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Moderate deviations for diffusions in a random Gaussian shear flow drift

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Abstract

We prove quenched and annealed moderate deviation principle in large time for random additive functional of Brownian motion $\int_0^t v(B_s)\,ds$, where B is a d-dimensional Brownian motion, and v is a stationary Gaussian field from $\mathbb{R}^d$ with value in $\mathbb{R}$, independent of the Brownian motion. The speed of the moderate deviations is linked to the decay of correlation of the random field. The results are proved in dimension $d \leqslant 3$. These random additive functionals are the central object in the study of diffusion processes with random drift $X_t = W_t + \int_0^t V(X_s)\,ds$, where V is a centered Gaussian shear flow random field independent of the Brownian W.

Résumé

Soit B un mouvement Brownien d-dimensionnel, et $(v(x), x \in \mathbb{R}^d)$ un champ Gaussien stationnaire centré indépendant de v. Nous prouvons un principe de déviations modérées en temps long pour la fonctionnelle additive aléatoire $\int_0^t v(B_s)\,ds$, lorsque $d \leqslant 3$. Ce principe est obtenu lorsqu'une réalisation du champ v est fixée, ou lorsqu'on moyenne sur l'aléa de v. La vitesse dans les déviations modérées dépend de la vitesse de décorrélation de v. Ces fonctionnelles additives sont l'objet central dans l'étude de diffusion dans des champs de vitesse aléatoires cisaillés.

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1. Introduction

In this paper, we continue our investigation of the large deviations properties in large time for diffusions (X_t) in random incompressible velocity fields

$$dX_t = dW_t + V(t, X_t)\,dt; \quad V \text{ random}; \quad \mathbb{E}(V) = 0; \quad \operatorname{div}(V) = 0.$$

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Such a process serves as a model for diffusion in a random media (incompressible turbulent flow, porous media ...). As such, it has been thoroughly studied, under various assumptions on the random drift V (see for instance [3,4, 24,7,8,14–19,26,27,32]).

We focus here on the Gaussian shear flow model: we assume that V is time independent, with the following simple spatial structure:

$$\text{For all } x = (x_1, x_2) \in \mathbb{R}^d \times \mathbb{R}, \quad V(x_1, x_2) = \big(0, v(x_1)\big),$$

where $(v(x_1); x_1 \in \mathbb{R}^d)$ is a centered stationary Gaussian field with value in $\mathbb{R}$, and with covariance

$$K(x-y) = \mathbb{E}\big(v(x)v(y)\big) = \int_{\mathbb{R}^d} \mathrm{e}^{ik(x-y)} \frac{\phi(k)}{\|k\|^{(d-\alpha)_+}}\, dk. \tag{1}$$

ϕ is a rapidly decreasing function at infinity, which plays the role of an ultraviolet cut-off. The parameter α is positive, and describes the decay of correlation at infinity. For $\alpha \geqslant d$, K is rapidly decreasing at infinity. For $0 < \alpha < d$, $K(x) \underset{\infty}{\asymp} \|x\|^{-\alpha}$. The famous Kolmogorov-41 law describing the statistical behavior of v for turbulent flows, would correspond to a negative value of $\alpha = -2/3$ (with an additional infrared cut-off), which is outside the scope of this paper. For this reason, and because time independence, the special case we are studying is more appropriate to describe diffusion in porous media. It was actually introduced in this setting by Matheron and de Marsily in [28].

Note that in presence of the shear flow structure, the diffusion X_t is given by

$$\begin{cases} X_{1,t} = W_{1,t}; \\ X_{2,t} = W_{2,t} + \int_0^t v(W_{1,s})\, ds. \end{cases} \tag{2}$$

Hence, the study of X_t for large time is reduced to the study of random additive functional of Brownian motion $Y_t \triangleq \int_0^t v(B_s)\, ds$. In [28], the mean square displacement $\tilde{E}_0[Y_t^2]$ is estimated for a field v which is δ-correlated, and for the annealed law $\tilde{E}_0$ (i.e. when we average over both the Brownian motion and the velocity field). Almost at the same time, Kesten and Spitzer [25] proved a "central limit theorem" for the discrete analogue of Y. For the model defined by (1) and (2), this study is done by Avellaneda and Majda [3], and by Horntrop and Majda [24]. All these papers exhibit super-diffusive behavior of Y, at least in a certain range of the parameters, and this is the reason why this kind of model has received so much attention.

The problem we address here, is the study of the moderate deviations of $\int_0^t v(B_s)\, ds$ in the model given by (1). More precisely, we look for rough asymptotics in large time for the probability of events like

$$\mathcal{A} \triangleq \left\{ \frac{1}{m(t)} \int_0^t v(B_s)\, ds > y \right\}, \quad y > 0,$$

with respect to the annealed measure $\tilde{P}_0$ and the quenched one P_0 (i.e. in frozen environment v), and for a scaling $m(t)$ such that $m(t) \gg 1$. We speak about "moderate deviations", since $m(t)$ is chosen to be negligible with respect to the large deviations normalizations given in [9,1]. More precisely, it is proved in [9] that

$$\tilde{P}_0\left[\frac{1}{t^{3/2}} \int_0^t v(B_s)\, ds > y \right] \approx \exp\big(-t I_a(y)\big), \tag{3}$$

while [1] establishes that

$$P_0\left[\frac{1}{t\sqrt{\log(t)}} \int_0^t v(B_s)\, ds > y \right] \approx \exp\big(-t I_q(y)\big). \tag{4}$$

It is to note that in (3) and (4), the scalings $t^{3/2}$ and $t\sqrt{\log(t)}$ do not depend on the covariance function K, but on the choice of a Gaussian statistic for v. On the other hand, the rate functions I_a and I_q depend on K. A formal computation from (3) and (4) using the behavior of I_a and I_q near the origin, would lead to

$$\begin{aligned}\tilde{P}_0\left[\frac{1}{m(t)}\int_0^t v(B_s)\,ds > y\right] &\approx \exp\left(-tI_a\left(\frac{m(t)}{t^{3/2}}y\right)\right)\\ &\approx \exp\left(-C_a(\alpha,d)\frac{m(t)^{\frac{4}{2+\alpha\wedge d}}}{t^{\frac{4-\alpha\wedge d}{2+\alpha\wedge d}}}|y|^{\frac{4}{2+\alpha\wedge d}}\right),\\ &\quad\text{for } t^{1-\frac{\alpha\wedge d}{4}} \ll m(t) \ll t^{3/2};\end{aligned} \tag{5}$$

$$\begin{aligned}P_0\left[\frac{1}{m(t)}\int_0^t v(B_s)\,ds > y\right] &\approx \exp\left(-tI_q\left(\frac{m(t)}{t\sqrt{\log(t)}}y\right)\right)\\ &\approx \exp\left(-C_q(\alpha,d)\frac{m(t)^{\frac{4}{\alpha\wedge d}}t^{1-\frac{4}{\alpha\wedge d}}}{\log(t)^{\frac{2}{\alpha\wedge d}}}|y|^{\frac{4}{\alpha\wedge d}}\right),\\ &\quad\text{for } t^{1-\frac{\alpha\wedge d}{4}}\sqrt{\log(t)} \ll m(t) \ll t\sqrt{\log(t)}.\end{aligned} \tag{6}$$

Actually, estimates (5) and (6) are the main results of this paper (see Theorems 2 and 4). They are proved for $d \leqslant 3$ and for $m(t)$ such that

$$t \leqslant m(t) \ll t^{3/2}, \quad \text{in the annealed case;} \tag{7}$$

$$t \leqslant m(t) \ll t\sqrt{\log(t)}, \quad \text{in the quenched one.} \tag{8}$$

They can be straightforwardly interpreted in terms of the diffusion X (cf. Corollaries 3 and 5), and show a super-diffusive behavior of this diffusion: taking for instance $m(t) = t$, Corollary 5 states that the probability for X to travel during a time t to a distance t, is much larger than in the usual diffusive case.

The constants $C_q(\alpha,d)$ and $C_a(\alpha,d)$ are given by variational formulas, and are non degenerate for $\alpha \wedge d \leqslant 4$. Note that in this domain of parameters, the quenched rate functional is always convex, whereas the annealed one is convex only for $\alpha \wedge d \leqslant 2$.

The proof of (5) and (6) has little to do with the formal computations made above. Let us describe it briefly in the annealed situation. First of all, note that because of the non-convexity of the annealed rate functional, it is hopeless to use the Gärtner–Ellis method (i.e. to pass by the log-Laplace transform) in order to obtain (5). Indeed, this strategy would lead to a rate functional, which is a Legendre transform, henceforth convex. Instead, we prove (5) by a contraction principle. The first remark is that by Brownian scaling invariance, the problem is to find the probability of $\{\langle L_{t/r^2}; v_t\rangle > y\}$ $(r > 0)$, where

- $L_t \triangleq \frac{1}{t}\int_0^t \delta_{B_s}\,ds$ is the Brownian occupation measure;
- $v_t(x) \triangleq \frac{t}{m}v(rx)$ is a kind of coarse-graining of the field, on a large scale r to be chosen later in such a way that $t/r^2 \gg 1$;
- and $\langle\cdot\,;\cdot\rangle$ is the duality bracket.

Now, the results of Donsker and Varadhan [12] give a large deviation principle (LDP) for L_{t/r^2} with speed t/r^2 and rate function $\mathcal{L}$. On the other hand, when $t \leqslant m$, we prove a LDP for v_t restricted to finite volume, with speed

$m^2 r^{\alpha\wedge d}/t^2$, and rate function L. Therefore, in the annealed case, a contraction principle should yield a LDP for Y_t, when one equals the speed rates for each marginal LDP, i.e. when

$$t/r^2 = \frac{m^2 r^{\alpha\wedge d}}{t^2}. \tag{9}$$

Moreover, the rate functional should be

$$\mathcal{I}(y) = \inf_{\mu,v}\left\{\mathcal{L}(\mu) + L(v);\ \langle\mu; v\rangle = y\right\}.$$

This leads to (5). Following this simple strategy, we are confronted to two technical problems. The first one is that L_{t/r^2} and v_r satisfy LDP in weak topologies where the function $(\mu, u) \mapsto \langle\mu; u\rangle$ is not continuous. We have thus to smoothen the Brownian occupation measure. We succeed in this regularization when $d \leqslant 3$.

The second one is that L_{t/r^2} does not satisfy a full LDP, i.e. the LDP upper bound is only valid for compact sets. We have thus to proceed to a compactification. The method we have chosen, has been developed by Donsker and Varadhan in [13] to study the large deviations for the volume of the Wiener sausage. It consists in replacing the Brownian motion on $\mathbb{R}^d$, by the Brownian motion on a torus of large radius. In [13], this projection on a torus clearly decreases the volume of the Wiener sausage. In our situation, such a monotony is no more obvious, and in order to make this comparison possible, we impose an additional assumption on the covariance function ($\|\phi\|_\infty = \phi(0)$), which we believe to be only an artefact of the method.

In the quenched case, the power function in the rate functional is convex in the domain of parameters where $C_q(\alpha, d)$ is positive. Therefore, the Gärtner–Ellis method is here appropriate to obtain the quenched upper bound. Denoting by $\sigma(R)$ the Brownian exit time of a ball of radius R, and using Brownian scaling invariance, our problem is then more or less equivalent to look for rough asymptotics of the log-Laplace transform restricted to $\sigma(Rt/r^2) > t/r^2$:

$$\begin{aligned}&\frac{r^2}{t}\log E_0\left[\exp\left(\frac{t}{r^2}\alpha\langle L_{t/r^2}; v_t\rangle\right);\ \sigma\left(Rt/r^2\right) > t/r^2\right]\\ &\quad= \frac{r^2}{t}\log E_0\left[\exp\left(\alpha\int_0^{t/r^2} v_t(B_s)\,ds\right);\ \sigma\left(Rt/r^2\right) > t/r^2\right].\end{aligned} \tag{10}$$

By Feynman–Kac formula, this behavior is related to the (quenched) behavior of the principal eigenvalue $\lambda(\alpha v_t, B(0, Rt/r^2))$ of the random operator $\frac{1}{2}\Delta + \alpha v_t$, with Dirichlet conditions on the boundary of $B(0, Rt/r^2)$. Similar quantities have been thoroughly studied both in the annealed and in the quenched setting, for different scalings, and for different kinds of potential v: see for instance Sznitman [34], Merkl and Wüthrich [29,31,30] for the case of a Poissonian potential; Gärtner and Molchanov [22,23], Biskup and König [5] for the i.i.d case; Gärtner and König [20], Gärtner, König and Molchanov [21] for more general potentials, including Gaussian ones. Except for [34], all these papers are based on a lemma whose first version appeared in Gärtner and König [20], and whose great merit is to enable the compactification in fairly general situations. This lemma asserts that $\lambda(\alpha v_t, B(0, Rt/r^2))$ is comparable with $\min_j \lambda(\alpha v_t, B(x_j, A))$ where the $B(x_j, A)$ are balls of fixed size A covering $B(0, Rt/r^2)$. Now using the LDP for v_t, it can be proved that this minimum has an a.s. limit, as soon as r is chosen so that

$$\frac{t}{r^2} = \exp\left(\frac{m^2 r^{\alpha\wedge d}}{t^2}\right). \tag{11}$$

This leads to the upper bound in (6).

Note that for r satisfying (11), we are actually interested in asymptotics for

$$E_0\left[\exp\left(\frac{\alpha}{\log(t)^{2/\alpha\wedge d}}\left(\frac{m}{t}\right)^{\frac{4}{\alpha\wedge d}-1}\int_0^t v(B_s)\,ds\right)\right]. \tag{12}$$

We are thus in a different asymptotic regime than the above cited papers, which except for [29,31,30], give asymptotics for

$$E_0\left[\exp\left(\alpha\int_0^t v(B_s)\,ds\right)\right],$$

in relation with the parabolic Anderson model.

The quenched lower bound is obtained by forcing the Brownian motion to stay in a region where the field v is performing a large deviation.

We end this introduction with two remarks. The first one concerns the ranges of the scaling $m(t)$ given by (7) and (8). The expression "moderate deviations" is usually used to designate the deviations for all the normalizations between the central limit theorem, and the large deviations. At least for the annealed case and $d=1$, the central limit normalization is $t^{1-\alpha\wedge 2/4}$ (see [3,4,24]). Hence our technics do not cover all the range of possible normalizations. The restriction $m(t)\geqslant t$ comes from the LDP for v_t, which is no more valid if $m(t)\ll t$. Note also that even the formal computation (5) do not cover all the possible normalizations between the central limit theorem and the large deviations, since $\alpha\wedge 1\leqslant \alpha\wedge 2$.

The second remark is to mention that the case where v consists of bounded and i.i.d, and $m(t)=t$, is treated in [2]. The case at hand in this paper, differs from [2] in essentially two directions: the introduction of correlations, and the unbondedness of the field. The main effect of correlations is to change the speed for the LDP of v_t. The unboudedness causes some difficulties in the regularization procedure, which result in the restriction $d\leqslant 3$. We believe however that Eqs. (5) and (6) should be true, whenever they make sense, i.e. whenever the constants are non-degenerate ($\alpha\wedge d\leqslant 4$).

The paper is organized as follows. In Section 2, we introduce the notations and state the main results. Section 3 is devoted to the LDP satisfied by v_t. In Section 4, we prove the LDP for the annealed case, while the quenched LDP is addressed in Section 5. Finally, Section 6 gathers some technical lemmas.

2. Notations and results

We begin with some notations used throughout the paper. When x and y are real, $x_+=\max(x,0)$, and $x\wedge y=\min(x,y)$.

When D is a subset of $\mathbb{R}^d$, we denote by $L^p(D)$ the space of measurable functions such that $\int_D |f(x)|^p\,dx<\infty$. The norm in $L^p(D)$ will be denoted by $\|\cdot\|_{p,D}$ or simply by $\|\cdot\|_p$ when $D=\mathbb{R}^d$. The conjugate element of p is denoted by p' ($\frac{1}{p'}=1-\frac{1}{p}$), and $\langle\cdot;\cdot\rangle$ is used for the duality bracket between L^p and $L^{p'}$, or more generally for the duality between measures and functions. When it makes sense, $*$ is the convolution operator.

When $f\in L^2(\mathbb{R}^d)$, $\hat f$ is its Fourier transform. When f is in the Schwartz space of rapidly decreasing functions and $\alpha\in\,]0,d[$, $\mathcal{R}_\alpha(f)$ is the convolution operator by the Riesz potential,

$$\mathcal{R}_\alpha(f)(x)=\int_{\mathbb{R}^d}\frac{f(y)}{|x-y|^{d-\alpha}}\,dy. \tag{13}$$

We recall some standard results on the operators $\mathcal{R}_\alpha$, which can be found for instance in [33] (see Lemma 2, p. 117, and Theorem 1, p. 119). First of all, when f and g are Schwartz functions,

$$\langle f;\mathcal{R}_\alpha(g)\rangle=\int_{\mathbb{R}^d}\frac{\hat f(k)\bar{\hat g}(k)}{|k|^\alpha}\,dk. \tag{14}$$

Moreover, $\mathcal{R}_\alpha$ can be extended to a continuous operator between L^p spaces:

$$\forall p \in \left]1, \frac{d}{\alpha}\right[, \ \forall f \in L^p(\mathbb{R}^d), \quad \left\|\mathcal{R}_\alpha(f)\right\|_q \leqslant C\|f\|_p, \quad \text{for } \frac{1}{q} = \frac{1}{p} - \frac{\alpha}{d}. \tag{15}$$

$\mathcal{R}_0$ will by analogy denote the identity operator.

Finally, a family $(Z_t, t \in \mathbb{R}^+)$ of random variables with values in the topological space $\mathcal{Z}$ is said to satisfy a full LDP, with speed $v(t)$ ($v(t) \gg 1$), and good rate function I if and only if

1. $I : \mathcal{Z} \mapsto \mathbb{R}^+$ has compact level sets.
2. LD lower bound.

   For all open set G of $\mathcal{Z}$, $\quad \liminf_{t\to\infty} \frac{1}{v(t)} \log P[Z_t \in G] \geqslant -\inf_{z\in G} I(z).$
3. LD upper bound.

   For all closed set F of $\mathcal{Z}$, $\quad \limsup_{t\to\infty} \frac{1}{v(t)} \log P[Z_t \in F] \leqslant -\inf_{z\in F} I(z).$

The LDP is said to be a weak one, if I is only lower semicontinuous, and the upper bound is only valid for compact subsets of $\mathcal{Z}$.

The Gaussian field. Let $(v(x), x \in \mathbb{R}^d)$ be a centered stationary Gaussian field with values in $\mathbb{R}$, defined on a probability space $(\mathcal{X}, \mathcal{G}, \mathbb{P})$. $\mathbb{E}$ will denote the expectation with respect to $\mathbb{P}$, so that the covariance function of v is defined by $K(x-y) \triangleq \mathbb{E}[v(x)v(y)]$.

We assume that v has a spectral density D, which is smooth, except at the origin, and which is rapidly decreasing at infinity. We will write

$$D(k) \triangleq \frac{\phi(k)}{|k|^{(d-\alpha)_+}}, \tag{16}$$

where $\alpha > 0$, and $\phi : \mathbb{R}^d \mapsto \mathbb{R}^+$ is even, smooth and rapidly decreasing. Without loss of generality, we assume that $\phi(0) = 1$.

D being integrable, $K(x) \triangleq \int_{\mathbb{R}^d} \mathrm{e}^{ikx} D(k)\, dk$ is continuous, and tends to zero at infinity. K attains its maximal value at 0. Actually, ϕ being rapidly decreasing, K is infinitely differentiable, with bounded derivatives. Hence, $\mathbb{E}[(v(x)-v(y))^2] = 2(K(0) - K(x-y)) \leqslant C\|x-y\|^2$, and it follows from Kolmogorov continuity criterion that v admits a continuous version.

The parameter α is linked to the decay of K at infinity. For $\alpha \geqslant d$, K is rapidly decreasing at infinity. Note that in this situation, K is integrable, and that by Fourier inverse transform $\int K(x)\, dx = D(0) = \phi(0) = 1$. For $0 < \alpha < d$, $K(x) \underset{\infty}{\asymp} \|x\|^{-\alpha}$, so that K is not in $L^1(\mathbb{R}^d)$.

For $A > 0$ and $r > 0$, let $Q(A) \triangleq [-A; A]^d$, and $v_t^A(x) = \frac{t}{m(t)} v(rx) \mathbb{1}_{Q(A)}(x)$. v_t^A will be viewed as a random variable with values in $L^2(Q(A))$ endowed with the weak topology defined by duality with test functions of $L^2(Q(A))$. A key result in all the sequel, is the following large deviation principle.

Theorem 1. *Assume that t, m and r are linked in such a way that $t \leqslant m(t)$ and $r(t) \gg 1$. When $t \to \infty$, for all $A > 0$, v_t^A satisfies a full LDP on $L^2(Q(A))$ with speed $m^2 r^{\alpha\wedge d}/t^2$, and good rate function*

$$L_A(u) \triangleq \sup_{f \in L^2(Q(A))} \left\{ \langle u, f\rangle - \frac{1}{2} \int_{\mathbb{R}^d} \frac{|\hat{f}(k)|^2}{\|k\|^{(d-\alpha)_+}}\, dk \right\}, \tag{17}$$

where $\hat{f}$ denotes the L^2-Fourier transform of f.

The diffusion in random Gaussian shear flow. Let $\{B_t, t \in \mathbb{R}^+\}$ be a d-dimensional Brownian motion, independent of the random field v. E_x denotes expectation under the Wiener measure starting from x. For $t > 0$, let $L_t \triangleq \frac{1}{t}\int_0^t \delta_{B_s}\,ds$ be the Brownian occupation measure. The main result of this paper concerns moderate deviations estimates for $Y_t \triangleq \frac{1}{m(t)}\int_0^t v(B_s)\,ds = \langle L_t, \frac{t}{m}v\rangle$ under the "quenched" measure P_0, and the "annealed" one $\tilde{P}_0 \triangleq \mathbb{P}\otimes P_0$. What is meant by "moderate deviations", is estimates of events such as $\{|Y_t - y| \leqslant \varepsilon\}$, under the quenched and annealed measures.

In establishing these moderate deviations results, we need the large deviations results of Donsker and Varadhan [12] about L_t viewed as a random variable with value in the space of probability measures $\mathcal{M}^1(\mathbb{R}^d)$. $\mathcal{M}^1(\mathbb{R}^d)$ is endowed with the topology of weak convergence defined by duality with bounded and continuous test functions. In this topological space, L_t satisfies a weak large deviation principle with speed t and rate function $\mathcal{L}$ defined by

$$\mathcal{L}(\mu) = \begin{cases} \frac{1}{2}\int_{\mathbb{R}^d} \|\nabla\sqrt{f}\|^2\,dx & \text{if } d\mu = f\,dx, \\ +\infty & \text{otherwise.} \end{cases} \tag{18}$$

We introduce now some notations related to the Brownian motion. First of all, when $\mathcal{D}$ is a domain of $\mathbb{R}^d$, $\sigma(\mathcal{D})$ is the exit time of $\mathcal{D}$. If $\mathcal{D} = Q(A)$, $\sigma(\mathcal{D})$ will be denoted by $\sigma(A)$. When V is a bounded measurable function on $\mathcal{D}$, $\lambda(V,\mathcal{D})$ is the principal eigenvalue of the Schrödinger operator $\frac{1}{2}\Delta + V$, with Dirichlet condition on the boundary of $\mathcal{D}$:

$$\begin{aligned}\lambda(V,\mathcal{D}) &= \inf_{f\in C_c^\infty(\mathcal{D})}\left\{-\left\langle f; \frac{1}{2}\Delta f + Vf\right\rangle;\ \|f\|_{2,\mathcal{D}} = 1\right\} \\ &= \inf_{f\in C_c^\infty(\mathcal{D})}\left\{\frac{1}{2}\int \|\nabla f\|^2\,dx - \int Vf^2\,dx;\ \|f\|_{2,\mathcal{D}} = 1\right\} \\ &= \inf_{\mu\in\mathcal{M}_1^0(\mathcal{D})}\big\{\mathcal{L}(\mu) - \langle\mu; V\rangle\big\}\end{aligned}$$

where $\mathcal{M}_1^0(\mathcal{D})$ denotes the space of probability measures with compact support in $\mathcal{D}$.

In relation with Y, we define the diffusion in the random shear flow drift. For $x \in \mathbb{R}^{d+1}$, let $x_1 \in \mathbb{R}^d$ and $x_2 \in \mathbb{R}$ be defined by the decomposition $x = (x_1, x_2)$. Let V be the random field on $\mathbb{R}^{d+1}$ with values in $\mathbb{R}^{d+1}$ defined by

$$V(x) = V(x_1, x_2) = \big(0, v(x_1)\big). \tag{19}$$

Let $W_t = (B_t, Z_t)$ $(Z_t \in \mathbb{R}, t \in \mathbb{R}^+)$ be a standard Brownian motion in $\mathbb{R}^{d+1}$ independent of V, and let X be the solution of the stochastic differential equation

$$dX_t = dW_t + V(X_s)\,ds, \quad X_0 = 0. \tag{20}$$

It is plain that $X_{1,t} = B_t$ and $X_{2,t} = Z_t + \int_0^t v(B_s)\,ds = Z_t + m(t)Y_t$. Moreover Z and Y are independent, so that estimates on Y lead straightforwardly to estimates on X.

The annealed moderate deviation principle

Theorem 2. *Assume that $d \leqslant 3$, that $t \leqslant m(t) \ll t^{3/2}$, and that ϕ reaches its maximal value at* 0.

There exists a constant $C_a(\alpha, d) \in\,]0, +\infty[$ given by the variational formulas (34), (35), (36), *such that under the annealed measure $\tilde{P}_0$, Y_t satisfies a full LDP in $\mathbb{R}$, with speed $v_a(t) \triangleq m^{\frac{4}{2+\alpha\wedge d}}/t^{\frac{4-\alpha\wedge d}{2+\alpha\wedge d}}$ and rate function $C_a(\alpha, d)|y|^{\frac{4}{2+\alpha\wedge d}}$.*

Remark. The additional assumption on ϕ is only needed in the LD upper bound, and we think it is unnecessary. Let us enlighten a little more this last claim. As already explained in the introduction, this assumption is needed to make possible the compactification method of Donsker and Varadhan. But assume for a moment that we are in the domain $\alpha \wedge d \leqslant 2$, where the rate functional is convex. Then we can use the Gärtner–Ellis method to obtain the LD upper bound. Proceeding in such a way, we obtain the LD upper bound without the assumption on ϕ.

As a corollary of the annealed moderate deviations for Y, we obtain the annealed moderate deviations for the diffusion X.

Corollary 3. *Assume that $d \leqslant 3$, that $t \leqslant m(t) \ll t^{3/2}$, and that ϕ reaches its maximal value at 0.*
For $x = (x_1, x_2) \in \mathbb{R}^{d+1}$, let $I(x)$ be defined by

$$I(x_1, x_2) = \begin{cases} C_a(\alpha, d)|x_2|^{4/(2+\alpha\wedge d)}, & \text{if } x_1 = 0; \\ +\infty, & \text{otherwise.} \end{cases} \tag{21}$$

Under the annealed measure $\tilde{P}_0$, $\frac{1}{m(t)} X_t$ satisfies a full LDP in $\mathbb{R}^{d+1}$, with speed $v_a(t)$ and rate function I.

Remark. Here again, the additional assumption on ϕ is only needed in the upper bound.

Proof. For all $\delta > 0$,

$$\tilde{P}_0\left[\left\| \frac{X_t}{m(t)} - \begin{pmatrix} 0 \\ Y_t \end{pmatrix} \right\| \geqslant \delta\right] = P_0\left[\left\| \begin{pmatrix} B_t/m(t) \\ Z_t/m(t) \end{pmatrix} \right\| \geqslant \delta\right].$$

Hence, $\forall \delta > 0$,

$$\lim_{t\to\infty} \frac{1}{v_a(t)} \log \tilde{P}_0\left[\left\| \frac{X_t}{m(t)} - \begin{pmatrix} 0 \\ Y_t \end{pmatrix} \right\| \geqslant \delta\right] = -\infty.$$

We have thus proved that $\frac{X_t}{m(t)}$ and $\binom{0}{Y_t}$ are exponentially equivalent (cf. Definition 4.2.10 of [11]), and Theorem 2 implies Corollary 3 (see Theorem 4.2.13 of [11]). □

The quenched moderate deviation principle

Theorem 4. *Assume that $d \leqslant 3$, and that $t \leqslant m(t) \ll t\sqrt{\log(t)}$. There exists a constant $C_q(\alpha, d) \in]0, +\infty[$ given by the variational formula* (53)*, such that under the quenched measure P_0, Y_t satisfies a full LDP in $\mathbb{R}$, with speed $v_q(t) \triangleq t(m/t\sqrt{\log(t)})^{4/(\alpha\wedge d)}$, and rate function $C_q(\alpha, d)|y|^{4/(\alpha\wedge d)}$.*

Again, we deduce from the quenched moderate deviations for Y, a similar statement for the diffusion X.

Corollary 5. *For $x = (x_1, x_2) \in \mathbb{R}^{d+1}$, let $J(x)$ be defined by*

$$J(x_1, x_2) = \begin{cases} C_q(\alpha, d)|x_2|^{\frac{4}{\alpha\wedge d}}, & \text{if } x_1 = 0; \\ +\infty, & \text{otherwise.} \end{cases} \tag{22}$$

Assume that $d \leqslant 3$, and that $t \leqslant m(t) \ll t\sqrt{\log(t)}$. Under the quenched measure P_0, $\frac{1}{m(t)} X_t$ satisfies a full LDP in $\mathbb{R}^{d+1}$, with speed $v_q(t)$ and rate function J.

Proof. The result follows again from the exponential equivalence under the quenched measure, of Y_t and $\frac{X_t}{m(t)}$. □

3. Large deviations for the Gaussian field

The aim of this section is to prove Theorem 1, i.e. the LDP for

$$v_t^A(x) \triangleq \frac{t}{m} v(rx) \mathbb{1}_{Q(A)}(x).$$

We begin with the lower bound. For $u_0 \in L_2(Q(A))$, I a finite set, $f_i \in L^2(Q(A))$, and $\varepsilon > 0$, set

$$\mathcal{V}\big(u_0; (f_i)_{i\in I}, \varepsilon\big) \triangleq \big\{u \in L^2\big(Q(A)\big);\ \forall i \in I, \big|\langle u, f_i\rangle - \langle u_0, f_i\rangle\big| \leqslant \varepsilon\big\}.$$

These sets form a basis of the weak topology on $L^2(Q(A))$. The lower bound follows then from

Lemma 6. *Let t, m and r be linked in such a way that $t \leqslant m(t)$, and $r(t) \gg 1$. For all $A > 0$, for all finite set I, for all u_0, f_i in $L^2(Q(A))$,*

$$\lim_{\varepsilon\to 0} \varliminf_{t\to\infty} \frac{t^2}{m^2 r^{\alpha\wedge d}} \log \mathbb{P}\big[v_t^A \in \mathcal{V}\big(u_0; (f_i)_{i\in I}, \varepsilon\big)\big] \geqslant -L_A(u_0).$$

Proof. We can assume without loss of generality that the functions f_i are linearly independent in $L^2(Q(A))$. Let $N \triangleq |I|$, and Z the element of $\mathbb{R}^N$, whose ith coordinate is $\langle f_i, v_t^A\rangle$. Z is a centered Gaussian vector, with covariance matrix σ_t^2 given by

$$\big(\sigma_t^2\big)_{i,j} = \frac{t^2}{m^2} \int \int_{Q(A)\times Q(A)} K\big(r(x-y)\big) f_i(x) f_j(y)\, dx\, dy.$$

Note that for $\alpha \geqslant d$, K is rapidly decreasing. Hence, $r^d \int K(r(\cdot - y)) f_j(y)\, dy$ converges to f_j in $L^2(\mathbb{R}^d)$ when $r \to \infty$, and $\lim_{t\to\infty} \frac{m^2 r^d}{t^2} (\sigma_t^2)_{i,j} = \langle f_i, f_j\rangle = \langle \hat{f}_i, \hat{f}_j\rangle$ (remind that $\int K(x)\, dx = \phi(0) = 1$).

On the other side, when $\alpha \in]0, d[$,

$$\frac{m^2 r^\alpha}{t^2} \big(\sigma_t^2\big)_{i,j} = \int_{\mathbb{R}^d} \frac{\phi(k/r)}{\|k\|^{d-\alpha}} \hat{f}_i(k) \bar{\hat{f}}_j(k)\, dk.$$

It follows then from Lemma 21 in Section 6 and dominated convergence that

$$\lim_{t\to\infty} \frac{m^2 r^\alpha}{t^2} \big(\sigma_t^2\big)_{i,j} = \int \hat{f}_i(k) \bar{\hat{f}}_j(k) \frac{dk}{\|k\|^{d-\alpha}}.$$

We have thus proved that

$$\lim_{t\to\infty} \frac{m^2 r^{\alpha\wedge d}}{t^2} \big(\sigma_t^2\big)_{i,j} = \int \hat{f}_i(k) \bar{\hat{f}}_j(k) \frac{dk}{\|k\|^{(d-\alpha)_+}} \triangleq \big(\sigma_\infty^2\big)_{i,j}. \tag{23}$$

Note that by linear independence of the functions f_i, the limiting matrix is positive definite, and the same is true for σ_t^2 for sufficiently large t. Setting $z_0 = (\langle f_i, u_0\rangle)_{i\in I}$, and denoting for $z \in \mathbb{R}^d$, $\|z\|^2_{(\sigma_t^2)^{-1}} = {}^t z (\sigma_t^2)^{-1} z$, we have

$$\begin{aligned}
\mathbb{P}\big[v_t^A \in \mathcal{V}\big(u_0; (f_i)_{i\in I}, \varepsilon\big)\big] &= \mathbb{P}\big[\|Z - z_0\|_\infty \leqslant \varepsilon\big] \\
&= \int_{\|z-z_0\|_\infty \leqslant \varepsilon} \exp\bigg(-\frac{\|z\|^2_{(\sigma_t^2)^{-1}}}{2}\bigg) \frac{dz}{\sqrt{2\pi}^N \sqrt{\det(\sigma_t^2)}} \\
&\geqslant \exp\Bigg(-\frac{1}{2}\Bigg(\|z_0\|_{(\sigma_t^2)^{-1}} + \varepsilon \sqrt{\sum_{i,j} \big|\big(\sigma_t^2\big)^{-1}_{i,j}\big|}\Bigg)^2\Bigg) \frac{(2\varepsilon)^N}{\sqrt{2\pi}^N \sqrt{\det(\sigma_t^2)}}.
\end{aligned}$$

By (23), $\lim_{t\to\infty} \frac{t^2}{m^2 r^{\alpha\wedge d}} \log\det(\sigma_t^2) = 0$. Hence,

$$\begin{aligned}&\varliminf_{t\to\infty}\frac{t^2}{m^2r^{\alpha\wedge d}}\log\mathbb{P}\big[v_t^A\in\mathcal{V}\big(u_0;(f_i)_{i\in I},\varepsilon\big)\big]\\&\quad\geqslant-\frac12\varlimsup_{t\to\infty}\left(\|z_0\|_{(\frac{m^2r^{\alpha\wedge d}}{t^2}\sigma_t^2)^{-1}}+\varepsilon\sqrt{\sum_{i,j}\left|\left(\frac{m^2r^{\alpha\wedge d}}{t^2}\sigma_t^2\right)^{-1}_{i,j}\right|}\right)^2\\&\quad=-\frac12\left(\|z_0\|_{(\sigma_\infty^2)^{-1}}+\varepsilon\sqrt{\sum_{i,j}\left|(\sigma_\infty^2)^{-1}_{i,j}\right|}\right)^2.\end{aligned}$$

Letting ε go to zero leads to

$$\begin{aligned}&\lim_{\varepsilon\to0}\varliminf_{t\to\infty}\frac{t^2}{m^2r^{\alpha\wedge d}}\log\mathbb{P}\big[v_t^A\in\mathcal{V}\big(u_0;(f_i)_{i\in I},\varepsilon\big)\big]\\&\quad\geqslant-\frac12\|z_0\|^2_{(\sigma_\infty^2)^{-1}}=-\sup_{z\in\mathbb{R}^N}\left\{(z,z_0)-\frac12\|z\|^2_{\sigma_\infty^2}\right\}\\&\quad\geqslant-\sup_{z\in\mathbb{R}^N}\left\{\left(u_0,\sum_i z_if_i\right)-\frac12\int\frac{|\sum_i z_i\hat f_i|^2}{\|k\|^{(d-\alpha)_+}}\,dk\right\}\geqslant-L_A(u_0).\qquad\square\end{aligned}$$

The weak large deviations upper bound (i.e. the upper bound for compact subsets) is a consequence of Theorem 4.5.1 in [11] and of the following lemma.

Lemma 7. *Let t, m and r be linked in such a way that $t\leqslant m(t)$, and $r(t)\gg1$. For all $A>0$, for all $f\in L^2(Q(A))$,*

$$\lim_{t\to\infty}\frac{t^2}{m^2r^{\alpha\wedge d}}\log\mathbb{E}\left[\exp\left(\frac{m^2r^{\alpha\wedge d}}{t^2}\langle v_t^A,f\rangle\right)\right]=\frac12\int_{\mathbb{R}^d}\frac{|\hat f(k)|^2}{\|k\|^{(d-\alpha)_+}}\,dk.\tag{24}$$

Proof. This lemma is a straightforward consequence of the fact that $\langle v_t^A,f\rangle$ is a centered Gaussian random variable with variance σ_t^2, and of limit (23). $\square$

Theorem 1 is thus proved as soon as the exponential tightness is established. Since closed balls in $L^2(Q(A))$ are weakly relatively compact, it is enough to prove

Lemma 8. *Let t, m and r be linked in such a way that $t\leqslant m(t)$, and $r(t)\gg1$. For all $A>0$,*

$$\lim_{L\to\infty}\varlimsup_{t\to\infty}\frac{t^2}{m^2r^{\alpha\wedge d}}\log\mathbb{P}\big[\|v_t^A\|_2\geqslant L\big]=-\infty.\tag{25}$$

Moreover, L_A is a good rate function (i.e. has compact level sets).

Proof. The goodness of L_A is a consequence of the lower bound and of (25), which is therefore the only point to prove.

The operator $K_t: f\in L^2(Q(A))\mapsto K_t*f=\frac{t^2}{m^2}\int K(r(\cdot-y))f(y)\,dy$ is a trace-class operator in $L^2(Q(A))$, whose trace is

$$\operatorname{tr}(K_t)=\frac{t^2}{m^2}\int_{Q(A)}K\big(r(x-x)\big)\,dx=CA^dK(0)\frac{t^2}{m^2}.\tag{26}$$

Let then (f_i) be an orthonormal basis of $L^2(Q(A))$ of eigenfunctions of K_t, associated to the eigenvalues λ_i. Writing the decomposition of v_t^A on this basis yields

$$v_t^A = \sum_i \sqrt{\lambda_i} Z_i f_i,$$

where the random variables Z_i are i.i.d with common law $\mathcal{N}(0,1)$. Hence, $\|v_t^A\|_2^2 = \sum_i \lambda_i Z_i^2$, and $\forall \lambda < \frac{1}{2\lambda_{\max}(K_t)}$,

$$\mathbb{E}\big[\exp\big(\lambda \|v_t^A\|_2^2\big)\big] = \prod_i \frac{1}{\sqrt{1-2\lambda\lambda_i}}.$$

Taking $\lambda = \frac{1}{4\lambda_{\max}(K_t)}$, we get

$$\mathbb{P}\big[\|v_t^A\|_2^2 \geqslant L^2\big] \leqslant \exp\left(-\frac{L^2}{4\lambda_{\max}(K_t)}\right) \prod_i \frac{1}{\sqrt{1-\frac{\lambda_i}{2\lambda_{\max}(K_t)}}} \leqslant \exp\left(-\frac{L^2}{4\lambda_{\max}(K_t)}\right) \exp\left(\frac{\operatorname{tr}(K_t)}{2\lambda_{\max}(K_t)}\right),$$

since $\log(1-x) \geqslant -2x$ for all $x \in [0,1/2]$. Now,

$$\begin{aligned} \frac{m^2 r^{\alpha\wedge d}}{t^2} \lambda_{\max}(K_t) &= r^{\alpha\wedge d} \sup_{f, \|f\|_2=1} \int K\big(r(x-y)\big) f(x) f(y)\, dx\, dy \\ &= \sup_{f,\|f\|_2=1} \int \frac{\phi(k/r)}{\|k\|^{(d-\alpha)_+}} \big|\hat{f}(k)\big|^2 dk \leqslant C(d,\alpha) \|\phi\|_\infty A^{(d-\alpha)_+}, \end{aligned}$$

by Lemma 21. By (26), we get then that $\forall L$ such that $L^2 > CK(0)A^d$,

$$\mathbb{P}\big[\|v_t^A\|_2^2 \geqslant L^2\big] \leqslant \exp\left(-\frac{m^2 r^{\alpha\wedge d}}{t^2 C_1}\left(L^2 - C_2 \frac{t^2}{m^2}\right)\right)$$

for some constants C_1, C_2 (depending on d, α, ϕ, A). Since $m(t) \geqslant t$, this ends the proof of Lemma 8. □

4. Annealed moderate deviations

This section is devoted to the proof of Theorem 2. Using scaling invariance of the Brownian motion, $\langle L_t; \frac{t}{m} v\rangle \stackrel{\text{(law)}}{=} \langle L_{t/r^2}; v_t\rangle$, where $v_t(x) \triangleq \frac{t}{m} v(rx)$. We set for convenience $\tau = t/r^2$.

In all the sequel, ψ is a smooth, non negative, rotationally invariant function with support in $Q(1)$, such that $\int \psi(x)\,dx = 1$; and for $\delta > 0$, set $\psi_\delta(x) \triangleq \psi(x/\delta)/\delta^d$.

Step 1. Smoothing the field

Lemma 9. *Let r, m and t be linked in such a way that $t \leqslant m \ll t^{3/2}$, and $\tau = \frac{m^2}{t^2} r^{\alpha\wedge d}$ (i.e. $r = (t^3/m^2)^{\frac{1}{2+\alpha\wedge d}} \gg 1$). For $d \leqslant 3$, and for all $\varepsilon > 0$,*

$$\overline{\lim_{\delta\to 0}}\, \overline{\lim_{t\to\infty}} \frac{1}{\tau} \log \tilde{P}_0\big[\big|\langle L_\tau; v_t - \psi_\delta * v_t\rangle\big| \geqslant \varepsilon\big] = -\infty.$$

Proof. Since

$$\limsup_{\tau\to\infty} \frac{1}{\tau} \log P_0\big(\sigma(R\tau) \leqslant \tau\big) = -R^2/2, \tag{27}$$

and since v and $-v$ have the same law, it is enough to prove that $\forall R > 0$, $\forall \varepsilon > 0$,

$$\varlimsup_{\delta\to 0}\varlimsup_{t\to\infty}\frac{1}{\tau}\log \tilde{P}_0\big[\langle L_\tau; v_t-\psi_\delta * v_t\rangle \geqslant \varepsilon;\ \sigma(R\tau)>\tau\big]=-\infty.$$

We begin to prove a quenched bound on the probability of the event

$$\mathcal{A}\triangleq\big\{\langle L_\tau; v_t-\psi_\delta * v_t\rangle \geqslant \varepsilon;\ \sigma(R\tau)>\tau\big\}.$$

For $A>0$ and $j\in\mathbb{Z}^d$, let $Q_j(A)$ be the box of center $x_j=2jA$ and radius A; i.e., $Q_j(A)=x_j+[-A,A]^d$. We partition $Q(R\tau)$ with such boxes. The following lemma, whose proof is given in Section 6, gives estimates of $P_0(\mathcal{A})$ in terms of $\max_j\|v_t\|_{2,Q_j(A)}$ where the maximum runs over the indices of boxes $Q_j(A)$ which intersect $Q(R\tau)$.

Lemma 10. *For $d\leqslant 3$, there exists constants C_1, C_2 (depending only on d,ψ) such that $\mathbb{P}$-a.s., for all $\delta, R, A, \varepsilon>0$,*

$$\begin{aligned}&P_0\big[\langle L_\tau; v_t-\psi_\delta * v_t\rangle \geqslant \varepsilon; \sigma(R\tau)>\tau\big]\\&\quad\leqslant C_1\left(1+\frac{\tau^{d/2}}{A^d}+\frac{\|v_t\|_{\infty,Q(R\tau)}^{d/2}}{\varepsilon^{d/2}}\right)\exp\left\{-\tau\frac{C_2\varepsilon^{4/(d+1)}}{(\sqrt{\delta}\max_j\|v_t\|_{2,Q_j(A+\delta)})^{4/(d+1)}}\right\}\mathrm{e}^{\tau C_1/A^2},\end{aligned}$$

where the maximum runs over the indices j of the boxes $Q_j(A)$ which intersect $Q(R\tau)$.

Apply now Hölder inequality to get $\forall p>1$,

$$\tilde{P}_0(\mathcal{A})\leqslant C_1\mathrm{e}^{\tau\frac{C_1}{A^2}}\mathbb{E}\left[\left(1+\frac{\tau^{d/2}}{A^d}+\frac{\|v_t\|_{\infty,Q(R\tau)}^{d/2}}{\varepsilon^{d/2}}\right)^{p'}\right]^{1/p'}\mathbb{E}\left[\exp\left\{-\tau\frac{pC_2\varepsilon^{4/(d+1)}}{(\sqrt{\delta}\max_j\|v_t\|_{2,Q_j(A+\delta)})^{4/(d+1)}}\right\}\right]^{1/p}.$$

The proof of Lemma 9 is then completed if we show

$$\forall p>0,\ \forall R>0,\quad \limsup_{t\to\infty}\frac{1}{\tau}\log\mathbb{E}\big[\|v_t\|_{\infty,Q(R\tau)}^p\big]\leqslant 0; \tag{28}$$

$$\forall A>0,\ \forall p>0,\quad \limsup_{\gamma\to\infty}\limsup_{t\to\infty}\frac{1}{\tau}\log\mathbb{E}\big[\mathrm{e}^{-\tau\gamma/\max_j\|v_t\|_{2,Q_j(A)}^p}\big]=-\infty. \tag{29}$$

Limit (28) is an easy consequence of Lemma 23 in Section 6. Turning to (29), note that for all $L>0$,

$$\begin{aligned}\mathbb{E}\big[\mathrm{e}^{-\tau\gamma/\max_j\|v_t\|_{2,Q_j(A)}^p}\big]&\leqslant\mathbb{P}\big[\max_j\|v_t\|_{2,Q_j(A)}\geqslant L\big]+\mathrm{e}^{-\tau\frac{\gamma}{L^p}}\\&\leqslant\sum_j\mathbb{P}\big[\|v_t\|_{2,Q_j(A)}\geqslant L\big]+\mathrm{e}^{-\tau\frac{\gamma}{L^p}}\\&\leqslant C\left(\frac{R\tau}{A}\right)^d\mathbb{P}\big[\|v_t\|_{2,Q(A)}\geqslant L\big]+\mathrm{e}^{-\tau\frac{\gamma}{L^p}},\end{aligned}$$

by stationarity. Hence, for all $L>0$,

$$\varlimsup_{t\to\infty}\frac{1}{\tau}\log\mathbb{E}\big[\mathrm{e}^{-\tau\gamma/\max_j\|v_t\|_{2,Q_j(A)}^p}\big]\leqslant\max\left\{-\frac{\gamma}{L^p};\ \varlimsup_{t\to\infty}\frac{1}{\tau}\log\mathbb{P}\big[\|v_t\|_{2,Q(A)}\geqslant L\big]\right\}.$$

(29) follows from Lemma 8 by sending first γ to ∞, and then L to ∞. □

Step 2. Annealed lower bound Lemma 9 states the exponential equivalence between $\langle L_\tau; v_t\rangle$, and $\langle L_\tau; \psi_\delta * v_t\rangle$. Hence, the problem is reduced to find annealed deviations for $\langle L_\tau; \psi_\delta * v_t\rangle$.

Lemma 11. *For $u \in L^2(\mathbb{R}^d)$, set*

$$L(u) \triangleq \sup_{f\in L^2(\mathbb{R}^d)} \left\{\langle u; f\rangle - \frac{1}{2}\int \frac{|\hat{f}(k)|^2}{\|k\|^{(d-\alpha)_+}}\,dk\right\} = \frac{1}{2}\int \left|\hat{u}(k)\right|^2 \|k\|^{(d-\alpha)_+}\,dk. \tag{30}$$

For $\delta > 0$ and $y \in \mathbb{R}^d$, define

$$\mathcal{I}_\delta(y) = \inf\left\{\mathcal{L}(\mu) + L(u);\ \mu \in \mathcal{M}_1(\mathbb{R}^d),\ u \in L^2(\mathbb{R}^d),\ \langle \mu; \psi_\delta * u\rangle = y\right\}. \tag{31}$$

Let r, m and t be such that $t \leqslant m \ll t^{3/2}$, and $\tau = \frac{m^2}{t^2} r^{\alpha\wedge d}$. For all $\delta > 0$, for all $y \in \mathbb{R}^d$, and all $\varepsilon > 0$,

$$\liminf_{t\to\infty} \frac{1}{\tau} \log \tilde{P}_0\left[\left|\langle L_\tau; \psi_\delta * v_t\rangle - y\right| < \varepsilon\right] \geqslant -\inf\left\{\mathcal{I}_\delta(z);\ |z-y| < \varepsilon\right\}.$$

Proof. Let $A > 0$ be fixed.

$$\tilde{P}_0\left[\left|\langle L_\tau; \psi_\delta * v_t\rangle - y\right| < \varepsilon\right] \geqslant \tilde{P}_0\left[\left|\langle L_\tau; \psi_\delta * v_t\rangle - y\right| < \varepsilon;\ \sigma(A) > \tau\right].$$

On $\{\sigma(A) > \tau\}$, only the values of v_t on $Q(A+\delta)$ are relevant, so that we can replace v_t by $v_t^{A+\delta}$ in the right hand side of the above inequality. Let $\mu_0 \in \mathcal{M}_1^0(Q(A))$ and $u_0 \in L^2(Q(A+\delta))$ be such that $|\langle \mu_0; \psi_\delta * u_0\rangle - y| < \varepsilon$. By continuity of the function $(\mu, u) \in \mathcal{M}_1^0(Q(A)) \times L^2(Q(A+\delta)) \mapsto \langle \mu; \psi_\delta * u\rangle$ (see Lemma 24 in Section 6), one can find weak neighborhoods $\mathcal{V}_1(u_0)$ and $\mathcal{V}_2(\mu_0)$ such that

$$u \in \mathcal{V}_1(u_0) \text{ and } \mu \in \mathcal{V}_2(\mu_0) \quad \Rightarrow \quad \left|\langle \mu; \psi_\delta * u\rangle - y\right| < \varepsilon.$$

One get then by independence of v and B

$$\tilde{P}_0\left[\left|\langle L_\tau; \psi_\delta * v_t\rangle - y\right| < \varepsilon\right] \geqslant P_0\left[\sigma(A) > \tau,\ L_\tau \in \mathcal{V}_2(\mu_0)\right]\mathbb{P}\left[v_t^{A+\delta} \in \mathcal{V}_1(u_0)\right].$$

It follows now from the large deviations results on L_τ and $v_t^{A+\delta}$ that for all $A > 0$,

$$\liminf_{t\to\infty} \frac{1}{\tau} \log \tilde{P}_0\left[\left|\langle L_\tau; \psi_\delta * v_t\rangle - y\right| < \varepsilon\right] \geqslant -\mathcal{L}(\mu_0) - L_{A+\delta}(u_0).$$

Taking the supremum on μ_0 and u_0 leads to a lower bound with the rate functional

$$\mathcal{I}_{A,\delta}(y) \triangleq \inf\left\{\mathcal{L}(\mu) + L_{A+\delta}(u);\ \mu \in \mathcal{M}_0^1\left(Q(A)\right),\ u \in L^2\left(Q(A+\delta)\right),\ \langle \mu; \psi_\delta * u\rangle = y\right\};$$

i.e. $\liminf_{t\to\infty} \frac{1}{\tau} \log \tilde{P}_0[|\langle L_\tau; \psi_\delta * v_t\rangle - y| < \varepsilon] \geqslant -\inf\{\mathcal{I}_{A,\delta}(z); |z-y| < \varepsilon\}$. We send now A to ∞. It is easy to see that the infimum defining $\mathcal{I}_\delta$ can be restricted to probability measures with compact support. More precisely,

$$\inf\left\{\mathcal{I}_\delta(z); |z-y| < \varepsilon\right\} = \inf\left\{\mathcal{L}(\mu) + L(u);\ \mu \in \mathcal{M}_1^0(\mathbb{R}^d),\ u \in L^2(\mathbb{R}^d),\ \left|\langle \mu; \psi_\delta * u\rangle - y\right| < \varepsilon\right\}.$$

Therefore,

$$\limsup_{A\to\infty} \inf\left\{\mathcal{I}_{A+\delta}(z);\ |z-y| < \varepsilon\right\} \leqslant \inf\left\{\mathcal{I}_\delta(z);\ |z-y| < \varepsilon\right\}. \tag{32}$$

Indeed, let $\mu \in \mathcal{M}_1^0(\mathbb{R}^d)$, and $u \in L^2(\mathbb{R}^d)$ be such that $|\langle \mu; \psi_\delta * u\rangle - y| < \varepsilon$. Let A be such that $\mu \in \mathcal{M}_1^0(Q(A))$. Then $\langle \mu; \psi_\delta * u\rangle = \langle \mu; \psi_\delta * (u|_{Q(A+\delta)})\rangle$, and

$$L(u) \geqslant \sup\left\{\langle u; f\rangle - \frac{1}{2}\int \frac{|\hat{f}(k)|^2}{\|k\|^{(d-\alpha)_+}}\,dk;\ f \in L^2\left(Q(A+\delta)\right)\right\} = L_{A+\delta}(u|_{Q(A+\delta)}),$$

which implies (32). □

From Lemma 11, we deduce the lower bound for $\langle L_\tau; v_t\rangle$.

Corollary 12. *Let r, m and t be such that $t \leqslant m \ll t^{3/2}$, and $\tau = \frac{m^2}{t^2} r^{\alpha\wedge d}$. Assume $d \leqslant 3$. Then, for all $y \in \mathbb{R}$, and all $\varepsilon > 0$,*

$$\liminf_{t\to\infty} \frac{1}{\tau} \log \tilde{P}_0\big[\big|\langle L_\tau; v_t\rangle - y\big| < \varepsilon\big] \geqslant -C_a(\alpha, d)|y|^{\frac{4}{2+\alpha\wedge d}}, \tag{33}$$

where $C_a(\alpha, d) \in]0, +\infty[$ is given by the variational formulas

$$C_a(\alpha, d) = \inf\big\{\mathcal{L}(\mu) + L(u);\ \mu \in \mathcal{M}_1(\mathbb{R}^d),\ u \in L^2(\mathbb{R}^d),\ \langle\mu; u\rangle = 1\big\} \tag{34}$$

$$= \inf\left\{\mathcal{L}(\mu) + \frac{L(u)}{\langle\mu; u\rangle^2};\ \mu \in \mathcal{M}_1(\mathbb{R}^d),\ u \in L^2(\mathbb{R}^d)\right\} \tag{35}$$

$$= \left(\frac{\alpha\wedge d}{2}\right)^{2/(2+\alpha\wedge d)} \left(1 + \frac{2}{\alpha\wedge d}\right)\left(\inf_{u,\mu}\left\{\frac{L^2(u)\mathcal{L}(\mu)^{\alpha\wedge d}}{\langle\mu; u\rangle^4}\right\}\right)^{1/(2+\alpha\wedge d)}. \tag{36}$$

Proof. Set

$$\mathcal{I}(y) = \inf\big\{\mathcal{L}(\mu) + L(u);\ \mu \in \mathcal{M}_1(\mathbb{R}^d);\ u \in L^2(\mathbb{R}^d);\ \langle\mu; u\rangle = y\big\}. \tag{37}$$

By Lemmas 9 and 11, it is enough to prove that:

$$\text{(i)}\ \limsup_{\delta\to 0} \inf\big\{\mathcal{I}_\delta(z);\ |z - y| < \varepsilon\big\} \leqslant \mathcal{I}(y); \tag{38}$$

$$\text{(ii)}\ \mathcal{I}(y) = C_a(\alpha, d)|y|^{4/(2+\alpha\wedge d)}. \tag{39}$$

We begin to show (38). Take μ and u, such that $\langle\mu; u\rangle = y$ and $\mathcal{L}(\mu) + L(u) < \infty$. By Lemma 22, there exists C such that

$$\big|\langle\mu; \psi_\delta * u\rangle - y\big| = \big|\langle\mu; \psi_\delta * u\rangle - \langle\mu; u\rangle\big| \leqslant C\sqrt{\delta}\,\|u\|_2\mathcal{L}(\mu)^{(d+1)/4}.$$

Hence, for $\delta \leqslant \delta_0(\varepsilon, u, \mu)$, $|\langle\mu; \psi_\delta * u\rangle - y| < \varepsilon$, and for such δ,

$$\inf\big\{\mathcal{I}_\delta(z);\ |z - y| < \varepsilon\big\} \leqslant \mathcal{L}(\mu) + L(u).$$

μ and u being arbitrary, this leads to (38).

Let us now prove (39). First of all, making the change of variable $u \mapsto y\lambda u$ ($\lambda \in \mathbb{R}$), and noting that $L(\lambda y u) = \lambda^2 y^2 L(u)$, it is easy to see that

$$\mathcal{I}(y) = \inf_{u,\mu}\left\{\mathcal{L}(\mu) + y^2\frac{L(u)}{\langle\mu; u\rangle^2}\right\}. \tag{40}$$

We now apply dilations. For $\lambda > 0$, μ such that $\mathcal{L}(\mu) < \infty$, and $u \in L^2(\mathbb{R}^d)$, set

$$d\mu_\lambda(x) = \lambda^d \frac{d\mu}{dx}(\lambda x)\,dx, \quad \text{and} \quad u_\lambda(x) = u(\lambda x).$$

Then $\langle\mu_\lambda; u_\lambda\rangle = \langle\mu; u\rangle$, $\mathcal{L}(\mu_\lambda) = \lambda^2\mathcal{L}(\mu)$, and $L(u_\lambda) = \lambda^{-\alpha\wedge d}L(u)$. Doing these changes of variables in (40), then optimizing in λ yields (39), with $C_a(\alpha, d)$ given by expression (36). The two other expressions (34) and (35) are obtained by taking $y = 1$ in (37) and (40). It remains now to prove that $C_a(\alpha, d) \in]0, \infty[$, and this is the statement of Lemma 25 of Section 6. □

Step 3. Annealed upper bound

Lemma 13. *Let r, m and t be such that $t \leqslant m \ll t^{3/2}$, and $\tau = \frac{m^2}{t^2} r^{\alpha\wedge d}$. Assume that ϕ reaches its maximal value at 0. For all $y \in \mathbb{R}^+$, and all $\delta > 0$, let*

$$I_\delta(y) \triangleq \inf\left\{\mathcal{L}(\mu) + \frac{y^2}{2\langle \mathcal{R}_{(d-\alpha)_+}(\psi_\delta * \mu); \psi_\delta * \mu\rangle};\ \mu \in \mathcal{M}_1(\mathbb{R}^d)\right\}. \tag{41}$$

Then,

$$\limsup_{t\to\infty} \frac{1}{\tau} \log \tilde{P}_0\big[\langle L_\tau; \psi_\delta * v_t\rangle \geqslant y\big] \leqslant -I_\delta(y).$$

Proof. P_0-a.s., for all $a > 0$,

$$\begin{aligned}\mathbb{P}\big[\langle L_\tau; \psi_\delta * v_t\rangle \geqslant y\big] &\leqslant \mathrm{e}^{-\tau a y}\mathbb{E}\left[\exp\left(a\int_0^\tau \psi_\delta * v_t(B_s)\,ds\right)\right]\\ &= \mathrm{e}^{-\tau a y}\exp\left(\tau\frac{a^2}{2}\iint r^{\alpha\wedge d}K\big(r(x-y)\big)\psi_\delta * L_\tau(x)\psi_\delta * L_\tau(y)\,dx\,dy\right).\end{aligned}$$

Note that the function $\psi_\delta * L_\tau$ is in $L^p(\mathbb{R}^d)$ for all p, since the same is true for ψ_δ. Now, for any $f \in \bigcap_p L_p(\mathbb{R}^d)$,

$$\begin{aligned}\iint r^{\alpha\wedge d}K\big(r(x-y)\big)f(x)f(y)\,dx\,dy &= \int \frac{\phi(k/r)}{\|k\|^{(d-\alpha)_+}}\big|\hat{f}(k)\big|^2\,dk\\ &\leqslant \int \frac{|\hat{f}(k)|^2}{\|k\|^{(d-\alpha)_+}}\,dk \quad \text{since } 0 \leqslant \phi \leqslant 1,\\ &= \big\langle \mathcal{R}_{(d-\alpha)+}(f); f\big\rangle.\end{aligned}$$

Now, for $\alpha < d$, and for positive function f,

$$\begin{aligned}\big\langle \mathcal{R}_{(d-\alpha)+}(f); f\big\rangle &= \iint \frac{f(x)f(y)}{\|x-y\|^\alpha}\,dx\,dy\\ &= \iint_{Q(A)\times Q(A)} \sum_{i,j\in\mathbb{Z}^d} \frac{f(x+2iA)f(y+2jA)}{\|x-y+2(i-j)A\|^\alpha}\,dx\,dy\\ &\leqslant \iint_{Q(A)\times Q(A)} \frac{f_A(x_A)f_A(y_A)}{d_A(x_A,y_A)^\alpha}\,dx\,dy\end{aligned}$$

where for $x \in \mathbb{R}^d$, x_A denotes the projection of x on the torus $\mathcal{T}(A)$ of radius A; d_A is the Riemanian metric on the torus $\mathcal{T}(A)$,

$$d_A(x_A, y_A) = \min\big\{\|x-y-2jA\|;\ j \in \mathbb{Z}^d\big\};$$

and f_A is the periodized function $f_A(x_A) = \sum_{j\in\mathbb{Z}^d} f(x+2jA)$.

On the other side, for $\alpha \geqslant d$, it is clear that if f is positive,

$$\int f^2(x)\,dx = \int_{Q(A)} \sum_{j\in\mathbb{Z}^d} f(x+2jA)^2\,dx \leqslant \int_{Q(A)} \left(\sum_j f(x+2jA)\right)^2 dx = \int_{Q(A)} f_A^2(x_A)\,dx.$$

Applying all the preceding to $f = \psi_\delta * L_\tau$, we are led to

$$\iint r^{\alpha\wedge d} K\big(r(x-y)\big)\psi_\delta * L_\tau(x)\psi_\delta * L_\tau(y)\,dx\,dy$$
$$\leqslant \begin{cases} \int_{\mathcal{T}(A)} (\psi_\delta * L_\tau)_A^2(x_A)\,dx_A, & \text{if } \alpha \geqslant d; \\ \iint_{\mathcal{T}(A)} \frac{(\psi_\delta * L_\tau)_A(x_A)(\psi_\delta * L_\tau)_A(y_A)}{d_A(x_A,y_A)^\alpha}\,dx_A\,dy_A, & \text{if } \alpha < d. \end{cases}$$

But, for $\delta < 2A$,

$$(\psi_\delta * L_\tau)_A(x_A) = \frac{1}{\tau}\int_0^\tau \sum_{i\in\mathbb{Z}^d} \psi_\delta\big(\|x + 2iA - B_s\|\big)\,ds = \frac{1}{\tau}\int_0^\tau \psi_\delta\big(d_A(x_A, B_s^A)\big)\,ds,$$

where B_s^A is the Brownian on the torus $\mathcal{T}(A)$. Let L_τ^A be the occupation measure of B^A, and $\psi_\delta^A : (x_A, y_A) \mapsto \psi_\delta(d_A(x_A, y_A))$. We have proved that P_0-a.s., for all $a > 0$, for all $A > 0$,

$$\mathbb{P}\big[\langle L_\tau; \psi_\delta * v_t\rangle \geqslant y\big] \leqslant \mathrm{e}^{-\tau a y} \exp\left(\tau \frac{a^2}{2} F_{\delta,A}\big(L_\tau^A\big)\right); \tag{42}$$

with

$$F_{\delta,A} : \mu \in \mathcal{M}_1\big(\mathcal{T}(A)\big) \mapsto \begin{cases} \int_{\mathcal{T}(A)} (\psi_\delta^A * \mu)^2(x_A)\,dx_A & \text{for } \alpha \geqslant d; \\ \iint_{\mathcal{T}(A)} \frac{(\psi_\delta^A * \mu)(x_A)(\psi_\delta^A * \mu)(y_A)}{d_A(x_A,y_A)^\alpha}\,dx_A\,dy_A & \text{for } 0 < \alpha < d. \end{cases} \tag{43}$$

Taking the optimal a in (42), then integrating with respect to P_0, yields

$$\tilde{P}_0\big[\langle L_\tau; \psi_\delta * v_t\rangle \geqslant y\big] \leqslant E_0\left[\exp\left(-\tau \frac{y^2}{2F_{\delta,A}(L_\tau^A)}\right)\right].$$

We are now in a favorable position to apply Varadhan integral lemma. Indeed, L_τ^A satisfies a full LDP in $\mathcal{M}_1(\mathcal{T}(A))$ (endowed with the weak convergence) with speed τ and good rate function $\mathcal{L}_A$. Moreover the function $\mu \in \mathcal{M}_1(\mathcal{T}(A)) \mapsto -\frac{y^2}{2F_{\delta,A}(\mu)}$ is obviously bounded above by 0, and is u.s.c. (see Lemma 26 in Section 6). It follows then from Lemma 4.3.6 in [11] that for all $A > 0$

$$\begin{aligned} &\limsup_{t\to\infty} \frac{1}{\tau} \log \tilde{P}_0\big[\langle L_\tau; \psi_\delta * v_t\rangle \geqslant y\big] \leqslant -I_{\delta,A}(y) \\ &\text{where } I_{\delta,A}(y) \triangleq \inf\left\{\frac{y^2}{2F_{\delta,A}(\mu)} + \mathcal{L}_A(\mu);\ \mu \in \mathcal{M}^1\big(\mathcal{T}(A)\big)\right\}. \end{aligned} \tag{44}$$

Take now the limit $A \to \infty$. The result follows from Lemma 27. □

We let now δ go to 0. Since v and $-v$ have the same law, Theorem 2 is an obvious consequence of the following lemma.

Lemma 14. *Let r, m and t be such that $t \leqslant m \ll t^{3/2}$, and $\tau = \frac{m^2}{t^2} r^{\alpha\wedge d}$. Assume that $d \leqslant 3$ and that ϕ reaches its maximal value at* 0. *For all $y \in \mathbb{R}^+$*

$$\limsup_{t\to\infty} \frac{1}{\tau} \log \tilde{P}_0\big[\langle L_\tau; v_t\rangle \geqslant y\big] \leqslant -C_a(\alpha, d)|y|^{\frac{4}{2+\alpha\wedge d}},$$

where $C_a(\alpha, d)$ is defined in Corollary 12.

Proof. Let $0 < \varepsilon < y$. For all $\delta > 0$,

$$\tilde{P}_0\big[\langle L_\tau; v_t\rangle \geqslant y\big] \leqslant \tilde{P}_0\big[\langle L_\tau; \psi_\delta * v_t\rangle \geqslant y - \varepsilon\big] + \tilde{P}_0\big[\big|\langle L_\tau; v_t - \psi_\delta * v_t\rangle\big| > \varepsilon\big].$$

Thus, by Lemma 9, the only thing to show is that for all $y \in \mathbb{R}^+$,

$$\liminf_{\delta\to 0} I_\delta(y) \geqslant C_a(\alpha,d)\, y^{\frac{4}{2+\alpha\wedge d}}.$$

To this end, we are first going to prove that

$$\liminf_{\delta\to 0} I_\delta(y) \geqslant \inf\left\{\mathcal{L}(\mu) + \frac{y^2}{2\langle \mathcal{R}_{(d-\alpha)_+}(\frac{d\mu}{dx});\, \frac{d\mu}{dx}\rangle};\ \mu \in \mathcal{M}^1(\mathbb{R}^d)\right\}. \tag{45}$$

Fix $L > \liminf_{\delta\to 0} I_\delta(y)$. For a sequence (δ_n) converging to 0, one gets probabilities $d\mu_n = f_n\,dx$ satisfying for all n,

$$\mathcal{L}(\mu_n) + \frac{y^2}{2\langle \mathcal{R}_{(d-\alpha)_+}(\psi_{\delta_n} * f_n);\, \psi_{\delta_n} * f_n\rangle} < L.$$

But, for all $\delta > 0$ and all μ such that $\mathcal{L}(\mu) < \infty$ (f will denote the density of μ)

$$\begin{aligned}
&\left|\left\langle \mathcal{R}_{(d-\alpha)_+}(\psi_\delta * f);\, \psi_\delta * f\right\rangle - \left\langle \mathcal{R}_{(d-\alpha)_+}(f);\, f\right\rangle\right| \\
&\quad \leqslant 2\|f\|_{q'} \left\|\mathcal{R}_{(d-\alpha)_+}(\psi_\delta * f - f)\right\|_q \\
&\quad \leqslant 2\|f\|_{q'} \|\psi_\delta * f - f\|_p, \quad \text{if } p \in \left]1; \frac{d}{(d-\alpha)_+}\right[\text{ and } \frac{1}{q} = \frac{1}{p} - \frac{(d-\alpha)_+}{d} \text{ by (15)}, \\
&\quad \leqslant C\delta^{1/p'} \mathcal{L}(\mu)^{\frac{\alpha\wedge d}{2} + \frac{1}{2p'}}, \quad \text{if } p \in [1;2] \text{ and } q' \in \left[1, \frac{d}{(d-2)_+}\right[\text{ by (68) and Lemma 22.}
\end{aligned}$$

The above sequence of inequalities holds as soon as one can find p satisfying all the above requirements. It can be easily checked that this is indeed the case for $d \leqslant 3$. Hence, we obtain for the sequence μ_n,

$$\left|\left\langle \mathcal{R}_{(d-\alpha)_+}(\psi_{\delta_n} * f_n);\, \psi_{\delta_n} * f_n\right\rangle - \left\langle \mathcal{R}_{(d-\alpha)_+}(f_n);\, f_n\right\rangle\right| \leqslant C\delta_n^{1/p'} L^{\frac{\alpha\wedge d}{2} + \frac{1}{2p'}},$$

and this implies that

$$\inf\left\{\mathcal{L}(\mu) + \frac{y^2}{2\langle \mathcal{R}_{(d-\alpha)_+}(\frac{d\mu}{dx});\, \frac{d\mu}{dx}\rangle}\right\} \leqslant \liminf_{n\to\infty}\left\{\mathcal{L}(\mu_n) + \frac{y^2}{2\langle \mathcal{R}_{(d-\alpha)_+}(\psi_{\delta_n} * f_n);\, \psi_{\delta_n} * f_n\rangle}\right\} \leqslant L.$$

This ends the proof of (45).

By the action of dilation $d\mu \mapsto d\mu_\lambda = \lambda^d\, d\mu(\lambda x)$, it is easy to see that the infimum in (45) is equal to $\tilde{C}_a(\alpha,d)\, y^{4/(2+\alpha\wedge d)}$, where by definition, $\tilde{C}_a(\alpha,d)$ is the value of the infimum for $y = 1$. It remains now to check that $\tilde{C}_a(\alpha,d) = C_a(\alpha,d)$. For that purpose, note that for μ such that $\mathcal{L}(\mu) < \infty$ ($d\mu = f\,dx$)

$$\frac{1}{2}\left\langle \mathcal{R}_{(d-\alpha)_+}(f);\, f\right\rangle = \sup\left\{\langle \mu; u\rangle - L(u);\, u \in L^2(\mathbb{R}^d)\right\}.$$

Since for all $\lambda \in \mathbb{R}$, and all u, $L(\lambda u) = \lambda^2 L(u)$, this supremum is also equal to $\sup\{\frac{\langle \mu; u\rangle^2}{4L(u)};\ u \in L^2(\mathbb{R}^d)\}$. Hence $\tilde{C}_a(\alpha,d) = C_a(\alpha,d)$ (where we use expression (35) of $C_a(\alpha,d)$). □

5. Quenched moderate deviations

In this section, we prove Theorem 4. The proof goes as follows. We begin to regularize the Brownian occupation measure for $d \leqslant 3$ (Section 5.1). We turn next to the upper bound (Section 5.2), which is obtained by the Gärtner–Ellis method. The computation of the log-Laplace transform is made possible using the localization lemma of [20], and the large deviations of the field. Finally, we obtain in Section 5.3 the lower bound, by forcing the Brownian motion to stay during time interval $[0,\tau]$ in a spatial region of size $r = (\frac{t^2}{m^2}\log(\tau))^{1/\alpha\wedge d}$, where the field is performing a large deviation.

5.1. Smoothing the field

Lemma 15. *Let r, m, and t be such that $t \leqslant m \ll t\sqrt{\log(t)}$, and $r^{\alpha\wedge d}\frac{m^2}{t^2} = \log(\tau)$ (i.e. $r \approx ((t^2\log(t))/m^2)^{1/\alpha\wedge d} \gg 1$). For $d \leqslant 3$, $\mathbb{P}$-a.s., for all $\varepsilon > 0$,*

$$\limsup_{\delta\to 0}\limsup_{t\to\infty}\frac{1}{\tau}\log P_0\big[\big|\langle L_\tau; v_t - \psi_\delta * v_t\rangle\big| \geqslant \varepsilon\big] = -\infty.$$

Proof. Using estimate (27), Lemma 10 and Lemma 23 (actually (70)), it is sufficient to prove that

$$\mathbb{P}\text{-a.s.},\ \forall A > 0, \quad \limsup_{t\to\infty}\max_j \|v_t\|_{2,Q_j(A)} < \infty, \tag{46}$$

the maximum involving $\approx (R\tau/A)^d$ terms. But, by Lemma 8, $\forall A > 0$, there exists C and L_0 (depending on A) such that for $L \geqslant L_0$,

$$\begin{aligned}\mathbb{P}\big[\max_j \|v_t\|_{2,Q_j(A)} \geqslant L\big] &\leqslant \sum_j \mathbb{P}\big[\|v_t\|_{2,Q_j(A)} \geqslant L\big]\\ &\leqslant C\tau^d\,\mathbb{P}\big[\|v_t\|_{2,Q(A)} \geqslant L\big]\\ &\leqslant C\tau^d \mathrm{e}^{-\frac{r^{\alpha\wedge d}m^2}{t^2}\frac{L^2}{C}} = C\tau^{d-\frac{L^2}{C}}.\end{aligned}$$

Thus, choosing L large enough, it follows from Borel–Cantelli lemma that $\mathbb{P}$-a.s., $\forall A > 0$, there exists $C(A)$, such that $\limsup_{t\to\infty}\max_j \|v_t\|_{2,Q_j(A)} \leqslant C$. □

5.2. Quenched upper bound

As in the annealed case, we begin with an upper bound for the regularized version of $\langle L_\tau; v_t\rangle$.

Lemma 16. *Let r, m, and t be such that $t \leqslant m \ll t\sqrt{\log(t)}$, and $r^{\alpha\wedge d}\frac{m^2}{t^2} = \log(\tau)$. $\mathbb{P}$-a.s., $\forall a \in \mathbb{Q}^+$, $\forall \delta \in \mathbb{Q}^+$, $\forall y > 0$,*

$$\begin{aligned}&\limsup_{t\to\infty}\frac{1}{\tau}\log P_0\big[\langle L_\tau; \psi_\delta * v_t\rangle \geqslant y\big]\\ &\quad\leqslant -ay - \inf\big\{\mathcal{L}(\mu) - a\langle\mu; \psi_\delta * u\rangle;\ \mu \in \mathcal{M}_1(\mathbb{R}^d),\ u \in L^2(\mathbb{R}^d), L(u) \leqslant d\big\}.\end{aligned}$$

Proof. Exactly as in the proof of Lemma 10, it is possible to find some (deterministic) constant C, such that $\forall A > 0, \forall R > 0$,

$$\begin{aligned}P_0\big[\langle L_\tau; \psi_\delta * v_t\rangle \geqslant y\big] &\leqslant P_0\big[\sigma(R\tau) \leqslant \tau\big] + P_0\big[\langle L_\tau; \psi_\delta * v_t\rangle \geqslant y;\ \sigma(R\tau) > \tau\big]\\ &\leqslant P_0\big[\sigma(R\tau) \leqslant \tau\big] + \mathrm{e}^{-\tau a y}\mathrm{e}^{\tau\frac{C}{A^2}}\varepsilon^{-\tau\min_j \lambda(a(\psi_\delta * v_t); Q_j(A))},\end{aligned} \tag{47}$$

where the minimum runs over the indices of boxes $Q_j(A)$ intersecting $Q(R\tau)$. We have thus to study the a.s. behavior of the minimum of eigenvalues. By stationarity, $\forall x \in \mathbb{R}$,

$$\mathbb{P}\big[\min_j \lambda\big(a(\psi_\delta * v_t); Q_j(A)\big) \leqslant x\big] \leqslant C\left(\frac{R\tau}{A}\right)^d \mathbb{P}\big[\lambda\big(a(\psi_\delta * v_t); Q(A)\big) \leqslant x\big].$$

Now, the function $u \in L^2(Q(A+\delta)) \mapsto \lambda(a(\psi_\delta * u); Q(A))$ is continuous for the weak topology of L^2 (see Lemma 28 in Section 6). Hence, by the large deviations upper bound for the field v_t, one gets that for $\tau = \mathrm{e}^{r^{\alpha\wedge d}m^2/t^2}$

$$\overline{\lim_{t\to\infty}}\frac{1}{\tau}\log\mathbb{P}\Big[\min_j\lambda\big(a(\psi_\delta*v_t);\,Q_j(A)\big)\leqslant x\Big]\leqslant d-\inf_u\big\{L_{A+\delta}(u);\,\lambda\big(a(\psi_\delta*u),\,Q(A)\big)\leqslant x\big\}.$$

So, as soon as x satisfies

$$d<\inf_u\big\{L_{A+\delta}(u);\,\lambda\big(a(\psi_\delta*u),\,Q(A)\big)\leqslant x\big\},\tag{48}$$

Borel–Cantelli lemma applied along sequences of the form $\tau_n^\eta=\mathrm{e}^{\eta n}$ leads to

$$\liminf_{n\to\infty}\min_j\lambda\big(a(\psi_\delta*v_{t_n});\,Q_j(A)\big)\geqslant x.\tag{49}$$

We want now to take in (49) the optimal allowed x, i.e. we want to invert the relation (48) with respect to x. Note that if

$$x<\inf_u\big\{\lambda\big(a(\psi_\delta*u),\,Q(A)\big);\;L_{A+\delta}(u)\leqslant d\big\},\tag{50}$$

then each u such that $\lambda(a(\psi_\delta*u),Q(A))\leqslant x$ is necessarily such that $L_{A+\delta}(u)>d$. In other words, if x satisfies (50), then

$$\inf_u\big\{L_{A+\delta}(u);\;\lambda\big(a(\psi_\delta*u),\,Q(A)\big)\leqslant x\big\}\geqslant d,$$

with strict inequality if the infimum is reached, which is actually the case by goodness of $L_{A+\delta}$ and continuity of $u\mapsto\lambda(a(\psi_\delta*u);Q(A))$. Hence, if x satisfies (50), it also satisfies (48) and therefore (49). Thus, $\forall a,\delta,A,R$, $\mathbb{P}$-a.s.,

$$\begin{aligned}\underline{\lim}_{t\to\infty}\min_j\lambda\big(a(\psi_\delta*v_t);\;Q_j(A)\big)&\geqslant\inf_u\big\{\lambda\big(a(\psi_\delta*u),\,Q(A)\big);\;L_{A+\delta}(u)\leqslant d\big\}\\&=\inf_{u,\mu}\big\{\mathcal{L}(\mu)-a\langle\mu;\psi_\delta*u\rangle;\;L_{A+\delta}(u)\leqslant d\big\},\end{aligned}\tag{51}$$

where the $\underline{\lim}$ is taken along sequences $\tau_n=\mathrm{e}^{\eta n}$. One can then deduce the same result for general τ, since for $\tau\in[\tau_n;\tau_{n+1}[$,

$$\begin{aligned}\min_{j\in J}\lambda\big(a(\psi_\delta*v_t);\,Q_j(A)\big)&\geqslant(r/r_{n+1})^2\min_{j\in J_{n+1}}\lambda\big(a(r_{n+1}/r)^2\psi_{\delta r/r_{n+1}}*v_{t_{n+1}};\,Q_j(Ar/r_{n+1})\big)\\&\geqslant(r/r_{n+1})^2\min_{j\in J_{n+1}}\lambda\big(a\psi_\delta*v_{t_{n+1}};\,Q_j(Ar/r_{n+1})\big)\\&\quad-a\left(\left(\frac{r_{n+1}^2}{r^2}-1\right)\|\psi_\delta\|_2+\big\|\psi_\delta-\psi_{\frac{\delta r}{r_{n+1}}}\big\|_2\right)\max_{j\in J_{n+1}}\|v_{t_{n+1}}\|_{2,Q_j(\frac{Ar}{r_{n+1}}+\delta)},\end{aligned}$$

where J_{n+1} is the set of indices such that $Q_j(Ar/r_{n+1})$ intersect $Q(R\tau_{n+1}r/r_{n+1})$. The result follows now from the fact that r_{n+1}/r tends to 1. The details are left to the reader.

Putting (47) and (51) together, and letting R tend to infinity along sequences, yields

$$\begin{aligned}&\limsup_{t\longrightarrow\infty}\frac{1}{\tau}\log P_0\big[\langle L_\tau,\psi_\delta*v_t\rangle\geqslant y\big]\\&\quad\leqslant-ay+\frac{C}{A^2}-\inf_{\mu\in\mathcal{M}_1^0(Q(A)),u\in L^2(Q(A+\delta))}\big\{\mathcal{L}(\mu)-a\langle\mu;\psi_\delta*u\rangle;\;L_{A+\delta}(u)\leqslant d\big\}\\&\quad\leqslant-ay+\frac{C}{A^2}-\inf_{u,\mu}\big\{\mathcal{L}(\mu)-a\langle\mu;\psi_\delta*u\rangle;\;\mu\in\mathcal{M}_1(\mathbb{R}^d);\;u\in L^2(\mathbb{R}^d),L(u)\leqslant d\big\}.\end{aligned}$$

We take now the limit $A\to\infty$ to get Lemma 16. □

Letting δ go to zero in Lemma 16 leads to

Lemma 17. *Let r, m, and t be such that $t \leqslant m \ll t\sqrt{\log(t)}$, and $r^{\alpha\wedge d}\frac{m^2}{t^2} = \log(\tau)$. Assume that $d \leqslant 3$. Then, $\mathbb{P}$-a.s., $\forall y \geqslant 0$,*

$$\limsup_{t\to\infty} \frac{1}{\tau} \log P_0\big[\langle L_\tau; v_t\rangle \geqslant y\big] \leqslant -C_q(\alpha, d)\, y^{4/\alpha\wedge d}, \tag{52}$$

where $C_q(\alpha, d) \in]0, +\infty[$ is given by

$$C_q(\alpha, d) = \left(\inf_{u,\mu}\left\{\frac{L^2(u)\mathcal{L}(\mu)^{\alpha\wedge d}}{d^2\langle\mu; u\rangle^4}\right\}\right)^{1/\alpha\wedge d}. \tag{53}$$

Proof. We begin to prove that

$$\liminf_{\delta\to 0}\inf_{u,\mu}\big\{\mathcal{L}(\mu) - a\langle\mu; \psi_\delta * u\rangle;\ L(u) \leqslant d\big\} \geqslant \inf_{u,\mu}\big\{\mathcal{L}(\mu) - a\langle\mu; u\rangle;\ L(u) \leqslant d\big\}, \tag{54}$$

and we can assume that the lim inf in the above expression is finite. Let then $l > \liminf_{\delta\to 0}\inf_{u,\mu}\{\mathcal{L}(\mu) - a\langle\mu; \psi_\delta * u\rangle;\ L(u) \leqslant d\}$. Note that since $L(-u) = L(u)$,

$$\inf_{u,\mu}\big\{\mathcal{L}(\mu) - a\langle\mu; \psi_\delta * u\rangle;\ L(u) \leqslant d\big\} = \inf_{u,\mu}\big\{\mathcal{L}(\mu) - a\big|\langle\mu; \psi_\delta * u\rangle\big|;\ L(u) \leqslant d\big\}, \tag{55}$$

$$\inf_{u,\mu}\big\{\mathcal{L}(\mu) - a\langle\mu; u\rangle;\ L(u) \leqslant d\big\} = \inf_{u,\mu}\big\{\mathcal{L}(\mu) - a\big|\langle\mu; u\rangle\big|;\ L(u) \leqslant d\big\}. \tag{56}$$

Let now (δ_n) be a sequence realizing the lim inf in (54), and let u_n and μ_n be such that $L(u_n) \leqslant d$, and $\mathcal{L}(\mu_n) - a|\langle\mu_n; \psi_{\delta_n} * u_n\rangle| \leqslant l$. Set $f_n = \frac{d\mu_n}{dx}$. Then,

$$\begin{aligned}
\mathcal{L}(\mu_n) &\leqslant l + a\big|\langle\mu_n; \psi_{\delta_n} * u_n\rangle\big| \\
&\leqslant l + a\sqrt{L(u_n)}\big\|\mathcal{R}_{\frac{(d-\alpha)_+}{2}}(\psi_{\delta_n} * f_n)\big\|_2 \\
&\leqslant l + a\sqrt{d}\|\psi_{\delta_n} * f_n\|_p \quad \text{for } \frac{1}{p} = 1 - \frac{\alpha\wedge d}{2d} \text{ by (15)} \\
&\leqslant l + a\sqrt{d}\|f_n\|_p \\
&\leqslant l + Ca\sqrt{d}\mathcal{L}(\mu_n)^{\frac{\alpha\wedge d}{4}} \quad \text{by (68).}
\end{aligned}$$

For $d \leqslant 3$, $\alpha\wedge d < 4$, we deduce from the above bounds that $\sup_n \mathcal{L}(\mu_n) < \infty$. Hence,

$$\begin{aligned}
\big|\langle\mu_n; \psi_{\delta_n} * u_n\rangle - \langle\mu_n; u_n\rangle\big| &\leqslant L(u_n)^{1/2}\big\|\mathcal{R}_{\frac{(d-\alpha)_+}{2}}(f_n - \psi_{\delta_n} * f_n)\big\|_2 \\
&\leqslant C\sqrt{d}\|f_n - \psi_{\delta_n} * f_n\|_p \quad \text{for } \frac{1}{p} = 1 - \frac{\alpha\wedge d}{2d}, \\
&\leqslant C\mathcal{L}(\mu_n)^{(d+1)/2p'}\delta_n^{1/p'} \quad \text{by Lemma 22,} \\
&\leqslant C\delta_n^{1/p'}.
\end{aligned} \tag{57}$$

It follows that

$$\begin{aligned}
\inf\big\{\mathcal{L}(\mu) - a\big|\langle\mu; u\rangle\big|;\ L(u) \leqslant d\big\} &\leqslant \mathcal{L}(\mu_n) - a\big|\langle\mu_n; u_n\rangle\big| \\
&\leqslant \mathcal{L}(\mu_n) - a\big|\langle\mu_n; \psi_{\delta_n} * u_n\rangle\big| + Ca\delta_n^{1/p'} \leqslant l + Ca\delta_n^{1/p'}.
\end{aligned}$$

This proves (54). We deduce from (54), Lemmas 15 and 16 that $\mathbb{P}$-a.s., $\forall a \in \mathbb{Q}^+$, $\forall y > 0$,

$$\limsup_{t\to\infty} \frac{1}{\tau} \log P_0\big[\langle L_\tau; v_t\rangle \geqslant y\big] \leqslant -ay - \inf_{u,\mu}\big\{\mathcal{L}(\mu) - a\big|\langle\mu; u\rangle\big|;\ L(u) \leqslant d\big\}. \tag{58}$$

The change of variable $u \mapsto \lambda u$ yields now

$$\inf_{u,\mu}\big\{\mathcal{L}(\mu) - a\big|\langle\mu; u\rangle\big|;\ L(u) \leqslant d\big\} = \inf_{u,\mu}\left\{\mathcal{L}(\mu) - a\sqrt{\frac{d}{L(u)}}\,\big|\langle\mu; u\rangle\big|\right\}.$$

Hence, by dilations,

$$\inf_{u,\mu}\big\{\mathcal{L}(\mu) - a\big|\langle\mu; u\rangle\big|;\ L(u) \leqslant d\big\} = -\tilde{C}_q(\alpha, d)a^{\frac{4}{4-\alpha\wedge d}}, \tag{59}$$

where

$$\tilde{C}_q(\alpha, d) = \left(1 - \frac{\alpha\wedge d}{4}\right)\left[d^2\left(\frac{\alpha\wedge d}{4}\right)^{\alpha\wedge d}\sup_{u,\mu}\left\{\frac{\langle\mu; u\rangle^4}{L^2(u)\mathcal{L}(\mu)^{\alpha\wedge d}}\right\}\right]^{\frac{1}{4-\alpha\wedge d}}.$$

By Lemma 25, $\tilde{C}_q(\alpha, d) \in]0, \infty[$. Lemma 17 is now obtained by taking the infimum with respect to $a \in \mathbb{Q}^+$ in (58). □

5.3. *Quenched lower bound*

Step 1. A.s. behavior of the field

Lemma 18. *Let r, m, and t be such that $t \leqslant m \ll t\sqrt{\log(t)}$, and $r^{\alpha\wedge d}\frac{m^2}{t^2} = \log(\tau)$. For all $\delta > 0$, $\varepsilon > 0$, $A > 0$, and for all $u_0 \in L^2(Q(A+\delta))$ such that $L_{A+\delta}(u_0) < d$, $\mathbb{P}$-a.s., for τ sufficiently large (along a subsequence $\tau_n^\eta = n^\eta$, $\eta > 0$), one can find a box $Q_j(A)$ intersecting $Q(\tau/\log(\tau))$ such that*

$$\big\|\psi_\delta * v_t - \psi_\delta * u_0(\cdot - x_j)\big\|_{\infty, Q_j(A)} \leqslant \varepsilon.$$

Proof. Let H be the complex Hilbert space $L^2(\mathbb{R}^d, D(k)\,dk)$. For $F \subset \mathbb{R}^d$, let H_F be the subspace of H spanned by the functions $\{e^{i\cdot x}; x \in F\}$, and let e_F be the orthogonal projection on H_F. Set

$$\begin{aligned} d(t) &= \sup\big\{\|e_{F_1}e_{F_2}\|;\ F_1, F_2 \text{ closed subsets of } \mathbb{R}^d,\ \mathrm{dist}(F_1, F_2) \geqslant t\big\} \\ &= \sup\big\{(f, g)_H;\ f \in H_{F_1},\ g \in H_{F_2},\ \|f\|_H = \|g\|_H = 1;\ F_1, F_2 \text{ closed subsets of } \mathbb{R}^d, \mathrm{dist}(F_1, F_2) \geqslant t\big\}. \end{aligned}$$

From our assumptions, $d(t)$ is rapidly decreasing when $\alpha \geqslant d$, and $d(t)$ decreases like $t^{-\alpha}$ for $\alpha < d$. Let $\mathcal{F}_i$ be the σ-algebra $\sigma(v(x); x \in F_i)$. Then, for f_1 (respectively f_2) bounded $\mathcal{F}_1$-measurable (respectively $\mathcal{F}_2$-measurable), one has (see for instance Lemma 5.12 in [10])

$$\mathbb{E}\big(|f_1||f_2|\big) \leqslant \prod_{i=1}^{2}\|f_i\|_{1+d(\mathrm{dist}(F_1, F_2))}. \tag{60}$$

Let $J = \{j \in \mathbb{Z}^d;\ j/2 \in \mathbb{Z}^d,\ Q_j(A) \subset Q(\tau/\log(\tau))\}$, so that for $j_1, j_2 \in J$,

$$\mathrm{dist}\big(Q_{j_1}(A), Q_{j_2}(A)\big) \geqslant 2A.$$

For $j \in J$, set $A_j = \{v;\ \|\psi_\delta * v_t - \psi_\delta * u_0(\cdot - x_j)\|_{\infty, Q_j(A)} > \varepsilon\}$. A_j is measurable w.r.t. $\sigma(v(x);\ x \in rQ_j(A+\delta))$. Adapting the proof in [10] of the hypermixing property of Gaussian field to the d-dimensional case leads to

$$\mathbb{P}\left[\bigcap_{j\in J}A_j\right] \leqslant \prod_{j\in J}\mathbb{P}[A_j]^{1/\prod_{l=1}^{k(\tau)}(1+d(2^l r(A-\delta)))^{2^d}}, \quad \text{where } 2^{k(\tau)} = \frac{\tau}{\log(\tau)A}.$$

By stationarity, we obtain then

$$\mathbb{P}\Big[\bigcap_{j\in J} A_j\Big] \leqslant \mathbb{P}\big[\|\psi_\delta * v_t - \psi_\delta * u_0\|_{\infty,Q(A)} > \varepsilon\big]^{|J|[\prod_{l=1}^{k(\tau)}(1+d(2^l r(A-\delta)))^{-2^d}]}$$
$$\approx \mathbb{P}\big[\|\psi_\delta * v_t - \psi_\delta * u_0\|_{\infty,Q(A)} > \varepsilon\big]^{|J|},$$

for t sufficiently large. Now, $\{u \in L^2(Q(A+\delta));\ \|\psi_\delta * u - \psi_\delta * u_0\|_{\infty,Q(A)} \leqslant \varepsilon\}$ is a weak neighborhood of u_0. By the large deviations for v_t^A, if β is such that $L_{A+\delta}(u_0) < \beta < d$, we get for t sufficiently large that

$$\mathbb{P}\big[\forall j \in J,\ \big\|\psi_\delta * v_t - \psi_\delta * u_0(\cdot - x_j)\big\|_{\infty,Q_j(A)} > \varepsilon\big] \leqslant \big(1 - \mathrm{e}^{-\frac{r^{\alpha\wedge d} m^2}{t^2}\beta}\big)^{C(\frac{R\tau}{\log(\tau)A})^d}.$$

The result is now a consequence of Borel–Cantelli lemma. □

Step 2. Lower bound

Lemma 19. *Assume that r, m, and t are such that $t \leqslant m \ll t\sqrt{\log(t)}$, and $r^{\alpha\wedge d}\frac{m^2}{t^2} = \log(\tau)$. For all $\delta > 0$, $\varepsilon > 0$, $A > 0$, $\forall y \in \mathbb{R}$, $\mathbb{P}$-a.s.,*

$$\liminf_{t\to\infty} \frac{1}{\tau} \log P_0\big[\big|\langle L_\tau; \psi_\delta * v_t\rangle - y\big| \leqslant \varepsilon\big] \geqslant -\mathcal{J}_{A,\delta}(y),$$

where

$$\mathcal{J}_{A,\delta}(y) \triangleq \inf\big\{\mathcal{L}(\mu);\ \mu \in \mathcal{M}_1^0\big(Q(A)\big);\ u \in L^2\big(Q(A+\delta)\big);\ L_{A+\delta}(u) < d;\ \langle \mu, \psi_\delta * u\rangle = y\big\},$$

and the lim inf *is taken along subsequences $\tau_n^\eta = n^\eta$ ($\eta > 0$).*

Proof. Fix $\eta > 0$. Let $\mu_0 \in \mathcal{M}_1^0(Q(A))$ and $u_0 \in L^2(Q(A+\delta))$ be such that $L_{A+\delta}(u_0) < d$, $\langle \mu_0, \psi_\delta * u_0\rangle = y$ and $\mathcal{L}(\mu_0) \leqslant \mathcal{J}_{A,\delta}(y) + \eta$. Since $L(u_0) < d$, $\mathbb{P}$-a.s. one can find for τ sufficiently large along a sequence, a box $Q_{j_0}(A)$ in $Q(\tau/\log(\tau))$ where $\|\psi_\delta * v_t - \psi_\delta * u_0(\cdot - x_{j_0})\|_{\infty,Q_{j_0}(A)} \leqslant \varepsilon/4$. The lower bound is then obtained by forcing the Brownian motion to go fast in this box, to remain there for the rest of the time, and to look there like $\mu_0(\cdot - x_{j_0})$. We introduce therefore $\mathcal{V}(\mu_0)$ a weak neighborhood of μ_0 in $\mathcal{M}_1(Q(A))$, such that

$$\mu \in \mathcal{V}(\mu_0) \Rightarrow \big|\langle \mu; \psi_\delta * u_0\rangle - y\big| < \varepsilon/4.$$

In what follows, θ_s is the shift along Brownian trajectories ($\theta_s(\omega) = \omega(s + \cdot)$). Then,

$$P_0\big[\big|\langle L_\tau; \psi_\delta * v_t\rangle - y\big| \leqslant \varepsilon\big]$$
$$\geqslant P_0\Bigg[\Bigg|\frac{1}{\tau}\int_0^{\frac{\tau}{\log(\tau)}} \psi_\delta * v_t(B_s)\Bigg| \leqslant \frac{\varepsilon}{4};\ \big|B_{\frac{\tau}{\log(\tau)}} - x_{j_0}\big| \leqslant 1;$$
$$\sigma\big(Q_{j_0}(A)\big)\circ\theta_{\frac{\tau}{\log(\tau)}} \geqslant \tau - \frac{\tau}{\log(\tau)};\ L_{\tau-\frac{\tau}{\log(\tau)}}\circ\theta_{\frac{\tau}{\log(\tau)}} \in x_{j_0} + \mathcal{V}(\mu_0)\Bigg]$$
$$\geqslant P_0\Bigg[\Bigg|\frac{1}{\tau}\int_0^{\frac{\tau}{\log(\tau)}} \psi_\delta * v_t(B_s)\Bigg| \leqslant \frac{\varepsilon}{4};\ \big|B_{\frac{\tau}{\log(\tau)}} - x_{j_0}\big| \leqslant 1\Bigg] \inf_{|x|\leqslant 1} P_0\Bigg[L_{\tau-\frac{\tau}{\log(\tau)}} \in \mathcal{V}(\mu_0);\ \sigma(A) \geqslant \tau - \frac{\tau}{\log(\tau)}\Bigg],$$

by Markov property, and translation invariance of the Brownian motion. Hence

$$\liminf_{t\to\infty} \frac{1}{\tau} \log P_0\big[\big|\langle L_\tau; \psi_\delta * v_t\rangle - y\big| \leqslant \varepsilon\big]$$

$$\geqslant -\mathcal{L}(\mu_0) + \liminf_{t\to\infty} \frac{1}{\tau} \log P_0 \left[\left| \frac{1}{\tau} \int_0^{\frac{\tau}{\log(\tau)}} \psi_\delta * v_t(B_s) \right| \leqslant \frac{\varepsilon}{4};\ \left| B_{\frac{\tau}{\log(\tau)}} - x_{j_0} \right| \leqslant 1 \right].$$

It remains now to show that

$$\liminf_{t\to\infty} \frac{1}{\tau} \log P_0 \left[\left| \frac{1}{\tau} \int_0^{\frac{\tau}{\log(\tau)}} \psi_\delta * v_t(B_s) \right| \leqslant \frac{\varepsilon}{4};\ \left| B_{\frac{\tau}{\log(\tau)}} - x_{j_0} \right| \leqslant 1 \right] \geqslant 0, \tag{61}$$

to end the proof of Lemma 19. But

$$P_0 \left[\left| \frac{1}{\tau} \int_0^{\frac{\tau}{\log(\tau)}} \psi_\delta * v_t(B_s) \right| \leqslant \frac{\varepsilon}{4};\ \left| B_{\frac{\tau}{\log(\tau)}} - x_{j_0} \right| \leqslant 1 \right]$$

$$\geqslant P_0 \left[\left| \frac{1}{\tau} \int_0^{\frac{\tau}{\log(\tau)}} \psi_\delta * v_t(B_s) \right| \leqslant \frac{\varepsilon}{4};\ \left| B_{\frac{\tau}{\log(\tau)}} - x_{j_0} \right| \leqslant 1;\ \sigma\left(\frac{\tau}{\log(\tau)}\right) > \frac{\tau}{\log(\tau)} \right].$$

On $\sigma(\tau/\log(\tau)) > \tau/\log(\tau)$,

$$\left| \frac{1}{\tau} \int_0^{\tau/\log(\tau)} \psi_\delta * v_t(B_s)\, ds \right| \leqslant \frac{1}{\log(\tau)} \|v_t\|_{\infty, Q(\tau/\log(\tau))} \leqslant C \frac{\sqrt{\log(r\tau/\log(\tau))}}{\log(\tau)} \frac{t}{m}$$

by (70). Hence this quantity is less than $\varepsilon/4$ for large t. Moreover, since $|x_{j_0}| \leqslant \tau/\log(\tau)$, $P_0(|B_{\tau/\log(\tau)} - x_{j_0}| \leqslant 1) \leqslant \mathrm{e}^{-C\tau/\log(\tau)}$, and (61) follows now from (27). □

Inverting "$\forall y \in \mathbb{R}$" and "$\mathbb{P}$-a.s." in Lemma 19, we get the lower bound for all $y \in \mathbb{Q}$. This in turn implies the lower bound for all y since

$$\limsup_{y_n \to y} \mathcal{J}_{A,\delta}(y_n) \leqslant \mathcal{J}_{A,\delta}(y). \tag{62}$$

Indeed, by the change of variable $u \mapsto \lambda u$, one can see that

$$\mathcal{J}_{A,\delta}(y) = \inf_{u,\mu} \left\{ \mathcal{L}(\mu);\ |y| < \sqrt{\frac{d}{L_{A+\delta}(u)}} \left| \langle \mu; \psi_\delta * u \rangle \right| \right\},$$

from which (62) is easily deduced.

At this point, we have thus shown that for $\frac{m^2 r^{\alpha\wedge d}}{t^2} = \log(\tau)$, $\forall \delta > 0$, $\forall A > 0$, $\forall \varepsilon > 0$, $\mathbb{P}$-a.s., $\forall y \in \mathbb{R}$,

$$\liminf_{t\to\infty} \frac{1}{\tau} \log P_0\left[\left| \langle L_\tau; \psi_\delta * v_t \rangle - y \right| < \varepsilon \right]$$
$$\geqslant -\inf\left\{ \mathcal{L}(\mu);\ \mu \in \mathcal{M}_1^0\big(Q(A)\big),\ u \in L^2\big(Q(A+\delta)\big),\ L(u) < d,\ \left| \langle \mu; \psi_\delta * u \rangle - y \right| < \varepsilon \right\}.$$

We want now to take the limit $A \to \infty$ in $\mathbb{Q}$. For $\delta > 0$ and y in $\mathbb{R}$, set

$$\mathcal{J}_\delta(y) \triangleq \inf\left\{ \mathcal{L}(\mu);\ \mu \in \mathcal{M}_1(\mathbb{R}^d),\ u \in L^2(\mathbb{R}^d),\ L(u) < d,\ \langle \mu, \psi_\delta * u \rangle = y \right\}.$$

It is easy to see that $\inf\{\mathcal{J}_\delta(z); |z - y| < \varepsilon\}$ can be restricted to probability measures with compact support. Since $L_A(u|_A) \leqslant L(u)$, it follows then that

$$\limsup_{A\to\infty} \inf\left\{ \mathcal{J}_{A,\delta}(z);\ |z-y| < \varepsilon \right\} \leqslant \inf\left\{ \mathcal{J}_\delta(z); |z-y| < \varepsilon \right\}.$$

From this remark, we deduce that for $\frac{m^2 r^{\alpha\wedge d}}{t^2} = \log(\tau)$, $\forall\delta > 0$, $\forall\varepsilon > 0$, $\mathbb{P}$-a.s., $\forall y \in \mathbb{R}$,

$$\liminf_{t\to\infty} \frac{1}{\tau} \log P_0\big[\big|\langle L_\tau; \psi_\delta * v_t\rangle - y\big| < \varepsilon\big] \geqslant -\inf\big\{\mathcal{J}_\delta(z);\ |z-y| < \varepsilon\big\}, \tag{63}$$

the lim inf being always taken along subsequences. We let now δ go to 0 in $\mathbb{Q}$ to obtain

Lemma 20. *Let r, m, and t be such that $t \leqslant m \ll t\sqrt{\log(t)}$, and $r^{\alpha\wedge d}\frac{m^2}{t^2} = \log(\tau)$. Assume that $d \leqslant 3$. $\mathbb{P}$-a.s., $\forall y \in \mathbb{R}$, $\forall\varepsilon > 0$,*

$$\liminf_{t\to\infty} \frac{1}{\tau} \log P_0\big[\big|\langle L_\tau; v_t\rangle - y\big| \leqslant \varepsilon\big]$$

$$\geqslant -\inf\big\{\mathcal{L}(\mu);\ \mu \in \mathcal{M}_1(\mathbb{R}^d);\ u \in L^2(\mathbb{R}^d);\ L(u) < d;\ \langle\mu, u\rangle = y\big\} \tag{64}$$

$$= -C_q(\alpha, d)|y|^{\frac{4}{\alpha\wedge d}}, \tag{65}$$

where the lim inf *is taken along sequences t_n such that $\tau_n = n^\eta$ $(\eta > 0)$.*

Proof. The first inequality (64) uses the same kind of arguments as the one used previously (see (57)). To compute the infimum, we apply successively the change of variable $u \mapsto \lambda u$, and then dilations. □

We have thus proved the lower bound along sequences t_n such that $\tau_n = n^\eta$. The result for general t is left to the reader.

6. Technical lemmas

Lemma 21. *For all $\alpha > 0$, there exists a constant $C = C(\alpha, d)$ such that for all $A > 0$, for all $f \in L^2(Q(A))$,*

$$\int_{\mathbb{R}^d} \frac{|\hat f(k)|^2}{\|k\|^{(d-\alpha)_+}}\,dk \leqslant C A^{(d-\alpha)_+} \int f^2(x)\,dx. \tag{66}$$

Proof. For $\alpha \geqslant d$, the result is just a consequence of Parseval equality. Let us assume that $\alpha \in\,]0, d[$. Then, $\forall\lambda > 0$,

$$\int_{\mathbb{R}^d} \frac{|\hat f(k)|^2}{\|k\|^{d-\alpha}}\,dk \leqslant \int_{\|k\|\leqslant\lambda} \frac{|\hat f(k)|^2}{\|k\|^{d-\alpha}}\,dk + \int_{\|k\|\geqslant\lambda} \frac{|\hat f(k)|^2}{\|k\|^{d-\alpha}}\,dk \leqslant C(d)\|\hat f\|_\infty^2 \int_0^\lambda r^{\alpha-1}\,dr + \frac{1}{\lambda^{d-\alpha}}\|f\|_2^2.$$

Now, $\|\hat f\|_\infty \leqslant \|f\|_1 \leqslant A^{d/2}\|f\|_2$. Taking the optimal λ in the preceding inequality yields the result. □

Proof of Lemma 10. Before proving Lemma 10, we are going to establish the following lemma, which will be of constant use throughout the paper.

Lemma 22. *Assume $d \leqslant 3$. There exists $C > 0$, such that $\forall\mu \in \mathcal{M}_1(\mathbb{R}^d)$ such that $\mathcal{L}(\mu) < \infty$,*

$$\forall p \in [1, 2],\qquad \left\|\psi_\delta * \frac{d\mu}{dx} - \frac{d\mu}{dx}\right\|_p \leqslant C\delta^{1/p'}\mathcal{L}(\mu)^{(d+1)/2p'}.$$

Proof. Let f be the density of μ with respect to Lebesgue measure. Set $g = \sqrt{f}$. Then $g \in H^1(\mathbb{R}^d)$, and $\|g\|_2 = 1$. By Sobolev embedding theorem, $g \in L^q(\mathbb{R}^d)$ for all $q \in [2, +\infty]$ when $d = 1$; for all $q \in [2, +\infty[$ when $d = 2$;

and for all $q \in [2, \frac{2d}{d-2}]$ when $d \geqslant 3$. Moreover, for all admissible q, we have (see for instance the proofs of Theorem VIII.7 and of Corollaries IX.10 and IX.11 in [6])

$$\|g\|_q \leqslant C\|g\|_2^{1-d(\frac{1}{2}-\frac{1}{q})}\|\nabla g\|_2^{d(\frac{1}{2}-\frac{1}{q})}. \tag{67}$$

Hence, taking into account that $\|g\|_2 = 1$ and that $\|\nabla g\|_2^2 = 2\mathcal{L}(\mu)$, we get

$$\|f\|_q \leqslant C\mathcal{L}(\mu)^{d/2q'}, \quad \text{where } \begin{cases} q \in [1, \infty] & \text{if } d = 1, \\ q \in [1, \infty[& \text{if } d = 2, \\ q \in [1, \frac{d}{d-2}] & \text{if } d \geqslant 3. \end{cases} \tag{68}$$

By Hölder inequality, we get then

$$\|\nabla f\|_r \leqslant C\mathcal{L}(\mu)^{1/2}\|f\|_{r/(2-r)}^{1/2} \leqslant C\mathcal{L}(\mu)^{\frac{1}{2}(1+\frac{d}{r'})}, \quad \text{for } \begin{cases} r \leqslant 2 & \text{if } d = 1; \\ r < 2 & \text{if } d = 2; \\ r \leqslant \frac{d}{d-1} & \text{if } d \geqslant 3. \end{cases} \tag{69}$$

Hence f belongs to the Sobolev space $W^{1,r}$, and it follows (see for instance Theorem 5, p. 155 in [33]) that

$$\int \frac{\|f(\cdot + t) + f(\cdot - t) - 2f(\cdot)\|_r^2}{\|t\|^{d+2}}\, dt \leqslant C\|\nabla f\|_r^2.$$

Therefore, setting $\Delta_t(y) \triangleq f(y+t) + f(y-t) - 2f(y)$,

$$\begin{aligned} \|\psi_\delta \star f - f\|_2^2 &= \frac{1}{4}\iint dy_1\, dy_2\, \psi(y_1)\psi(y_2)\int dx\, \Delta_{\delta y_1}(x)\Delta_{\delta y_2}(x) \\ &\leqslant \frac{1}{4}\int dy\, \psi(y)\|\Delta_{\delta y}\|_r \int dy\, \psi(y)\|\Delta_{\delta y}\|_{r'} \\ &\leqslant \|f\|_{r'}\left(\int dy\, \psi^2(y)\|y\|^{d+2}\right)^{1/2}\left(\int dy \frac{\|\Delta_{\delta y}\|_r^2}{\|y\|^{d+2}}\right)^{1/2} \\ &\leqslant C\delta\|f\|_{r'}\|\nabla f\|_r \\ &\leqslant C\delta\mathcal{L}(\mu)^{\frac{d}{2r'}+\frac{1}{2}+\frac{d}{2r}}, \quad \text{by (68) and (69).} \end{aligned}$$

The above sequence of inequalities is valid for $1 < r \leqslant 2$ if $d = 1$, and $1 < r < 2$ for $d = 2$. If $d \geqslant 3$, we have to choose r such that $r \leqslant \frac{d}{d-1}$ and $r' \leqslant \frac{d}{d-2}$. This is equivalent to say that $\frac{d}{2} \leqslant r \leqslant \frac{d}{d-1}$. Hence $d = 3$ and $r = 3/2$. At this point, we have established Lemma 22 for $p = 2$. The case $p \in [1, 2]$ comes from the interpolation inequality, and the fact that $\|\psi_\delta * f - f\|_1 \leqslant 2$. □

We return now to the proof of Lemma 10. Remind that

$$\mathcal{A} = \left\{(v, B);\ \frac{1}{\tau}\int_0^\tau (v_t - \psi_\delta * v_t)(B_s)\, ds \geqslant \varepsilon;\ \sigma(R\tau) > \tau\right\}.$$

Let Φ_A be any periodic function with period cell $Q(A)$. $\forall \gamma > 0$,

$$P_0(\mathcal{A}) \leqslant \mathrm{e}^{-\gamma\tau\varepsilon}E_0\left[\exp\left(\int_0^\tau \big(\gamma(v_t - \psi_\delta * v_t) - \Phi_A\big)(B_s)\, ds\right);\ \sigma(R\tau) > \tau\right]\mathrm{e}^{\tau\|\Phi_A\|_\infty}.$$

Using spectral estimates on Schrödinger semigroup (see for instance Theorem 1.2, p. 93 in [34]), we obtain

$$P_0(\mathcal{A}) \leqslant C\big[1 + \big(\tau\big(\gamma\|v_t\|_{\infty, Q(R\tau)} + \|\Phi_A\|_\infty\big)\big)^{d/2}\big]\mathrm{e}^{-\tau(\gamma\varepsilon - \|\Phi_A\|_\infty)}\mathrm{e}^{-\tau\lambda(\gamma(v_t - \psi_\delta * v_t) - \Phi_A;\, Q(R\tau))}.$$

We now use a lemma borrowed from [20], or more precisely the version of this lemma which is Lemma 4.6 of [5]. According to this lemma one can find a periodic function Φ_A such that $\|\Phi_A\|_\infty \leqslant C/A^2$ for a constant C depending only on d, and such that for all bounded measurable function V on $Q(R\tau)$,

$$\lambda\big(V-\Phi_A;\ Q(R\tau)\big) \geqslant \min_j \lambda\big(V;\ Q_j(A)\big).$$

Therefore, $\mathbb{P}$-a.s., $\forall\gamma>0$,

$$P_0(\mathcal{A}) \leqslant C\left[1+\left(\tau\left(\gamma\|v_t\|_{\infty,Q(R\tau)}+\frac{1}{A^2}\right)\right)^{d/2}\right]\mathrm{e}^{-\tau(\gamma\varepsilon-\frac{C}{A^2})}\mathrm{e}^{-\tau\min_j\lambda(\gamma(v_t-\psi_\delta*v_t);Q_j(A))}.$$

Now,

$$\lambda\big(\gamma(v_t-\psi_\delta*v_t);Q_j(A)\big)=\inf\left\{\mathcal{L}(\mu)-\gamma\int v_t\left(\frac{d\mu}{dx}-\psi_\delta*\frac{d\mu}{dx}\right);\ \mu\in\mathcal{M}_1^0\big(Q_j(A)\big)\right\}.$$

It follows then from Lemma 22 that $\mathbb{P}$-a.s., $\forall\gamma>0$,

$$\begin{aligned}&\lambda\big(\gamma(v_t-\psi_\delta*v_t);Q_j(A)\big)\\ &\quad\geqslant \inf\big\{\mathcal{L}(\mu)-\gamma\|v_t\|_{2,Q_j(A+\delta)}C\delta^{1/2}\mathcal{L}(\mu)^{(d+1)/4};\ \mu\in\mathcal{M}_1^0\big(Q_j(A)\big)\big\}\\ &\quad\geqslant \inf_{x\geqslant 0}\big\{x-C\gamma\|v_t\|_{2,Q_j(A+\delta)}\delta^{1/2}x^{(d+1)/4}\big\}\\ &\quad=-F_d\big(\gamma\delta^{1/2}\|v_t\|_{2,Q_j(A+\delta)}\big),\end{aligned}$$

where for $d\leqslant 2$, $F_d(x)\triangleq Cx^{4/(3-d)}$; $F_3(x)\triangleq +\infty\mathbb{1}_{x>C}$, and C is a constant which depends only on d, ψ. Hence, $\mathbb{P}$-a.s., $\forall\delta,\gamma,A,R$,

$$P_0(\mathcal{A}) \leqslant C\left[1+\left(\tau\left(\gamma\|v_t\|_{\infty,Q(R\tau)}+\frac{1}{A^2}\right)\right)^{d/2}\right]\mathrm{e}^{-\tau(\gamma\varepsilon-\frac{C}{A^2}-F_d(\gamma\delta^{1/2}\max_j\|v_t\|_{2,Q_j(A)}))}.$$

We now optimize the term in the exponential with respect to γ, i.e. we choose $\gamma\propto\varepsilon^{(3-d)/(d+1)}(\delta^{1/2}\times\max_j\|v_t\|_{2,Q_j(A+\delta)})^{-4/(1+d)}$. This produces an inequality of the form

$$\begin{aligned}P_0(\mathcal{A}) &\leqslant C_1\left[1+\frac{\tau^{d/2}}{A^d}+\frac{\|v_t\|^{d/2}_{\infty,Q(R\tau)}}{\varepsilon^{d/2}}\left(\frac{\tau\varepsilon^{4/(d+1)}}{(\delta^{1/2}\max_j\|v_t\|_{2,Q_j(\delta+A)})^{4/(d+1)}}\right)^{d/2}\right]\\ &\qquad\times\exp\left\{-C_2\tau\frac{\varepsilon^{4/(d+1)}}{(\delta^{1/2}\max_j\|v_t\|_{2,Q_j(\delta+A)})^{4/(d+1)}}\right\}\mathrm{e}^{C_1\frac{\tau}{A^2}}.\end{aligned}$$

Absorbing the polynomial term in the exponential term, leads now to the result of Lemma 10. □

Lemma 23. *Let m,r,t be such that $r\tau=\frac{t}{r}\gg 1$.*

$$\mathbb{P}\text{-a.s.},\quad \limsup_{t\to\infty}\frac{1}{\sqrt{2dK(0)\log(rR\tau)}}\frac{m}{t}\|v_t\|_{\infty,Q(R\tau)}\leqslant 1, \tag{70}$$

$$\forall p>0\ \exists C\big(p,d,K(0)\big)\ \text{s.t.}\quad \mathbb{E}\big[\|v_t\|^p_{\infty,Q(R\tau)}\big]\leqslant C\frac{t^p}{m^p}\big(1+(rR\tau)^d\big). \tag{71}$$

Proof. $\|v_t\|_{\infty,Q(R\tau)}=\frac{t}{m}\|v\|_{\infty,Q(rR\tau)}$. Since $\sup_{x\in Q(rR\tau)}E[v(x)^2]=K(0)$, and

$$\mathbb{E}\big[\big(v(x)-v(y)\big)^2\big]=2\big(K(0)-K(x-y)\big)\leqslant C\|x-y\|^2,$$

standard estimates on Gaussian processes (see for instance Theorem 2.4 in [35]) state that there exists u_0 such that $\forall u \geqslant u_0$,

$$\mathbb{P}\big[\|v\|_{\infty,Q(rR\tau)} \geqslant u\big] \leqslant C\frac{(R\tau r u)^d}{K(0)^{d/2}} P\left[X \geqslant \frac{u}{\sqrt{K(0)}}\right], \quad \text{where } X \sim \mathcal{N}(0,1).$$

It follows now from Borel–Cantelli lemma that $\mathbb{P}$-a.s.,

$$\limsup_{t\to\infty} \frac{1}{\sqrt{2dK(0)\log(rR\tau)}} \|v\|_{\infty,Q(rR\tau)} \leqslant 1,$$

which is just (70). Moreover,

$$\mathbb{E}\big[\|v\|^p_{\infty,Q(rR\tau)}\big] = p\int_0^\infty u^{p-1}\mathbb{P}\big[\|v\|_{\infty,Q(rR\tau)} \geqslant u\big]\,du \leqslant C\big(1+(R\tau r)^d\big),$$

for a constant C depending on d, p, $K(0)$. This implies (71). □

Lemma 24. *For all $A, B, \delta > 0$, $(\mu, u) \in \mathcal{M}_1(Q(A)) \times L^2(Q(B)) \mapsto \langle \mu, \psi_\delta * u\rangle$ is continuous in the product of weak topologies.*

Proof. Let (μ_n) be a sequence weakly converging to $\mu \in \mathcal{M}_1(Q(A))$, and (u_n) a sequence weakly converging to u in $L^2(Q(B))$. One has $\sup_n \|u_n\|_2 < \infty$, and this implies that the sequence $\psi_\delta * u_n$ is an equicontinuous and uniformly bounded sequence of continuous functions converging pointwise to $\psi_\delta * u$. By Arzela–Ascoli theorem, $\lim_{n\to\infty} \|\psi_\delta * u_n - \psi_\delta * u\|_\infty = 0$. The result follows now from the inequality

$$\big|\langle \mu; \psi_\delta * u\rangle - \langle \mu_n; \psi_\delta * u_n\rangle\big| \leqslant \big|\langle \mu - \mu_n, \psi_\delta * u\rangle\big| + \|\psi_\delta * u - \psi_\delta * u_n\|_\infty. \qquad \square$$

Lemma 25. *For $\alpha > 0$ and $d \in \mathbb{N}$ such that $\alpha \wedge d \leqslant 4$,*

$$0 < \inf\left\{\frac{L^2(u)\mathcal{L}(\mu)^{\alpha\wedge d}}{\langle \mu; u\rangle^4};\ \mu \in \mathcal{M}_1(\mathbb{R}^d),\ u \in L^2(\mathbb{R}^d)\right\} < \infty.$$

Proof. It is clear that the infimum is finite, and we have just to prove that it is strictly positive for $\alpha \wedge d \leqslant 4$. Now, for all $u \in L^2(\mathbb{R}^d)$ and all μ such that $\mathcal{L}(\mu) < \infty$, setting $f = \frac{d\mu}{dx}$,

$$\begin{aligned}
\langle \mu; u\rangle &\leqslant \sqrt{L(u)}\big\|\mathcal{R}_{\frac{(d-\alpha)_+}{2}}(f)\big\|_2 \\
&\leqslant C\sqrt{L(u)}\|f\|_p \quad \text{for } \frac{1}{p} = 1 - \frac{\alpha\wedge d}{2d} \text{ by (15)}, \\
&\leqslant C\sqrt{L(u)}\mathcal{L}(f)^{\frac{d}{2p'}} = C\sqrt{L(u)}\mathcal{L}(f)^{\frac{\alpha\wedge d}{4}} \quad \text{for } \alpha\wedge d \leqslant 4 \text{ by (68)}. \qquad \square
\end{aligned}$$

Lemma 26. *For all $A > 0$ and all $\delta > 0$, $F_{\delta,A}$ (defined by (43)) is continuous, when $\mathcal{M}_1(\mathcal{T}(A))$ is endowed with the weak convergence.*

Proof. Let (μ_n) be a sequence in $\mathcal{M}_1(\mathcal{T}(A))$ which converges weakly to μ. For all x_A in $\mathcal{T}(A)$, $\psi^A_\delta * \mu_n(x_A)$ converges to $\psi^A_\delta * \mu(x_A)$, and $\sup_n \|\psi^A_\delta * \mu_n\|_\infty \leqslant \|\psi_\delta\|_\infty$. The result follows from dominated convergence, and the fact that for $\alpha \in]0, d[$, $\int_{\mathcal{T}(A)}\int_{\mathcal{T}(A)} \frac{dx_A\,dy_A}{d_A(x_A,y_A)^\alpha} < \infty$. □

Lemma 27. *For $\delta > 0$, and $A > 0$, let I_δ and $I_{\delta,A}$ be defined by (41) and (44). Then, for all $y \in \mathbb{R}$ and all $\delta > 0$,*

$$\limsup_{A\to\infty} I_{\delta,A}(y) \geqslant I_\delta(y).$$

Proof. Let $L > \limsup_{A\to\infty} I_{\delta,A}(y)$. For sufficiently large A, let $\mu_A \in \mathcal{M}_1(\mathcal{T}(A))$ be such that $\mathcal{L}_A(\mu_A) + \frac{y^2}{2F_{\delta,A}(\mu_A)} < L$. Let f_A be the density of μ_A. f_A is viewed as a periodic function with period $2A$. Note that translating f_A does not change the value of $\mathcal{L}_A(\mu_A)$ and of $F_{\delta,A}(\mu_A)$. It has been proved in [13] that this translation can be done in such a way that

$$\int_{\partial Q(A)} f_A(x)\,dx \leqslant C/\sqrt{A}, \quad \text{with } \partial Q(A) \triangleq \bigl\{x \in Q(A);\ \forall i \in \{1,\dots,d\}\ |x_i| \geqslant A - \sqrt{A}\bigr\}.$$

We truncate now μ_A in order to define a new probability $\tilde{\mu}_A$ on $\mathbb{R}^d$. Let T be a smooth function, $0 \leqslant T \leqslant 1$, $T = 0$ on $Q(A)^c$, $T = 1$ on $Q(A)\backslash\partial Q(A)$, and let $d\tilde{\mu}_A = \frac{T(x)f_A(x)}{\int T(x)f_A(x)\,dx}\,dx$. It is also proven in [13] that T can be chosen in such a way that $\mathcal{L}(\tilde{\mu}_A) \leqslant \mathcal{L}_A(\mu_A) + \frac{C}{\sqrt{A}}$.

It remains now to prove that

$$F_{\delta,A}(\mu_A) \leqslant \bigl\langle \mathcal{R}_{(d-\alpha)_+}(\psi_\delta * \tilde{\mu}_A);\ \psi_\delta * \tilde{\mu}_A\bigr\rangle + \mathrm{o}(A). \tag{72}$$

Set $K_\delta(x) \triangleq \psi_\delta * (\mathcal{R}_{(d-\alpha)_+})(\psi_\delta)(x)$ and

$$K_{\delta,A}(x_A, y_A) \triangleq \begin{cases} \int_{\mathcal{T}(A)}\int_{\mathcal{T}(A)} \frac{\psi_\delta(d_A(x_A,x'_A))\psi_\delta(d_A(y_A,y'_A))}{d_A(y'_A,x'_A)^\alpha}\,dx'_A\,dy'_A & \text{for } 0 < \alpha < d,\\ \int_{\mathcal{T}(A)} \psi_\delta(d_A(x_A,x'_A))\psi_\delta(d_A(y_A,x'_A))\,dx'_A & \text{for } \alpha \geqslant d; \end{cases}$$

so that

$$F_{\delta,A}(\mu_A) = \int_{\mathcal{T}(A)}\int_{\mathcal{T}(A)} K_{\delta,A}(x_A,y_A)\,d\mu_A(x_A)\,d\mu_A(y_A),$$

$$\bigl\langle \mathcal{R}_{(d-\alpha)_+}(\psi_\delta * \tilde{\mu}_A);\ \psi_\delta * \tilde{\mu}_A\bigr\rangle = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} K_\delta(x-y)\,d\tilde{\mu}_A(x)\,d\tilde{\mu}_A(y).$$

Since ψ_δ is an all L^p-spaces, $\|K_\delta\|_\infty < \infty$ by the continuity properties of the operator $\mathcal{R}_{(d-\alpha)_+}$. In the same way, $\sup_A \|K_{\delta,A}\|_\infty < \infty$. Hence

$$\begin{aligned}&\bigl\langle \mathcal{R}_{(d-\alpha)_+}(\psi_\delta * \tilde{\mu}_A); \psi_\delta * \tilde{\mu}_A\bigr\rangle - F_{\delta,A}(\mu_A)\\ &\quad\geqslant \int_{Q(A)}\int_{Q(A)} dx\,dy f_A(x)f_A(y)\bigl[T(x)T(y)K_\delta(x-y) - K_{\delta,A}(x_A,y_A)\bigr]\\ &\quad= T_1 + T_2 + T_3,\end{aligned}$$

where

$$T_1 = \int_{Q(A)}\int_{Q(A)} f_A(x)f_A(y)\bigl(K_\delta(x-y) - K_{\delta,A}(x_A,y_A)\bigr)\,dx\,dy,$$

$$T_2 = \int_{Q(A)}\int_{Q(A)} \bigl(T(x)-1\bigr)f_A(x)f_A(y)K_\delta(x-y)\,dx\,dy,$$

$$T_3 = \int_{Q(A)}\int_{Q(A)} T(x)\bigl(T(y)-1\bigr)f_A(x)f_A(y)K_\delta(x-y)\,dx\,dy.$$

Note that

$$|T_2| + |T_3| \leqslant 2\|K_\delta\|_\infty \int_{\partial Q(A)} f_A(x)\,dx \leqslant C/\sqrt{A}.$$

To evaluate T_1, we partition $Q(A) \times Q(A)$ in three domains

$$\mathcal{D}_1 = \big\{(x, y) \in Q(A) \times Q(A);\ \|x - y\| < A - 2\delta\big\};$$
$$\mathcal{D}_2 = \big\{(x, y) \in Q(A) \times Q(A);\ \|x - y\| \geqslant A - 2\delta,\ d_A(x_A, y_A) \leqslant \sqrt{A}\big\};$$
$$\mathcal{D}_3 = \big\{(x, y) \in Q(A) \times Q(A);\ \|x - y\| \geqslant A - 2\delta,\ d_A(x_A, y_A) > \sqrt{A}\big\}.$$

By definition, $\mathcal{D}_2 \subset \partial Q(A) \times \partial Q(A)$, so that

$$\left|\int_{\mathcal{D}_2} f_A(x) f_A(y)\big(K_\delta(x - y) - K_{\delta,A}(x_A, y_A)\big)\,dx\,dy\right| \leqslant C\Bigg(\int_{\partial Q(A)} f_A(x)\,dx\Bigg)^2 \leqslant C/A.$$

Note that $\|x - y\| = d_A(x_A, y_A)$ as soon as $\|x - y\| < A$. Therefore, on $\mathcal{D}_1$,

$$K_\delta(x - y) = \begin{cases} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{\psi_\delta(d_A(x_A, x'_A))\psi_\delta(d_A(y_A, y'_A))}{d_A(x'_A, y'_A)^\alpha}\,dx'\,dy' & \text{for } 0 < \alpha < d, \\ \int_{\mathbb{R}^d} \psi_\delta(d_A(x_A, x'_A))\psi_\delta(d_A(y_A, x'_A))\,dx' & \text{for } \alpha \geqslant d. \end{cases}$$

In any case, $K_\delta(x - y) \geqslant K_{\delta,A}(x_A, y_A)$ on $\mathcal{D}_1$.

Finally, note that $\sup_{\|x\| \geqslant A} K_\delta(x) \leqslant C/A^\alpha$ and $\sup_{d_A(x_A, y_A) \geqslant \sqrt{A}} K_{\delta,A}(x_A, y_A) \leqslant C/A^{\alpha/2}$. Hence,

$$\left|\iint_{\mathcal{D}_3} f_A(x) f_A(y)\big(K_\delta(x - y) - K_{\delta,A}(x_A, y_A)\big)\,dx\,dy\right| \leqslant C/A^{\alpha/2}.$$

This ends the proof of (72) and of Lemma 27. □

Lemma 28. *For all $A, B > 0$, for all $\delta > 0$, $u \in L^2(Q(B)) \mapsto \lambda(\psi_\delta * u, Q(A))$ is continuous for the weak topology of $L^2(Q(B))$.*

Proof. Let (u_n) be a sequence converging weakly to u. Then $(\psi_\delta * u_n)$ is a sequence of continuous functions, converging uniformly to $\psi_\delta * u$. Hence,

$$\lambda\big(\psi_\delta * u_n, Q(A)\big) = \inf_{\mu \in \mathcal{M}_0^1(Q(A))} \big\{\mathcal{L}(\mu) - \langle \mu; \psi_\delta * u_n\rangle\big\}$$

is the infimum of continuous functions, and is therefore u.s.c. To prove the lower-semicontinuity, let $l > \liminf_{n\to\infty} \lambda(\psi_\delta * u_n, Q(A))$, and let $\mu_n \in \mathcal{M}_1^0(Q(A))$ be such that $\mathcal{L}(\mu_n) - \langle \mu_n; \psi_\delta * u_n\rangle \leqslant l$. (μ_n) is a tight sequence, and it converges (at least along a subsequence) to a probability measure μ. $\mathcal{L}$ being l.s.c., $\liminf_{n\to\infty} \mathcal{L}(\mu_n) \geqslant \mathcal{L}(\mu)$. Moreover,

$$\big|\langle \mu_n, \psi_\delta * u_n\rangle - \langle \mu; \psi_\delta * u\rangle\big| \leqslant \big|\langle \mu_n - \mu; \psi_\delta * u\rangle\big| + \|\psi_\delta * u_n - \psi_\delta * u\|_\infty.$$

Therefore, $\langle \mu_n, \psi_\delta * u_n\rangle \to \langle \mu; \psi_\delta * u\rangle$. □

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