Available online at www.sciencedirect.com
SCIENCE DIRECT®
ELSEVIER
C. R. Acad. Sci. Paris, Ser. I 337 (2003) 159–164
COMPTES RENDUS
MATHEMATIQUE

Partial Differential Equations

Lifting of BV functions with values in S^1

Juan Dávila [a,1], Radu Ignat [b]

[a] *Departamento de Ingeniería Matemática, CMM (UMR CNRS), Universidad de Chile, Casilla 170/3, Correo 3, Santiago, Chile*
[b] *École normale supérieure, 45, rue d'Ulm, 75230 Paris cedex 05, France*

Received and accepted 19 June 2003

Presented by Haïm Brezis

Abstract

We show that for every $u \in \mathrm{BV}(\Omega; S^1)$, there exists a bounded variation function $\varphi \in \mathrm{BV}(\Omega; \mathbb{R})$ such that $u = \mathrm{e}^{\mathrm{i}\varphi}$ a.e. on Ω and $|\varphi|_{\mathrm{BV}} \leqslant 2|u|_{\mathrm{BV}}$. The constant 2 is optimal in dimension $n > 1$. ***To cite this article: J. Dávila, R. Ignat, C. R. Acad. Sci. Paris, Ser. I 337 (2003).***

Résumé

Relèvement des fonctions BV à valeurs sur le cercle S^1. On montre que pour tout $u \in \mathrm{BV}(\Omega; S^1)$, il existe une fonction à variation bornée $\varphi \in \mathrm{BV}(\Omega; \mathbb{R})$ telle que $u = \mathrm{e}^{\mathrm{i}\varphi}$ p.p. dans Ω et $|\varphi|_{\mathrm{BV}} \leqslant 2|u|_{\mathrm{BV}}$. La constante 2 est optimale en dimension $n > 1$. ***Pour citer cet article : J. Dávila, R. Ignat, C. R. Acad. Sci. Paris, Ser. I 337 (2003).***

Version française abrégée

Soit $\Omega \subset \mathbb{R}^n$ un ouvert et $u : \Omega \to S^1$ une fonction mesurable. Un relèvement de u est une fonction mesurable $\varphi : \Omega \to \mathbb{R}$ telle que

$$u(x) = \mathrm{e}^{\mathrm{i}\varphi(x)}$$

pour presque tout $x \in \Omega$. Une question naturelle est de savoir s'il existe un relèvement φ qui préserve la régularité de la fonction u. Par exemple, si Ω est simplement connexe et u est continue, alors on sait qu'on peut trouver un relèvement φ continu. Motivée par l'étude de l'équation de Ginzburg–Landau et la théorie du degré, on constate une recherche assidue sur ce problème de l'existence du relèvement dans les espaces de Sobolev, BMO et VMO (voir [3,4,7,6]).

Dans cette Note on étudie le cas des fonctions à variation bornée :

Théorème 0.1. *Soit $u \in \mathrm{BV}(\Omega; S^1)$. Alors il existe un relèvement $\varphi \in \mathrm{BV}(\Omega; \mathbb{R})$ de u tel que $|\varphi|_{\mathrm{BV}} \leqslant 2|u|_{\mathrm{BV}}$.*

E-mail addresses: jdavila@dim.uchile.cl (J. Dávila), Radu.Ignat@ens.fr (R. Ignat).
[1] The author was partially supported by Fondecyt Grant No. 1020815.

1631-073X/$ – see front matter
doi:10.1016/S1631-073X(03)00314-5

Si $n \geqslant 2$ la constante 2 est optimale ; on donne un exemple dans la Section 4. En dimension $n = 1$ on peut trouver un relèvement φ tel que $|\varphi|_{\mathrm{BV}} \leqslant \frac{\pi}{2}|u|_{\mathrm{BV}}$. L'existence du relèvement BV est montré aussi dans [8], mais sans contrôle sur $|\varphi|_{\mathrm{BV}}$.

L'idée de la démonstration du Théorème 0.1 est de considérer la fonction $L : S^1 \to \mathbb{R}$, $L(\mathrm{e}^{\mathrm{i}\theta}) = \theta$, $\forall -\pi \leqslant \theta < \pi$. Alors $\varphi = L(u)$ est un relèvement (mesurable) de u, ainsi que toutes les fonctions $L(\mathrm{e}^{\mathrm{i}\alpha}u) - \alpha$, $\forall \alpha \in \mathbb{R}$. En suite on montre que la fonction $\alpha \mapsto |L(\mathrm{e}^{\mathrm{i}\alpha}u)|_{\mathrm{BV}}$ est semi-continue inférieurement et qu'on a $\int_0^{2\pi} |L(\mathrm{e}^{\mathrm{i}\alpha}u)|_{\mathrm{BV}}\,\mathrm{d}\alpha \leqslant 4\pi\,|u|_{\mathrm{BV}}$.

Corollaire 0.2. *Soit $u \in \mathrm{BV}(\Omega; S^1)$. Alors il existe une suite $u_k \in C^\infty(\Omega; S^1) \cap \mathrm{BV}(\Omega)$ telle que $u_k \to u$ p.p. et* $\limsup_{k\to\infty} |u_k|_{\mathrm{BV}} \leqslant 2|u|_{\mathrm{BV}}$.

Remarque 1. Si on note $\mathrm{SBV}(\Omega, \mathbb{R}^m) = \{u \in \mathrm{BV}(\Omega; \mathbb{R}^m)\colon\ D^c u \equiv 0\}$ (où $D^c u$ est la partie Cantor de la différentielle Du), alors pour tout $u \in \mathrm{SBV}(\Omega; S^1)$ il existe un relèvement $\varphi \in \mathrm{SBV}(\Omega; \mathbb{R})$ tel que $|\varphi|_{\mathrm{BV}} \leqslant 2|u|_{\mathrm{BV}}$.

1. Introduction

Let $\Omega \subset \mathbb{R}^n$ be an open set and $u : \Omega \to S^1$ a measurable function. A lifting of u is a measurable function $\varphi : \Omega \to \mathbb{R}$ such that

$$u(x) = \mathrm{e}^{\mathrm{i}\varphi(x)}$$

for a.e. $x \in \Omega$. If u has some regularity one may ask whether or not φ can be chosen with some regularity as well. For example, if Ω is simply connected and u is continuous then it is well known that φ can be chosen to be continuous.

We are concerned in this Note with the case when u has bounded variation, and by this we mean that $u \in L^1_{\mathrm{loc}}(\Omega; \mathbb{R}^2)$, $|u(x)| = 1$ for a.e. $x \in \Omega$ and its BV seminorm is finite, i.e.,

$$|u|_{\mathrm{BV}} = \sup\left\{ \int_\Omega \sum_{i=1}^2 u_i \operatorname{div} g_i \,\mathrm{d}x\colon\ g_i \in C_0^\infty\big(\Omega; \mathbb{R}^n\big),\ \sum_{i=1}^2 |g_i|^2 \leqslant 1 \text{ in } \Omega \right\} < \infty,$$

where the norm in $\mathbb{R}^n$ is the Euclidean norm.

Remark 1. Throughout the paper we will say that $v \in \mathrm{BV}(\Omega; \mathbb{R}^m)$ if $v \in L^1_{\mathrm{loc}}(\Omega)$ and its standard BV seminorm $|v|_{\mathrm{BV}}$ is finite. We adopt this convention, because in the case that the Lebesgue measure of Ω is infinite, with the standard definition of BV where it is required that $v \in L^1(\Omega)$, there would not be any S^1-valued BV function.

Our main result is the following.

Theorem 1.1. *Let $u \in \mathrm{BV}(\Omega; S^1)$. Then there exists a lifting $\varphi \in \mathrm{BV}(\Omega; \mathbb{R})$ of u such that*

$$|\varphi|_{\mathrm{BV}} \leqslant 2|u|_{\mathrm{BV}}. \tag{1}$$

Remark 2. (1) If $n \geqslant 2$ the constant 2 appearing in (1) is optimal. We present an example in Section 4.

(2) The case of dimension $n = 1$ is simple, and in fact one can find a lifting φ with $|\varphi|_{\mathrm{BV}} \leqslant \frac{\pi}{2}|u|_{\mathrm{BV}}$.

(3) In [8], the authors show the existence of a lifting BV but they do not control its BV seminorm.

(4) If u belongs to the Sobolev space $W^{1,1}(\Omega; S^1)$ and $\Omega \subset \mathbb{R}^2$ is smooth, bounded and simply connected it was already known that u has a lifting $\varphi \in \mathrm{BV}(\Omega; \mathbb{R})$ which satisfies (1) (private communication of H. Brezis and P. Mironescu).

Regarding other function spaces there has been recently much research, specially motivated by the study of the Ginzburg–Landau equation. Firstly, Bethuel and Zheng [3] proved that if Ω is bounded and simply connected and $u \in W^{1,p}(\Omega; S^1)$ with $p \geqslant 2$ then u has a lifting $\varphi \in W^{1,p}(\Omega; \mathbb{R})$; this result is false in general if $n \geqslant 2$ and $1 \leqslant p < 2$. A complete description for the existence of the lifting in general Sobolev spaces $W^{s,p}(\Omega; S^1)$, $0 < s < \infty$ and $1 < p < \infty$, was given later by Bourgain, Brezis and Mironescu [4]. There are also results in the space BMO and VMO, see Coifman and Meyer [7] and Brezis and Nirenberg [6].

The idea for the proof of Theorem 1.1 is to consider the function $L : S^1 \to \mathbb{R}$ defined by

$$L\big(e^{i\theta}\big) = \theta \quad \forall -\pi \leqslant \theta < \pi. \tag{2}$$

Then $\varphi = L(u)$ is a lifting of u, in the sense that $e^{i\varphi(x)} = u(x)$ for all $x \in \Omega$. We would like to have $|\varphi|_{\mathrm{BV}} \leqslant 2|u|_{\mathrm{BV}}$, but this is far from true. It may even happen that $L(u)$ does not belong to BV (classical results for composition of functions assert only that if $f : S^1 \to \mathbb{R}$ is Lipschitz then $f(u)$ is BV). There is a way to remedy this situation. Indeed, observe that for fixed $\alpha \in \mathbb{R}$ the function $L(e^{i\alpha}u) - \alpha$ is also a lifting of u. We shall prove

Theorem 1.2. *The function $\alpha \mapsto |L(e^{i\alpha}u)|_{\mathrm{BV}}$ is lower semi-continuous and*

$$\int_0^{2\pi} \big|L\big(e^{i\alpha}u\big)\big|_{\mathrm{BV}}\, d\alpha \leqslant 4\pi |u|_{\mathrm{BV}}. \tag{3}$$

Remark 3. Inequality (3) can be viewed as a sort of co-area inequality. In particular it implies that for a.e. $\alpha \in \mathbb{R}$, $L(e^{i\alpha}u) \in \mathrm{BV}$. The constant 4π in (3) is sharp; see the example in Section 4.

To prove Theorem 1.1 we pick α_0 such that $|L(e^{i\alpha_0}u)|_{\mathrm{BV}}$ is minimum. By (3), $\varphi = L(e^{i\alpha_0}u) - \alpha_0$ is a lifting of u that satisfies (1).

Corollary 1.3. *Let $u \in \mathrm{BV}(\Omega; S^1)$. Then there exists a sequence $u_k \in C^\infty(\Omega; S^1) \cap \mathrm{BV}(\Omega)$ such that $u_k \to u$ a.e. and in L^1_{loc} and*

$$\limsup_{k\to\infty} |u_k|_{\mathrm{BV}} \leqslant 2|u|_{\mathrm{BV}}.$$

2. Preliminaries about the space BV

The material that we present next is standard and can be found in the book [2]. Let $v \in \mathrm{BV}(\Omega; \mathbb{R}^m)$. Its jump set $S(v)$ is defined by the requirement that $x \in \Omega \setminus S(v)$ if and only if there exists $\tilde{v}(x) \in \mathbb{R}^m$ such that $\tilde{v}(x) = \text{ap-lim}_{y\to x} v(y)$, that is:

$$\forall \varepsilon > 0 \quad \lim_{r\to 0} \frac{\mathrm{meas}(B_r(x) \cap \{y \in \Omega\colon |v(y) - \tilde{v}(x)| > \varepsilon\})}{\mathrm{meas}(B_r(x))} = 0.$$

It can be proved (see [2]) that for $\mathcal{H}^{n-1}$-a.e. $x \in S(v)$ there exist $v^+(x), v^-(x) \in \mathbb{R}^m$ and a unit vector $\nu_v(x)$ such that

$$\underset{y\to x,\ \langle y-x,\nu_v(x)\rangle>0}{\text{ap-lim}} v(y) = v^+(x), \qquad \underset{y\to x,\ \langle y-x,\nu_v(x)\rangle<0}{\text{ap-lim}} v(y) = v^-(x). \tag{4}$$

Dv is a matrix valued Radon measure which can be decomposed as $Dv = D^a v + D^j v + D^c v$, where $D^a v$ is defined as the absolutely continuous part of Dv with respect to the Lebesgue measure, while $D^j v$ and $D^c v$ are defined as

$$D^j v = Dv \llcorner S(v), \qquad D^c v = \big(Dv - D^a v\big)\llcorner\big(\Omega \setminus S(v)\big).$$

$D^j v$ is called the jump part and $D^c v$ the Cantor part of Dv. It can be proved that

$$D^j v = (v^+ - v^-) \otimes \nu_v \, \mathcal{H}^{n-1} \llcorner S(v).$$

Since we use just the local behavior of BV functions, throughout the paper we consider the precise representative $v^* : \Omega \mapsto \mathbb{R}^m$ of each $v \in \mathrm{BV}$, i.e.,

$$v^*(x) = \lim_{r \to 0} \frac{1}{\mathrm{meas}(B_r(x))} \int_{B_r(x)} v \, \mathrm{d}y$$

if this limit exists, and $v^*(x) = 0$ otherwise. Remark that v^* specifies the values of v except on a $\mathcal{H}^{n-1}$-negligible set.

It is well known that if $v \in \mathrm{BV}(\Omega; \mathbb{R}^m)$ and $f : \mathbb{R}^m \to \mathbb{R}$ is Lipschitz then $f \circ v$ belongs to BV, and Ambrosio and Dal Maso [1] proved a chain rule in this context. The following lemma is a slight modification of this chain rule for u in BV with values in S^1 (see also Theorem 3.99 in [2] for the case of scalar BV functions):

Lemma 2.1. *Let $\Omega \subset \mathbb{R}^n$ be an open set and $u \in \mathrm{BV}(\Omega; S^1)$. Let $f : S^1 \to \mathbb{R}$ be a Lipschitz function. Then $v = f \circ u$ belongs to $\mathrm{BV}(\Omega; \mathbb{R})$, f is differentiable at $u(x)$ for $(|D^a u| + |D^c u|)$-a.e. x and*

$$Dv = \vec{f}_\tau(u)\big(D^a u + D^c u\big) + \big(f(u^+) - f(u^-)\big)\nu_u \, \mathcal{H}^{n-1} \llcorner S(u), \tag{5}$$

where $\vec{f}_\tau$ denotes the tangential derivative of f.

3. Proof of Theorem 1.2

Let $u \in \mathrm{BV}(\Omega; S^1)$. For the proof of this theorem we consider a sequence of Lipschitz functions that approximate L (defined in (2)), and carry out the computations with this approximation. For small $\varepsilon > 0$ we let $L_\varepsilon : S^1 \to \mathbb{R}$ denote the following function

$$L_\varepsilon\big(\mathrm{e}^{\mathrm{i}\theta}\big) = \begin{cases} \theta & \text{if } 0 \leqslant \theta \leqslant \pi - \varepsilon, \\ \dfrac{\pi - \varepsilon}{\varepsilon}(\pi - \theta) & \text{if } \pi - \varepsilon \leqslant \theta \leqslant \pi + \varepsilon, \\ \theta - 2\pi & \text{if } \pi + \varepsilon \leqslant \theta \leqslant 2\pi. \end{cases}$$

Let $\alpha \in \mathbb{R}$ and define $\phi_{\alpha,\varepsilon} : S^1 \to \mathbb{R}$ by

$$\phi_{\alpha,\varepsilon}\big(\mathrm{e}^{\mathrm{i}\theta}\big) = L_\varepsilon\big(\mathrm{e}^{\mathrm{i}(\alpha+\theta)}\big).$$

Then $\phi_{\alpha,\varepsilon}$ is Lipschitz and therefore $\phi_{\alpha,\varepsilon}(u) \in \mathrm{BV}$. We use now the chain rule from Lemma 2.1 to compute the derivative of $\phi_{\alpha,\varepsilon}(u)$:

$$D\phi_{\alpha,\varepsilon}(u) = (\vec{L}_\varepsilon)_\tau\big(\mathrm{e}^{\mathrm{i}\alpha}u\big)\big(D^a u + D^c u\big) + \big(\phi_{\alpha,\varepsilon}(u^+) - \phi_{\alpha,\varepsilon}(u^-)\big)\nu_u \, \mathcal{H}^{n-1} \llcorner S(u).$$

Since the measures in the expression above are mutually singular, for the total variation of the corresponding measures we have

$$\big|D\phi_{\alpha,\varepsilon}(u)\big| \leqslant \big|(\vec{L}_\varepsilon)_\tau\big(\mathrm{e}^{\mathrm{i}\alpha}u\big)\big|\big(\big|D^a u\big| + \big|D^c u\big|\big) + \big|\phi_{\alpha,\varepsilon}(u^+) - \phi_{\alpha,\varepsilon}(u^-)\big| \, \mathcal{H}^{n-1} \llcorner S(u).$$

Integrating this total variation over Ω we get

$$\big|\phi_{\alpha,\varepsilon}(u)\big|_{\mathrm{BV}} \leqslant \int_\Omega \big|(\vec{L}_\varepsilon)_\tau\big(\mathrm{e}^{\mathrm{i}\alpha}u\big)\big| \, \mathrm{d}\big(\big|D^a u\big| + \big|D^c u\big|\big) + \int_{S(u)} \big|\phi_{\alpha,\varepsilon}(u^+) - \phi_{\alpha,\varepsilon}(u^-)\big| \, \mathrm{d}\mathcal{H}^{n-1}. \tag{6}$$

Observe that the map $\alpha \mapsto |\phi_{\alpha,\varepsilon}(u)|_{\mathrm{BV}}$ is lower semi-continuous because it is the supremum over a family of continuous functions in α. In particular $\alpha \mapsto |\phi_{\alpha,\varepsilon}(u)|_{\mathrm{BV}}$ is measurable. Integrating (6) with respect to α over $[0, 2\pi]$ we get

$$\int_0^{2\pi} \left|\phi_{\alpha,\varepsilon}(u)\right|_{\mathrm{BV}} \mathrm{d}\alpha \leqslant \int_0^{2\pi}\int_\Omega \left|(\vec{L}_\varepsilon)_\tau\left(\mathrm{e}^{\mathrm{i}\alpha}u\right)\right| \mathrm{d}\left(\left|D^a u\right| + \left|D^c u\right|\right) \mathrm{d}\alpha + \int_0^{2\pi}\int_{S(u)} \left|\phi_{\alpha,\varepsilon}(u^+) - \phi_{\alpha,\varepsilon}(u^-)\right| \mathrm{d}\mathcal{H}^{n-1}\,\mathrm{d}\alpha.$$

Let us consider the first term on the right-hand side above; by Fubini's theorem

$$\int_0^{2\pi}\int_\Omega \left|(\vec{L}_\varepsilon)_\tau\left(\mathrm{e}^{\mathrm{i}\alpha}u\right)\right| \mathrm{d}\left(\left|D^a u\right| + \left|D^c u\right|\right) \mathrm{d}\alpha = \int_\Omega\int_0^{2\pi} \left|(\vec{L}_\varepsilon)_\tau\left(\mathrm{e}^{\mathrm{i}\alpha}u\right)\right| \mathrm{d}\alpha\, \mathrm{d}\left(\left|D^a u\right| + \left|D^c u\right|\right).$$

But an easy computation shows that for any fixed x, $\int_0^{2\pi} |(\vec{L}_\varepsilon)_\tau(\mathrm{e}^{\mathrm{i}\alpha}u(x))|\,\mathrm{d}\alpha = 4(\pi-\varepsilon)$. So

$$\int_0^{2\pi}\int_\Omega \left|(\vec{L}_\varepsilon)_\tau\left(\mathrm{e}^{\mathrm{i}\alpha}u\right)\right| \mathrm{d}\left(\left|D^a u\right| + \left|D^c u\right|\right) \mathrm{d}\alpha = 4(\pi-\varepsilon)\left(\left|D^a u\right|(\Omega) + \left|D^c u\right|(\Omega)\right). \tag{7}$$

On the other hand, using the explicit formula for L_ε it is not hard to verify that if $|\theta_1-\theta_2| \leqslant \pi$ then

$$\int_0^{2\pi} \left|L_\varepsilon\left(\mathrm{e}^{\mathrm{i}(\alpha+\theta_1)}\right) - L_\varepsilon\left(\mathrm{e}^{\mathrm{i}(\alpha+\theta_2)}\right)\right| \mathrm{d}\alpha = 2\frac{\pi-\varepsilon}{\pi}|\theta_1-\theta_2|\left(2\pi - |\theta_1-\theta_2|\right) \leqslant 8(\pi-\varepsilon)\sin\left(|\theta_1-\theta_2|/2\right).$$

Observe that if $\mathrm{e}^{\mathrm{i}\theta_1} = u^+(x)$ and $\mathrm{e}^{\mathrm{i}\theta_2} = u^-(x)$ with $|\theta_1-\theta_2| \leqslant \pi$, then $|u^+(x) - u^-(x)| = 2\sin(|\theta_1-\theta_2|/2)$. Hence, for any fixed $x \in S(u)$ we obtain

$$\int_0^{2\pi} \left|\phi_{\alpha,\varepsilon}\left(u^+(x)\right) - \phi_{\alpha,\varepsilon}\left(u^-(x)\right)\right| \mathrm{d}\alpha \leqslant 4(\pi-\varepsilon)\left|u^+(x) - u^-(x)\right|.$$

Integrating over $S(u)$ and combining the result with (7) we establish that

$$\int_0^{2\pi} \left|\phi_{\alpha,\varepsilon}(u)\right|_{\mathrm{BV}} \mathrm{d}\alpha \leqslant 4(\pi-\varepsilon)|u|_{\mathrm{BV}}. \tag{8}$$

To finish the proof note that $\alpha \mapsto |L(\mathrm{e}^{\mathrm{i}\alpha}u)|_{\mathrm{BV}}$ is lower semi-continuous with values in $[0,\infty]$, because

$$\left|L\left(\mathrm{e}^{\mathrm{i}\alpha}u\right)\right|_{\mathrm{BV}} = \sup_{g \in C_0^\infty\ |g| \leqslant 1} \int_\Omega L\left(\mathrm{e}^{\mathrm{i}\alpha}u\right) \operatorname{div} g\, \mathrm{d}x$$

and for fixed g the map $\alpha \mapsto \int_\Omega L(\mathrm{e}^{\mathrm{i}\alpha}u) \operatorname{div} g\,\mathrm{d}x$ is lower semi-continuous. Also observe that for all except a countable set of $\alpha \in \mathbb{R}$ we have $\operatorname{meas}(\{y \in \Omega\colon u(y) = -\mathrm{e}^{-\mathrm{i}\alpha}\}) = 0$, and for these values of α, $L_\varepsilon(\mathrm{e}^{\mathrm{i}\alpha}u) \to L(\mathrm{e}^{\mathrm{i}\alpha}u)$ a.e. in Ω as $\varepsilon \to 0$. This implies that for a.e. α, $|L(\mathrm{e}^{\mathrm{i}\alpha}u)|_{\mathrm{BV}} \leqslant \liminf_{\varepsilon\to 0} |L_\varepsilon(\mathrm{e}^{\mathrm{i}\alpha}u)|_{\mathrm{BV}}$. Hence, using (8) and Fatou's lemma

$$\int_0^{2\pi} \left|L\left(\mathrm{e}^{\mathrm{i}\alpha}u\right)\right|_{\mathrm{BV}} \mathrm{d}\alpha \leqslant \liminf_{\varepsilon\to 0} \int_0^{2\pi} \left|L_\varepsilon\left(\mathrm{e}^{\mathrm{i}\alpha}u\right)\right|_{\mathrm{BV}} \mathrm{d}\alpha \leqslant 4\pi|u|_{\mathrm{BV}}. \qquad \square$$

Remark 4. Recall the space of special functions with bounded variation

$$\mathrm{SBV}\left(\Omega;\mathbb{R}^m\right) = \left\{u \in \mathrm{BV}\left(\Omega;\mathbb{R}^m\right) \mid D^c u \equiv 0 \text{ in } \Omega\right\}.$$

We say that $u \in \mathrm{SBV}(\Omega; S^1)$ if $u \in \mathrm{SBV}(\Omega; \mathbb{R}^2)$ and $|u(x)| = 1$ for a.e. $x \in \Omega$. Then each $u \in \mathrm{SBV}(\Omega; S^1)$ has a lifting $\phi \in \mathrm{SBV}(\Omega; \mathbb{R})$ satisfying (1). Indeed, by Theorem 1.1, there exists a lifting $\phi \in \mathrm{BV}(\Omega; \mathbb{R})$ such that (1) holds. By the chain rule for BV functions applied to the relation $u = \mathrm{e}^{\mathrm{i}\phi}$ we obtain

$$Du = \mathrm{i}u\big(D^a\phi + D^c\phi\big) + \big(\mathrm{e}^{\mathrm{i}\phi^+} - \mathrm{e}^{\mathrm{i}\phi^-}\big)\nu_\phi\, \mathcal{H}^{n-1} \llcorner S(\phi). \tag{9}$$

Since $D^c u = 0$ we see that $D^c\phi = 0$ and so $\phi \in \mathrm{SBV}$.

4. The constant 2 is optimal

The following result is a consequence of the paper [5]:

Lemma 4.1. *Let Ω be the unit disc in $\mathbb{R}^2$. Define $u : \Omega \setminus \{0\} \mapsto S^1$, $u(x) = \frac{x}{|x|}$ for every $x \in \Omega \setminus \{0\}$. Let $\phi \in \mathrm{BV}(\Omega; \mathbb{R})$ be a lifting of u. Then $|D\phi|(\Omega) \geqslant 4\pi = 2|u|_{\mathrm{BV}}$.*

Proof. Firstly remark that $u \in \mathrm{W}^{1,p}(\Omega)$ for all $p \in [1, 2)$ and $\int_\Omega |\nabla u|\,\mathrm{d}x = 2\pi$. Take $\phi_n \in W^{1,1} \cap C^\infty(\Omega; \mathbb{R})$ such that $\phi_n \to \phi$ a.e. on Ω and $\int_\Omega |\nabla\phi_n|\,\mathrm{d}x \to |D\phi|(\Omega)$ as $n \to \infty$. Set $u_n = \mathrm{e}^{\mathrm{i}\phi_n} \in W^{1,1} \cap C^\infty(\Omega; S^1)$. For every $r \in (0, 1)$ denote $S_r = \{x \in \mathbb{R}^2\colon\ |x| = r\}$. Up to a subsequence, for a.e. $r \in (0, 1)$ we have $u_n \to u\mathcal{H}^1$-a.e. in S_r and $\sup_n \int_{S_r} |\nabla u_n|\,\mathrm{d}\mathcal{H}^1$; for those r, by Lemma 18 of [5]

$$\liminf_{n\to\infty} \int_{S_r} |\nabla u_n \cdot \tau|\,\mathrm{d}\mathcal{H}^1 \geqslant \int_{S_r} |\nabla u \cdot \tau|\,\mathrm{d}\mathcal{H}^1 + 2\pi = \int_{S_r} |\nabla u|\,\mathrm{d}\mathcal{H}^1 + 2\pi$$

(here τ is the tangent vector in each point of S_r). Therefore, by Fatou's lemma,

$$|D\phi|(\Omega) = \liminf_{n\to\infty} \int_\Omega |\nabla u_n|\,\mathrm{d}x \geqslant \int_0^1 \liminf_{n\to\infty} \int_{S_r} |\nabla u_n|\,\mathrm{d}\mathcal{H}^1\,\mathrm{d}r \geqslant \int_\Omega |\nabla u| + 2\pi. \qquad \square$$

Remark 5. For dimension $n \geqslant 3$, we consider the cylinder $\Omega = B^2 \times (0, 1)^{n-2} \subset \mathbb{R}^n$ where B^2 is the unit disc in $\mathbb{R}^2$ and we repeat the above argument for the function $v(z, x_3, \ldots, x_N) = u(z)$.

Acknowledgement

The authors are deeply grateful to Prof. H. Brezis for suggesting the problem and his support.

References

[1] L. Ambrosio, G. Dal Maso, A general chain rule for distributional derivatives, Proc. Amer. Math. Soc. 108 (1990) 691–702.
[2] L. Ambrosio, N. Fusco, D. Pallara, Functions of Bounded Variation and Free Discontinuity Problems, Oxford University Press, Oxford, 2000.
[3] F. Bethuel, X.M. Zheng, Density of smooth functions between two manifolds in Sobolev spaces, J. Funct. Anal. 80 (1988) 60–75.
[4] J. Bourgain, H. Brezis, P. Mironescu, Lifting in Sobolev spaces, J. Anal. Math. 80 (2000) 37–86.
[5] J. Bourgain, H. Brezis, P. Mironescu, $H^{1/2}$ maps with values into the circle: minimal connections, lifting and the Ginzburg–Landau equation, in press.
[6] H. Brezis, L. Nirenberg, Degree theory and BMO. I. Compact manifolds without boundaries, Selecta Math. (N.S.) 1 (1995) 197–263.
[7] R.R. Coifman, Y. Meyer, Une généralisation du théorème de Calderón sur l'intégrale de Cauchy, in: Fourier Analysis (Proc. Sem., El Escorial, 1979), in: Asoc. Mat. Española, Madrid, 1980, pp. 87–116.
[8] M. Giaquinta, G. Modica, J. Soucek, Cartesian Currents in the Calculus of Variations, Vol. II, Springer, 1998.