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Symplectic resolutions for nilpotent orbits (II)

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Abstract

Based on our previous work, Fu (Invent. Math. 151 (2003) 167–186), we prove that, given any two projective symplectic resolutions Z_1 and Z_2 of a nilpotent orbit closure in a complex simple Lie algebra of classical type, Z_1 is deformation equivalent to Z_2. This provides support for a 'folklore' conjecture on symplectic resolutions for symplectic singularities. ***To cite this article: B. Fu, C. R. Acad. Sci. Paris, Ser. I 337 (2003).***

Résumé

Résolutions symplectiques pour les orbites nilpotentes (II). En utilisant notre résultat précédent, Fu (Invent. Math. 151 (2003) 167–186), nous montrons qu'étant données deux résolutions symplectiques projectives Z_1 et Z_2 d'une adhérence d'orbite nilpotente dans une algèbre de Lie simple classique, Z_1 est déformation équivalente à Z_2. En particulier, ceci vérifie une conjecture « folklore » sur les résolutions symplectiques pour les singularités symplectiques. ***Pour citer cet article : B. Fu, C. R. Acad. Sci. Paris, Ser. I 337 (2003).***

Version française abrégée

Soit G un groupe de Lie semi-simple complexe, connexe d'algèbre de Lie $\mathfrak{g}$. Soit $\mathcal{O}$ une orbite nilpotente dans $\mathfrak{g}$. Alors $\overline{\mathcal{O}}$ est lisse en codimension 1. Sur $\mathcal{O}$ il existe une forme symplectique holomorphe ω (forme de Kostant–Kirillov). Une résolution des singularités $\pi : Z \to \overline{\mathcal{O}}$ est *symplectique* si la 2-forme $\pi^*(\omega)$, définie a priori, sur $\pi^{-1}(\mathcal{O})$ s'étend en une 2-forme symplectique sur Z tout entier.

Dans [3], nous avons montré que pour une telle résolution symplectique projective, la variété Z est isomorphe à $T^*(G/P)$ pour un sous-groupe parabolique P de G. Le but de cette Note est de comparer deux telles résolutions.

Rappelons que deux variétés Z_1 et Z_2 sont déformation équivalentes s'il existe un morphisme plat $\mathcal{Z} \xrightarrow{f} S$ au-dessus d'une courbe connexe S tel que Z_1 et Z_2 sont isomorphes à deux fibres de f. Dans cette Note, nous montrons le théorème suivant :

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Théorème 0.1. *Soit $\mathcal{O}$ une orbite nilpotente dans une algèbre de Lie simple complexe classique. Supposons que $\overline{\mathcal{O}}$ possède deux résolutions symplectiques projectives $Z_1 \to \overline{\mathcal{O}}$ et $Z_2 \to \overline{\mathcal{O}}$. Alors Z_1 est déformation équivalente à Z_2.*

Comme corollaire, les adhérences des orbites nilpotentes dans une algèbre de Lie simple complexe du type classique vérifient la conjecture suivante :

Conjecture 0.2. *Soit X une variété irréductible avec singularité symplectique. Soient $Z_i \to X,\ i = 1, 2,$ deux résolutions symplectiques projectives de X. Alors Z_1 est déformation équivalente à Z_2.*

Quand X est projective, cette conjecture a été démontrée par Huybrechts (Theorem 4.6 [6]). Dans [7], Kaledin a démontré cette conjecture sous une hypothèse assez restrictive. Dans [4], nous avons vérifié cette conjecture pour certaines variétés avec singularités quotients. L'article [4] contient aussi d'autres résultats sur les résolutions symplectiques des orbites nilpotentes.

L'idée de la démonstration du Théorème 0.1 est la suivante. Nous montrons d'abord que pour un sous-groupe parabolique P de G avec le facteur de Levi L, $T^*(G/P)$ est déformation équivalente à G/L. Ensuite, en utilisant les résultats de Hesselink [5], nous montrons que pour deux résolutions symplectiques $T^*(G/P_i) \to \overline{\mathcal{O}},\ i = 1, 2,$ d'une adhérence d'orbite nilpotente dans une algèbre de Lie simple complexe classique, soit G/P_1 est isomorphe à G/P_2, soit P_1 et P_2 ont un facteur de Levi en commun, ce que achèvera la démonstration.

1. Introduction

Let X be a complex variety, which is smooth in codimension 1. Following [1], the variety X is said to have *symplectic singularities* if there exists a holomorphic symplectic 2-form ω on X_{reg} such that for any resolution of singularities $\pi : Z \to X$, the 2-form $\pi^*\omega$ defined a priori on $\pi^{-1}(X_{\text{reg}})$ can be extended to a holomorphic 2-form on Z. If, furthermore, the 2-form $\pi^*\omega$ extends to a holomorphic symplectic 2-form on the whole of Z for some resolution of X, then we say that X admits a *symplectic resolution*, and the resolution π is called *symplectic*.

One important class of examples of symplectic singularities consists of nilpotent orbit closures in a semi-simple complex Lie algebra. Projective symplectic resolutions for these singularities have been completely characterized in [3]. This Note studies the relationship between two such projective symplectic resolutions.

Recall that two varieties Z_1 and Z_2 are *deformation equivalent* if there exists a flat morphism $\mathcal{Z} \xrightarrow{f} S$ over a connected curve S such that Z_1 and Z_2 are isomorphic to two fibers of f. The purpose of this Note is to prove the following:

Theorem 1.1. *Let $\mathcal{O}$ be a nilpotent orbit in a complex simple Lie algebra $\mathfrak{g}$ of classical type. Suppose that $\overline{\mathcal{O}}$ admits two projective symplectic resolutions $Z_1 \to \overline{\mathcal{O}}$ and $Z_2 \to \overline{\mathcal{O}}$. Then Z_1 is deformation equivalent to Z_2.*

In particular, we see that Z_1 and Z_2 are diffeomorphic, thus they have the same topological invariants (Betti numbers, etc.), which is in some sense the McKay correspondence for nilpotent orbit closures. Another motivation of this result is to provide support for the following:

Conjecture 1.2. *Let X be an irreducible variety with symplectic singularities. Suppose that we have two projective symplectic resolutions: $Z_1 \to X$ and $Z_2 \to X$. Then Z_1 is deformation equivalent to Z_2.*

When X is projective, this conjecture has been proven by Huybrechts (Theorem 4.6 [6]). In [7], Kaledin proved the conjecture under a rather restrictive hypothesis [7, Condition 5.1]. In [4], we proved that any projective symplectic resolution for a symmetric product $S^{(n)}$ of a symplectic connected surface S is isomorphic to the

Douady–Barlet resolution $S^{[n]} \to S^{(n)}$. In particular, this also verifies the above conjecture. Some other results on uniqueness of symplectic resolutions for some quotient symplectic singularities and for some nilpotent orbit closures are obtained in [4].

2. Key lemma

I am indebted to M. Brion for pointing out the following lemma, which plays a key role in the proof of our theorem.

Lemma 2.1. *Let G be a semi-simple complex connected Lie group with Lie algebra $\mathfrak{g}$. Let P be a parabolic subgroup of G with Levi factor L. Then $T^*(G/P)$ is deformation equivalent to G/L.*

Proof. We first show that $T_o^*(G/P)$ is deformation equivalent to P/L as P-varieties, where o is a base point of G/P. Let $\mathfrak{p}_u$ be the Lie algebra of the unipotent radical P_u of P. Then $\mathfrak{p}_u$, equipped with the adjoint action of P, is identified with $T_o^*(G/P)$.

Let z be an element in the center of $\mathfrak{l} := \mathrm{Lie}(L)$ such that its centralizer in G is exactly L, then its centralizer in $\mathfrak{g}$ is exactly $\mathfrak{l}$. Consider the family $V := (tz + \mathfrak{p}_u)_{t\in\mathbb{C}} \xrightarrow{f} \mathbb{C}z \cong \mathbb{C}$ of sub-spaces in $\mathfrak{g}$. Note that each subspace $tz + p_u$ is stable under the adjoint action of L and the adjoint action of P_u (since $[z, \mathfrak{p}_u]$ is contained in $\mathfrak{p}_u$). Thus we get a family of P-varieties with $\mathfrak{p}_u$ being the special fiber.

Now we show that $tz + \mathfrak{p}_u$ is isomorphic to P/L for $t \neq 0$. In fact, the P-orbit of tz coincides with the P_u-orbit of tz, which is closed in $\mathfrak{g}$ (since P_u is unipotent), but $Ad(P_u)tz$ and $tz + \mathfrak{p}_u$ have the same tangent spaces at tz, thus $Ad(P_u)tz$ is also open in $tz + \mathfrak{p}_u$, which shows that $tz + \mathfrak{p}_u$ is exactly the P-orbit of tz. By our choice of z, the latter is isomorphic to P/L.

Now consider the family $G \times_P V \to \mathbb{C}$, which is given by $(g, v)P \mapsto f(v)$. Then the central fiber is $T^*(G/P)$ and other fibers are all isomorphic to G/L, which concludes the proof. □

Remark 1. Notice that if two parabolic subgroups P_1 and P_2 have a Levi factor L in common, then the two deformations $G \times_{P_1} V_1 \to \mathbb{C}$ and $G \times_{P_2} V_2 \to \mathbb{C}$ have the same fibers over $\mathbb{C} - \{0\}$.

Remark 2. The natural morphism

$$G \times_P V \xrightarrow{\tilde{\pi}} W_P \subset \mathfrak{g}, \quad (g, v) \mapsto Ad(g)v$$

gives a deformation of the Springer resolution $\pi : T^*(G/P) \to \overline{\mathcal{O}}$, where W_P is the image of $\tilde{\pi}$, depending a priori on the polarization P. Notice that $\tilde{\pi}_t$ is an isomorphism if $t \neq 0$.

3. Proof of Theorem 1.1

First let us recall the following theorem from [3]:

Theorem 3.1. *Let $\mathcal{O}$ be a nilpotent orbit in a semi-simple complex Lie algebra $\mathfrak{g}$ and G a connected Lie group with Lie algebra $\mathfrak{g}$. Suppose that $\overline{\mathcal{O}}$ admits a symplectic resolution $Z \to \overline{\mathcal{O}}$. Then Z is isomorphic to $T^*(G/P)$ for some parabolic subgroup P of G and under this isomorphism, the map $Z \simeq T^*(G/P) \to \overline{\mathcal{O}}$ becomes the natural collapsing of the zero section.*

Such a parabolic sub-group P is called a *polarization* of $\mathcal{O}$ in [5]. So to prove Theorem 1.1, we need to show that if we have two polarizations P_1, P_2 of $\mathcal{O}$ such that $T^*(G/P_1)$ and $T^*(G/P_2)$ are birational to $\mathcal{O}$, then $T^*(G/P_1)$

is deformation equivalent to $T^*(G/P_2)$. In fact, we will prove that either P_1 and P_2 have conjugate Levi factors or G/P_1 is isomorphic to G/P_2 (in some cases of $\mathfrak{g} = \mathfrak{so}_{2n}$). Then Lemma 2.1 will conclude the proof.

To this end, we will do a case-by-case check, using the results of Hesselink in [5]. Let $\mathbf{d}$=$[d_1, \ldots, d_N]$ be the partition corresponding to the orbit $\mathcal{O}$ (cf. Section 5.1 [2]). Let $\mathbf{s}$ = $[s_1, \ldots, s_k]$ be the dual partition of $\mathbf{d}$, where $s_i = \#\{j \mid d_j \geqslant i\}$.

3.1. *Case* $\mathfrak{g} = \mathfrak{sl}_n$

Let $V = \mathbb{C}^n$. A *flag* F of V is a sequence of sub-spaces $0 = F_0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_k = V$. Its *type* is the sequence $(p_1, \ldots, p_k)$ given by $p_i = \dim(F_i/F_{i-1})$. One shows easily that the stabilizer $P \subset G$ of F is a parabolic subgroup.

By Theorem 3.3 [5], any polarization P of $\mathcal{O}$ is a stabilizer of such a flag F with type $(s_{\sigma(1)}, \ldots, s_{\sigma(k)})$ for some permutation $\sigma \in \mathfrak{S}_k$, and they have conjugate Levi factors.

3.2. *Case* $\mathfrak{g} = \mathfrak{sp}_{2n}$

Let $V = \mathbb{C}^{2n}$ and ϕ a non-degenerate anti-symmetric bilinear form on V. A flag $F = (F_0, \ldots, F_k)$ is called *isotropic* if $F_i^{\perp} = F_{k-i}$ for $0 \leqslant i \leqslant k$, where $F_i^{\perp}$ is the orthogonal space of F_i with respect to the bilinear form ϕ. The type $(p_1, \ldots, p_k)$ of an isotropic flag satisfies $p_i = p_{k+1-i}$ for $1 \leqslant i \leqslant k$. By Lemma 4.3 [5], every parabolic subgroup P of G is the stabilizer of some isotropic flag, and two parabolic subgroups with the same flag type are conjugate under the action of G (Lemma 4.4 loc. cit.).

Here all congruences are modulo 2. For an even number q, let

$$Pai(2n, q) = \{\text{partitions } \pi \text{ of } 2n \mid \pi_j \equiv 1 \text{ if } j \leqslant q;\ \pi_j \equiv 0 \text{ if } j > q\}.$$

The union $Pai(2n) = \bigcup_q Pai(2n, q)$ parametrizes all conjugate classes of Levi factors of parabolic subgroups in G. For any q, there exists an injective Spaltenstein mapping (Proposition 6.5 [5])

$$S_q : Pai(2n, q) \to \mathcal{P}_1(2n),$$

where $\mathcal{P}_1(2n)$ is the set of partitions of $2n$ in which odd parts occur with even multiplicity (cf. Section 5.1 [2]).

By the proof of Proposition 3.21 [3], if P is a parabolic subgroup of G such that $T^*(G/P)$ gives a symplectic resolution for $\overline{\mathcal{O}}$, then one has $q = \#\{j \mid d_j \equiv 1\}$. The injectivity of the map S_q implies that any two such parabolic subgroups P_1 and P_2 have conjugate Levi factors.

3.3. *Case* $\mathfrak{g} = \mathfrak{so}_{2n+1}$

Let $V = \mathbb{C}^{2n+1}$ and let ϕ be a non-degenerate symmetric bilinear form on V. In the same way as in the case of $\mathfrak{g} = \mathfrak{sp}_{2n}$, one defines the notion of isotropic flags. The proof folows in a similar way to the case of $\mathfrak{g} = \mathfrak{sp}_{2n}$.

3.4. *Case* $\mathfrak{g} = \mathfrak{so}_{2n}$

Let $V = \mathbb{C}^{2n}$ and ϕ a non-degenerate symmetric bilinear form on V. The group H of automorphisms of V preserving ϕ has two components. The identity component of H is our Lie group $G \simeq \mathrm{SO}(2n)$.

By Lemma 4.4 [5], the class of parabolic subgroups of G with flag type $(p_1, \ldots, p_k)$ splits into two conjugacy classes (denoted by P_1 and P_2) under the action of G if and only if $k = 2t$ and $p_t \geqslant 2$. Furthermore, the two parabolic subgroups are conjugate under the action of H, i.e., there exists an element $h \in H$ such that $P_2 = hP_1h^{-1}$. Take an isotropic flag $F = (F_0, \ldots, F_k) \in G/P_1$, then $F' = hF = (hF_0, \ldots, hF_k)$ is an isotropic flag in G/P_2. This gives an isomorphism between G/P_1 and G/P_2.

So we need only to consider polarizations of $\mathcal{O}$ with different flag types. By the proof of Proposition 3.22 [3] and the proof of Lemma 4.6 [5], two such polarizations have conjugate Levi factors.

4. Concluding remarks

4.1. Though we believe that Theorem 1.1 is also true for exceptional Lie algebras, because of a lack of a complete description of the polarizations of their nilpotent orbits, we do not know how to check this.

4.2. One should bear in mind that for two symplectic resolutions $Z_i \to \overline{\mathcal{O}},\ i = 1, 2$, although Z_1 is deformation equivalent to $Z_2,\ Z_1$ and Z_2 may be non-isomorphic. An explicit example is given in [4].

4.3. By Remark 1, our theorem can be strengthened as follows: Let $Z_i \to \overline{\mathcal{O}},\ i = 1, 2$, be two symplectic resolutions for a nilpotent orbit closure in a simple complex classical Lie algebra. Then there exists two deformations $\mathcal{Z}_i \to \mathbb{C}$, such that $\mathcal{Z}_{i,0} \cong Z_i$ and $\mathcal{Z}_{1,t} \cong \mathcal{Z}_{2,t}$ for any $t \neq 0$. This can be viewed as an analogue of the result of Huybrechts (Theorem 4.6 [6]).

4.4. A stronger form of Conjecture 1 is stated in [4], where we conjectured that there exist deformations $F_i : \mathcal{Z}_i \to \mathcal{X}$ of the morphisms $\pi_i : Z_i \to X$, such that $F_{i,t}$ is an isomorphism for $t \neq 0$. In the case of nilpotent orbit closures in $\mathfrak{sl}_n$, we proved this in [4], by constructing explicitly the deformations. For the other cases, although by Remark 2 we have a deformation $\tilde{\pi}$ of π, we do not know whether the deformation spaces W_P of $\overline{\mathcal{O}}$ are isomorphic or not for two polarizations.

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