and $k \leqslant n$. Then for any $j \in \mathbb{Z}^d$ and $\alpha > 0$,*

$$E\big(\log \langle \mathbb{1}_{S_k \in B_j^\alpha} \rangle^{(n)}\big) \leqslant \frac{-n^{2\alpha-1}}{2} \sum_{i=1}^d (j_i - \varepsilon_i)^2,$$

where $\varepsilon_i = \operatorname{sgn}(j_i)$ $(= 0$ if $j_i = 0)$.

Proof. Let note $a_{k,j}^\alpha = \mathbb{E}(\mathbb{1}_{S_k \in B_j^\alpha} e^{\beta \sum_{i=1}^n g(i,S_i)})$, so that $\langle \mathbb{1}_{S_k \in B_j^\alpha} \rangle^{(n)} = a_{k,j}^\alpha / Z_n$. Let be $\lambda = \tilde{\lambda}/k$ with $\tilde{\lambda}_i = (j_i - \varepsilon_i) n^\alpha,\ 1 \leqslant i \leqslant d$; then let us define the martingale

$$M_p^{\lambda,k} = \begin{cases} e^{\lambda . S_p - p\|\lambda\|^2/2} & \text{if } p \leqslant k, \\ e^{\lambda . S_k - k\|\lambda\|^2/2} & \text{if } p > k, \end{cases}$$

where $x.y$ denotes the usual scalar product in $\mathbb{R}^d$ and $\|x\|$ the associated euclidean norm. Under the probability $\mathbb{Q}^{\lambda,k}$ associated to this martingale, $(S_p)_{p\geqslant 0}$ has the law of the following shifted random walk under $\mathbb{P}$:

$$\tilde{S}_p = S_p + \tilde{\lambda}\left(\frac{p}{k} \wedge 1\right).$$

It follows that:

$$a_{k,j}^\alpha = \mathbb{E}\big(e^{\frac{-1}{k}(\tilde{\lambda}.S_k + \|\tilde{\lambda}\|^2/2)} \mathbb{1}_{S_k \in B_j^\alpha - \tilde{\lambda}} e^{\beta \sum_{i=1}^n \tilde{g}(i,S_i)}\big), \tag{9}$$

where $\tilde{g}(i,x) = g(i, x + \tilde{\lambda}(i/k \wedge 1))$.

Now we notice that on the event $\{S_k \in B_j^\alpha - \tilde{\lambda}\}$, one has $\tilde{\lambda}.S_k \geqslant 0$: indeed if we write $S_k = (S_k^1, \ldots, S_k^d)$, then for any $1 \leqslant i \leqslant d$, $|S_k^i - j_i n^\alpha + \tilde{\lambda}_i| \leqslant n^\alpha$, hence:

- for $j_i \geqslant 1$, $\tilde{\lambda}_i = (j_i - 1)n^\alpha \geqslant 0$ and $0 \leqslant S_k^i \leqslant 2n^\alpha$,

- for $j_i \leqslant -1$, $\tilde\lambda_i = (j_i+1)n^\alpha \leqslant 0$ and $-2n^\alpha \leqslant S_k^i \leqslant 0$,
- for $j_i = 0$, $\tilde\lambda_i = 0$,

so that in all cases $\tilde\lambda_i S_k^i \geqslant 0$ and thus $\tilde\lambda.S_k \geqslant 0$. Therefore on the event $\{S_k \in B_j^\alpha - \tilde\lambda\}$,

$$\mathrm{e}^{\frac{-1}{k}(\tilde\lambda.S_k+\|\tilde\lambda\|^2/2)} \leqslant \mathrm{e}^{\frac{-\|\tilde\lambda\|^2}{2n}} \leqslant \mathrm{e}^{\frac{-n^{2\alpha-1}}{2}\sum_1^d (j_i-\varepsilon_i)^2},$$

and (9) leads to:

$$a_{k,j}^\alpha \leqslant \mathrm{e}^{\frac{-n^{2\alpha-1}}{2}\sum_1^d (j_i-\varepsilon_i)^2}\mathbb{E}\big(\mathbb{1}_{S_k\in B_j^\alpha-\tilde\lambda}\mathrm{e}^{\beta\sum_{i=1}^n \tilde g(i,S_i)}\big).$$

On the other hand, $Z_n \geqslant \mathbb{E}(\mathbb{1}_{S_k\in B_j^\alpha-\tilde\lambda}\mathrm{e}^{\beta\sum_{i=1}^n g(i,S_i)})$, and since by stationarity the environment $\tilde g$ has the same distribution as g, it follows that for all $j \in \mathbb{Z}^d$,

$$E\big(\log\langle \mathbb{1}_{S_k\in B_j^\alpha}\rangle^{(n)}\big) \leqslant \frac{-n^{2\alpha-1}}{2}\sum_{i=1}^d (j_i-\varepsilon_i)^2. \qquad \square$$

Let $\nu > 1/2$. We deduce from Proposition 4.2 and from Corollary 2.5 (with $B(k,n) = B_j^\alpha$) that, P-a.s., for $n \geqslant n_0$, $k \leqslant n$, and all $j \in \mathbb{Z}^d$:

$$\log\langle \mathbb{1}_{S_k\in B_j^\alpha}\rangle^{(n)} \leqslant 2n^\nu - \frac{n^{2\alpha-1}}{2}\sum_{i=1}^d (j_i-\varepsilon_i)^2.$$

So, P-a.s., for $n \geqslant n_0$, $\langle \mathbb{1}_{|S_k|\geqslant n^\alpha}\rangle^{(n)} \leqslant \sum_{j\in(2\mathbb{Z})^d\setminus\{0\}} \mathrm{e}^{2n^\nu - \frac{n^{2\alpha-1}}{2}\sum_{i=1}^d (j_i-\varepsilon_i)^2}$, and

$$\langle \mathbb{1}_{\{\max_{k\leqslant n}|S_k|\geqslant n^\alpha\}}\rangle^{(n)} \leqslant \sum_{k=1}^n \langle \mathbb{1}_{|S_k|\geqslant n^\alpha}\rangle^{(n)} \leqslant \sum_{j\in(2\mathbb{Z})^d\setminus\{0\}} n\mathrm{e}^{2n^\nu - \frac{n^{2\alpha-1}}{2}\sum_{i=1}^d (j_i-\varepsilon_i)^2}.$$

But by symmetry, for any $C > 0$,

$$\sum_{j\in(2\mathbb{Z})^d\setminus\{0\}} \mathrm{e}^{-C\sum_{i=1}^d (j_i-\varepsilon_i)^2} \leqslant 2d\sum_{j_1\geqslant 2}\mathrm{e}^{-C(j_1-1)^2}\prod_{i=2}^d\sum_{j_i\in 2\mathbb{Z}}\mathrm{e}^{-C(j_i-\varepsilon_i)^2}$$

and using that $\sum_{j\geqslant 2}\mathrm{e}^{-C(j-1)^2} \leqslant \sum_{j\geqslant 1}\mathrm{e}^{-Cj} = \frac{\mathrm{e}^{-C}}{1-\mathrm{e}^{-C}}$, we conclude that, P-a.s., for some constant $C(d) > 0$, and for $n \geqslant n_0$:

$$\langle \mathbb{1}_{\{\max_{k\leqslant n}|S_k|\geqslant n^\alpha\}}\rangle^{(n)} \leqslant C(d)n\mathrm{e}^{2n^\nu}\frac{\mathrm{e}^{-n^{2\alpha-1}/2}}{1-\mathrm{e}^{-n^{2\alpha-1}/2}}.$$

Thus for all $\alpha > \frac{\nu+1}{2}$, P-a.s.,

$$\langle \mathbb{1}_{\{\max_{k\leqslant n}|S_k|\geqslant n^\alpha\}}\rangle^{(n)} \underset{n\to\infty}{\to} 0.$$

This is true for all $\nu > 1/2$, which ends the proof. $\square$

Appendix A. Proof of Lemma 2.4

The beginning of the proof is exactly identical with the one made by Lesigne and Volný in [5, pp. 148–149]. Only the last ten lines differ.

Let us denote $(\mathcal{F}_n^i)_{1\leqslant i\leqslant n}$ the filtration of $(X_n^i)_{1\leqslant i\leqslant n}$. The hypothesis that it is a martingale difference sequence means that for each i, X_n^i is $\mathcal{F}_n^i$-measurable and, if $i\geqslant 2$, $\boldsymbol{E}[X_n^i \mid \mathcal{F}_n^{i-1}]=0$.

Let us fix $a>0$ and for $1\leqslant i\leqslant n$ define

$$Y_n^i = X_n^i \mathbb{1}_{|X_n^i|\leqslant an^{1/3}} - \boldsymbol{E}\big[X_n^i \mathbb{1}_{|X_n^i|\leqslant an^{1/3}} \bigm| \mathcal{F}_n^{i-1}\big]$$

and

$$Z_n^i = X_n^i \mathbb{1}_{|X_n^i|> an^{1/3}} - \boldsymbol{E}\big[X_n^i \mathbb{1}_{|X_n^i|> an^{1/3}} \bigm| \mathcal{F}_n^{i-1}\big],$$

and then define $M_n' = \sum_{i=1}^k Y_n^i$ and $M_n'' = \sum_{i=1}^k Z_n^i$. Since $(X_n^i)_{1\leqslant i\leqslant n}$ is a martingale difference sequence, $(Y_n^i)_{1\leqslant i\leqslant n}$ and $(Z_n^i)_{1\leqslant i\leqslant n}$ are martingale difference sequences and $X_n^i = Y_n^i + Z_n^i$ $(1\leqslant i\leqslant n)$.

Let us fix $t\in(0,1)$. For every $x>0$,

$$\boldsymbol{P}\big(|M_n|>nx\big) \leqslant \boldsymbol{P}\big(|M_n'|>nxt\big) + \boldsymbol{P}\big(|M_n''|>nx(1-t)\big). \tag{A.1}$$

Since $|Y_n^i|\leqslant 2an^{1/3}$ for $1\leqslant i\leqslant n$, Azuma's inequality implies:

$$\boldsymbol{P}\big(|M_n'|>nxt\big) = \boldsymbol{P}\left(\frac{|M_n'|}{2an^{1/3}} > \frac{nxt}{2an^{1/3}}\right) \leqslant 2\exp\left(-\frac{t^2x^2}{8a^2}n^{1/3}\right). \tag{A.2}$$

To control the second term in (A.1), we notice that $\boldsymbol{E}((M_n'')^2) = \sum_{i=1}^n \boldsymbol{E}(Z_n^i)^2$. For each $1\leqslant i\leqslant n$, if we note $F_n^i(x) = \boldsymbol{P}(|X_n^i|>x)$:

$$\begin{aligned}\boldsymbol{E}\big(Z_n^i\big)^2 &= \boldsymbol{E}\big(\big(X_n^i\mathbb{1}_{|X_n^i|>an^{1/3}}\big)^2\big) - \boldsymbol{E}\big(\boldsymbol{E}\big(X_n^i\mathbb{1}_{|X_n^i|>an^{1/3}} \bigm| \mathcal{F}_n^{i-1}\big)^2\big)\\ &\leqslant \boldsymbol{E}\big(\big(X_n^i\mathbb{1}_{|X_n^i|>an^{1/3}}\big)^2\big) = -\int_{an^{1/3}}^{+\infty} x^2\,dF_n^i(x).\end{aligned}$$

Since $\boldsymbol{E}\mathrm{e}^{|X_n^i|}\leqslant K$, $F_n^i(x)\leqslant K\mathrm{e}^{-x}$ for all $x\geqslant 0$, hence:

$$-\int_{an^{1/3}}^{+\infty} x^2\,dF_i(x) \leqslant Ka^2n^{2/3}\mathrm{e}^{-an^{1/3}} + 2K\int_{an^{1/3}}^{+\infty} x\mathrm{e}^{-x}\,dx = K\big(a^2n^{2/3}+2an^{1/3}+2\big)\mathrm{e}^{-an^{1/3}}.$$

It follows that $\boldsymbol{E}((M_n'')^2)\leqslant nK(a^2n^{2/3}+2an^{1/3}+2)\mathrm{e}^{-an^{1/3}}$, and:

$$\boldsymbol{P}\big(|M_n''|>nx(1-t)\big) \leqslant \frac{K}{x^2(1-t)^2}\big(a^2n^{-1/3}+2an^{-2/3}+2n^{-1}\big)\mathrm{e}^{-an^{1/3}}. \tag{A.3}$$

We choose $a=\frac{1}{2}(tx)^{2/3}$ so that $\frac{t^2x^2}{8a^2}=a$. From (A.1), (A.2) and (A.3), we deduce:

$$\boldsymbol{P}\big(|M_n|>nx\big) \leqslant \left(2+\frac{K}{(1-t)^2}f(t,x,n)\right)\exp\left(-\frac{1}{2}(tx)^{2/3}n^{1/3}\right), \tag{A.4}$$

with $f(t,x,n)=\frac{1}{4}t^{4/3}x^{-2/3}n^{-1/3}+t^{2/3}x^{-4/3}n^{-2/3}+2x^{-2}n^{-1}$. Now by taking $x=n^{\nu-1}$, we have:

$$\boldsymbol{P}\big(|M_n|>n^\nu\big) = \boldsymbol{P}\big(|M_n|>nx\big) \leqslant \left(2+\frac{K}{(1-t)^2}g(t,n)\right)\exp\left(-\frac{1}{2}t^{2/3}x^{2/3}n^{1/3}\right), \tag{A.5}$$

with $g(t,n)=f(t,n^{\nu-1},n)=\frac{1}{4}t^{4/3}n^{-(2\nu-1)/3}+t^{2/3}n^{-2(2\nu-1)/3}+2n^{-(2\nu-1)}$. Now we fix $\varepsilon>0$ and choose $t_0\in(0,1)$ such that $0<1-t_0^{2/3}<\varepsilon/2$. (A.5) implies that:

$$\boldsymbol{P}\big(|M_n|>n^\nu\big)\exp\left(\frac{1}{2}(1-\varepsilon)n^{(2\nu-1)/3}\right) \leqslant \left(2+\frac{K}{(1-t_0)^2}g(t_0,n)\right)\exp\left(-\frac{\varepsilon}{4}n^{(2\nu-1)/3}\right).$$

But, since $\nu > 1/2$, $(2 + \frac{K}{(1-t_0)^2} g(t_0, n)) \exp(-\frac{\varepsilon}{4} n^{(2\nu-1)/3}) \underset{n\to\infty}{\to} 0$. Therefore there exists $n_0(\varepsilon)$ such that, for all $n \geqslant n_0(\varepsilon)$,

$$\boldsymbol{P}\big(|M_n| > n^\nu\big) \leqslant \exp\biggl(-\frac{1}{2}(1-\varepsilon) n^{(2\nu-1)/3}\biggr). \tag{A.6}$$

When $\varepsilon = 1/2$ this is exactly the statement of the Lemma 2.4.

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