



Mathematical Physics

Future complete S^1 symmetric Einsteinian spacetimes, the unpolarized case

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Abstract

We prove the existence of vacuum S^1 symmetric Einsteinian, unpolarized, space times which are complete in the direction of the expansion, for small initial data. **To cite this article:** *Y. Choquet-Bruhat, C. R. Acad. Sci. Paris, Ser. I 337 (2003).*

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Résumé

Espaces temps einsteinians complets vers le futur. On démontre l'existence d'espaces temps génériques, à symétrie S^1 mais non polarisés, complets dans la direction de l'expansion, pour des données initiales petites. **Pour citer cet article :** *Y. Choquet-Bruhat, C. R. Acad. Sci. Paris, Ser. I 337 (2003).*

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Version française abrégée

Un espace temps $(V, {}^{(4)}g)$ à symétrie S^1 est $V = M \times R$, avec M un fibré en cercles sur une surface Σ que nous supposons orientable, compacte et de genre supérieur à 1. Une métrique lorentzienne ${}^{(4)}g$ invariante par S^1 s'écrit :

$${}^{(4)}g = e^{-2\gamma} g + e^{2\gamma} (dx^3 + A_\alpha dx^\alpha)^2, \quad (1)$$

$${}^{(3)}g = -N^2 dt^2 + g_{ab} (dx^a + v^a dt)(dx^b + v^b dt). \quad (2)$$

Les coefficients sont définis sur $\Sigma \times R$. Les équations d'Einstein du vide pour $(V, {}^{(4)}g)$ sont un système couplé d'une application d'ondes de $(M, {}^{(3)}g)$ dans le plan de Poincaré et des équations d'Einstein sur $\Sigma \times R$ pour ${}^{(3)}g$ avec sources cette application. Ces équations se décomposent en un système elliptique sur chaque $\Sigma \times \{t\}$ et un système différentiel ordinaire pour l'évolution de la structure conforme de ces surfaces.

A l'aide de majorations elliptiques et d'inégalités différentielles satisfaites par des énergies corrigées pour l'application d'ondes et ses dérivées premières, on démontre un théorème d'existence globale vers le futur, pour

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des énergies initiales petites (hypothèse qui implique la trivialité du fibré M). Le paramètre temporel est défini par $t = -\tau^{-1}$, où τ est la courbure extrinsèque moyenne de $\Sigma \times \{t\}$. Les espaces temps ainsi construits satisfont au critère de complétude donné dans [4]. Ils possèdent, quand t tend vers l'infini, l'attracteur :

$${}^{(4)}g = e^{-2\gamma_\infty}(-4 dt^2 + 2t^2 \sigma_\infty) + e^{2\gamma_\infty}(dx^3 + H_\infty)^2, \quad (3)$$

où γ_∞ est une constante, $R(\sigma_\infty) = -1$, H_∞ est une 1-forme harmonique sur Σ .

1. Introduction

I generalize the nonlinear stability theorem obtained in collaboration with V. Moncrief [7,8] for Einsteinian 4-manifolds $(V, {}^{(4)}g)$ with $V = M \times R$, M a circle bundle over a compact, orientable surface Σ of genus greater than 1, and ${}^{(4)}g$ a Lorentzian metric admitting a Killing symmetry along the (spacelike) circular fibers: the polarization condition is removed, i.e., the orthogonality of the fibers to the quotient 3-manifolds. The vacuum Einstein equations take then the form of a wave map system (instead of a linear wave equation) coupled to 2 + 1 gravity. I use results on wave maps from curved manifolds obtained in [2,3]. Some steps of the proof have been obtained independently, using other notations by V. Moncrief, I thank him for communicating his manuscript to me, and for numerous conversations on the subject.

2. S^1 invariant Einsteinian universes

The spacetime manifold V is a trivial fiber bundle with Lie group S^1 and base $\Sigma \times R$, with Σ a smooth orientable compact surface of genus greater than one. The spacetime metric is ${}^{(4)}g$ invariant under the action of S^1 . We denote by x^3 a coordinate on S^1 , set $x^0 = t \in R$, x^a , $a = 1, 2$, coordinates on Σ . The metric ${}^{(4)}g$ takes the form (1), with γ a function, A a 1-form, and ${}^{(3)}g$ a Lorentzian metric on $\Sigma \times R$ which we write as (2) in the usual 2 + 1 decomposition. It has been proved in [10,6] that the 3 + 1 Einstein equations reduce then to a 2 + 1 Einstein-wave map system, with a wave map from $(\Sigma \times R, {}^{(3)}g)$ into the Poincaré plane $P \equiv (R^2, G \equiv 2(d\gamma)^2 + (1/2)e^{-4\gamma}(d\omega)^2)$ for a pair of functions (γ, ω) . The so called twist potential ω , zero in the polarized case, is a scalar function such that

$$dA = \frac{1}{2} e^{-3\gamma} * d\omega, \quad (4)$$

it permits the construction of A when it satisfies initially (and then at all times) the integral condition

$$\int_{\Sigma} e^{-4\gamma} N^{-1} \partial_0 \omega \mu_g = 0. \quad (5)$$

The 2 + 1 Einstein equations are equivalent [10] to the constraints, equations for lapse and shift, and the L^2 orthogonality of the evolution part of these equations to traceless divergence free symmetric 2-tensors, all on each Σ_t .

We solve the constraints on each Σ_t by the conformal method: we set $g_{ab} = e^{2\lambda} \sigma_{ab}$, with σ is a smooth Riemannian metric on Σ , depending on t , which we choose such that $R(\sigma) = -1$: Σ is in the negative Yamabe class since its genus is greater than 1, hence its Euler number is $\chi < 0$. We impose to Σ_t that it has constant mean curvature τ . We choose the time parameter t such that $t = -\tau^{-1}$. Then t will increase from $t_0 > 0$ to infinity when, Σ_t expanding, $\tau(t)$ increases from $\tau_0 < 0$ to zero. The constraints reduce to, with $|\cdot|$ pointwise norm, D covariant derivation and Δ_σ the Laplacian in the metric σ , and h traceless:

$$D_b h_a^b = L_a, \quad L_a \equiv -D_a u \cdot \dot{u}, \tag{6}$$

$$\Delta_\sigma \lambda = f(x, \lambda) \equiv p_1 e^{2\lambda} - p_2 e^{-2\lambda} + p_3, \tag{7}$$

$$p_1 \equiv \frac{1}{4} \tau^2, \quad p_2 \equiv \frac{1}{2} (|\dot{u}|^2 + |h|^2), \quad p_3 \equiv -\frac{1}{2} (1 + |Du|^2), \tag{8}$$

$$\dot{u} = e^{2\lambda} u', \quad u' = N^{-1} \partial_0 u, \quad \partial_0 = \frac{\partial}{\partial t} - v^a \partial_a, \quad \partial_a = \frac{\partial}{dx^a}. \tag{9}$$

The lapse N satisfies on each Σ_t the linear elliptic (in N) equation:

$$\Delta N - \alpha N = -e^{2\lambda} \frac{\partial \tau}{\partial t}, \quad \alpha \equiv e^{-2\lambda} (|h|^2 + |\dot{u}|^2) + \frac{1}{2} e^{2\lambda} \tau^2. \tag{10}$$

The definition of h gives for the shift v the system:

$$(\mathcal{L}_\sigma v)_{ab} \equiv D_a v_b + D_b v_a - \sigma_{ab} D_c v^c = f_{ab}, \tag{11}$$

$$f_{ab} \equiv 2N e^{-2\lambda} h_{ab} + \partial_t \sigma_{ab} - \frac{1}{2} \sigma_{ab} \sigma^{cd} \partial_t \sigma_{cd}. \tag{12}$$

The conformal Killing operator \mathcal{L}_σ has injective symbol, it has kernel zero on manifolds of genus greater than 1. The kernel of the dual of \mathcal{L}_σ is the space of divergence free, traceless symmetric (TT) 2-tensors, i.e., such that $\sigma^{ab} T_{ab} = 0, D^a T_{ab} = 0$.

We suppose given a mapping $Q \mapsto \psi(Q), Q = (Q^I, I = 1, \dots, 6G - 6) \in T_{eich}$, Teichmüller space of classes of conformally inequivalent metrics on $\Sigma, \psi(Q) \in M_{-1}$, metrics with scalar curvature -1 [1,5,9]. Then the $\partial\psi/\partial Q^I$ are tensors with traceless transverse parts a known basis $X_I(Q)$ of TT tensors on $(\Sigma, \psi(Q))$. The metrics σ_t are determined by a curve $t \mapsto Q(t)$ in T_{eich} . The traceless tensor h admits the decomposition

$$h = q + r, \quad \text{with } q = P^I(t) X_I(Q(t)) \tag{13}$$

with r a tensor in the range of \mathcal{L}_σ . The pair $(Q(t), P(t))$ satisfies an ordinary differential system deduced from the integrability condition of (10) and the L^2 orthogonality of the evolution part of the 2 + 1 Einstein equations with TT tensors. Its coefficients have simple expressions (see [6,7]) in terms of u, λ, N, v and h :

$$X_{IJ} \frac{dQ^I}{dt} + Y_{IJ} P^I + Z_J = 0, \quad X_{IJ} \frac{dP^I}{dt} + \Phi_J \left(P, \frac{dQ}{dt} \right) = 0. \tag{14}$$

The following theorem is a refinement of a theorem of [6], Sobolev spaces on Σ are defined through the metric σ_0 .

Theorem 2.1 (Local in time existence). *The Cauchy data¹ on Σ_{t_0} for the Einstein equations with S^1 isometry group are a C^∞ metric $\sigma_0 \in M_{-1}$, a C^∞ TT tensor q_0 , and data for u and \dot{u} such that the integral condition (5) is satisfied. This Cauchy problem has, if $u(t_0, \cdot) = u_0 \in H_2, \dot{u}(t_0, \cdot) \equiv \dot{u}_0 \in H_1$ and if $T - t_0$ is small enough, a solution with $u \in C^0([t_0, T], H_2), \dot{u} \in C^1([t_0, T], H_1); \lambda, N, v \in C^0([t_0, T], W_3^p) \cap C^1([t_0, T], W_2^p), 1 < p < 2$, and $N > 0$ while $\sigma \in C^1([t_0, T], C^\infty)$ with σ_t uniformly equivalent to σ_0 . This solution is unique up to t parametrization of τ , choice of A_t , and choice of the mapping ψ .*

It results from the local existence theorem and a standard argument that the solution exists on $[t_0, \infty)$ if the curve $t \mapsto Q(t)$ remains in a compact subset of T_{eich} and the norms $\|u(t, \cdot)\|_{H_2}, \|\dot{u}(t, \cdot)\|_{H_1}$ do not blow up for any finite t .

¹ The given data determine the initial data for the other unknown.

3. Energies

We denote by $|\cdot|$ and $|\cdot|_g$, norms in the metric G , and g and G ; by $\|\cdot\|_g$ the $L^2(g)$ norm of $|\cdot|_g$. As in [7] we define the first energy by

$$\varepsilon^2 \equiv E(t) \equiv \frac{1}{2} \{ \|u'\|_g^2 + \|Du\|_g^2 + \|h\|_g^2 \} \tag{15}$$

and we find that $E(t)$ is a nonincreasing function of t if τ is negative:

$$\frac{dE(t)}{dt} = \frac{1}{2} \tau \int_t (|h|_g^2 + |u'|^2) N \mu_g. \tag{16}$$

We define the second energy by the following formula

$$\tau^2 \varepsilon_1^2 \equiv E^{(1)}(t) \equiv \frac{1}{2} \{ \|\hat{\Delta}_g u\|_g^2 + \|\hat{\nabla} u'\|_g^2 \}, \tag{17}$$

where $\hat{\nabla}$ denotes a covariant derivative in the metrics g and G , for t dependent sections of a fiber bundle $E^{p,q}$ with base Σ and fiber $\otimes^p T_x^* \Sigma \otimes^q T_{u(x)} P$.

4. Elliptic estimates

The equations for h, λ, N , and v are elliptic equations on (Σ_t, σ_t) , identical with those written in [7], except that in the coefficients $Du \cdot \dot{u}, |Du|^2, |\dot{u}|^2$ which appear in these equations u is now a wave map and not a scalar function. The estimates obtained in [7] in terms of ε and ε_1 will be valid if the new coefficients satisfy the same estimates in terms of our new ε and ε_1 . The following inequalities are independent of these estimates, they result from the maximum principle and the equations satisfied by N and λ :

$$0 \leq N_m \leq N \leq N_M \leq 2, \quad e^{-2\lambda_M} \leq e^{-2\lambda} \leq e^{-2\lambda_m} \leq \frac{1}{2} \tau^2. \tag{18}$$

Hypothesis H_λ . There exists a number $c_\lambda > 1$, independent of t , such that

$$\frac{1}{\sqrt{2}} e^{\lambda_M} |\tau| \leq c_\lambda. \tag{19}$$

We denote by C_λ any positive number depending continuously on $c_\lambda \in R_+$.

Hypothesis H_σ . The curve $t \mapsto Q(t)$ is contained in a compact subset of T_{eich} . Choosing a cross section $\psi : T_{eich} \rightarrow M_{-1}$ the metric σ_t is then uniformly equivalent to σ_0 . The norms $W_s^p(\sigma_t)$ and $W_s^p(\sigma_0) \equiv W_s^p$ are uniformly equivalent as well as Sobolev constants on (Σ, σ_t) and (Σ, σ_0) . We denote by C_σ any positive number depending only on (Σ, σ_0) and the considered compact subset of T_{eich} .

Hypothesis H_E . There exists a positive number c_E such that $\varepsilon + \varepsilon_1 \leq c_E$. We denote by C_E any positive number depending continuously on $c_E \in R^+$.

We denote by $|\cdot|$ a pointwise norm in the σ and G metrics, by $\|\cdot\|$ and $\|\cdot\|_p$ L^2 and L^p norms of $|\cdot|$ in the σ metric. We denote by \hat{D} a covariant derivative for gradients of maps $(\Sigma, \sigma) \rightarrow (R^2, G)$. The following lemma is obtained by integration by parts, the Ricci formula and $|Du \cdot Du| \leq |Du|^2$.

Lemma 4.1. *If $R(\sigma) = -1$ and $R(G) \leq 0$, it holds that:*

$$\|\hat{D}Du\|^2 \leq 2 e^{2\lambda_M} \tau^2 \varepsilon_1^2 + \varepsilon^2. \tag{20}$$

Using properties of covariant derivatives \widehat{D} , this lemma, definitions and Sobolev embeddings, one proves that $|Du|^2$ and $|\dot{u}|^2$ and their derivatives satisfy the same inequalities as those satisfied by the function u in the polarized case. We obtain, denoting by $C_{E,\lambda,\sigma}$ some continuous positive function of $c_E, c_\lambda, c_\sigma \in R^+$:

Theorem 4.2. (i) *It holds that:*

$$1 \leq \frac{1}{\sqrt{2}}|\tau|e^{\lambda M} \leq 1 + C_{E,\lambda,\sigma}(\varepsilon + \varepsilon_1); \tag{21}$$

(ii) *There exists $\eta_1 > 0$ such that the hypothesis H_λ is satisfied as soon as*

$$\varepsilon + \varepsilon_1 \leq \eta_1. \tag{22}$$

We obtain estimates in W_s^p , $1 < p < 2$, by using the elliptic theory, and adapting the proofs of [7] to our case. We thus prove that:

$$\begin{aligned} \|h\|_{L^\infty(g)} &\leq C_\lambda \tau^2 \|h\|_\infty \leq C_\sigma C_\lambda \tau^2 \|h\|_{W_2^p} \leq C_{E,\lambda,\sigma} |\tau| (\varepsilon + \varepsilon_1), \\ \|2 - N\|_{L^\infty} &\leq C_\sigma \|2 - N\|_{H_2} \leq C_{E,\lambda,\sigma} (\varepsilon^2 + \varepsilon \varepsilon_1). \end{aligned}$$

Hence there exists $\eta_2 > 0$ and $N_m > 0$, (independent of t) such that

$$N \geq N_m > 0, \quad \text{if } \varepsilon + \varepsilon_1 \leq \eta_2, \tag{23}$$

$$\|DN\|_{L^\infty(g)} \leq C_{\lambda,\sigma,E} |\tau| (\varepsilon^2 + \varepsilon \varepsilon_1). \tag{24}$$

Using the differential equation satisfied by Q , hence by $\sigma \equiv \psi(Q)$, the hypothesis H_σ and these elliptic estimates we find that

$$\left| \frac{dQ}{dt} \right| \leq C_\sigma C_E |\tau| (\varepsilon + \varepsilon_1) \quad \text{hence } |\partial_t \sigma| \leq C_\sigma C_E |\tau| (\varepsilon + \varepsilon_1), \tag{25}$$

while the equation satisfied by the shift implies that:

$$\|v\|_{W_3^p} \leq C_\sigma C_E |\tau| (\varepsilon + \varepsilon_1). \tag{26}$$

5. Second energy estimate

We define $\widehat{\partial}_0$ to be a differential operator mapping a t dependent section of a bundle $E^{p,q}$ into another such section by:

$$\widehat{\partial}_0 \widehat{\nabla}^p u^A = \bar{\partial}_0 \widehat{\nabla}^p u^A + G_{BC}^A \partial_0 u^B \widehat{\nabla}^p u^C, \quad \bar{\partial}_0 \equiv \partial_0 - L_\nu, \tag{27}$$

where L_ν is the Lie derivative with respect to ν . The wave map system can then be written:

$$-\widehat{\partial}_0 u'^A + \widehat{\nabla}^a (N \partial_a u^A) + N \tau u' = 0. \tag{28}$$

Since $\widehat{\partial}_0 G_{AB} = 0$ and $\widehat{\partial}_0 g_{ab} \equiv \bar{\partial}_0 g_{ab} \equiv -2Nk_{ab}$, it holds that

$$\partial_0 J_1 = \widehat{\partial}_0 \widehat{\Delta}_g u \cdot \widehat{\Delta}_g u, \quad J_1 \equiv \frac{1}{2} |\widehat{\Delta}_g u|^2, \tag{29}$$

$$\partial_0 J_0 \equiv g^{ab} \widehat{\partial}_0 \widehat{\nabla}_a u' \cdot \widehat{\nabla}_b u' + N \left(h_g^{ab} + \frac{1}{2} g^{ab} \tau \right) \partial_a u' \cdot \partial_b u', \quad J_0 \equiv \frac{1}{2} |\widehat{\nabla} u'|_g^2. \tag{30}$$

Commutation relations between $\widehat{\partial}_0$ and $\widehat{\nabla}$ proved in [3] and lengthy estimates using previous results and the wave map equation lead to the following *second energy inequality*:

$$\frac{dE^{(1)}}{dt} - 2\tau E^{(1)}(t) \leq |\tau|^3 C_{\sigma,\lambda,E}(\varepsilon + \varepsilon_1)^3. \quad (31)$$

These energy estimates are not sufficient to bound the domain of Teichmüller space where σ_t projects.

6. Corrected energies

The corrected first energy is, with α a positive constant, $u = (\gamma, \omega)$:

$$E_\alpha(t) = E(t) - \alpha\tau E_c(t), \quad E_c(t) \equiv \int_{\Sigma_t} (u - \bar{u}) \cdot u' \mu_g, \quad (32)$$

$$\bar{u} \equiv -\frac{1}{4\pi\chi} \int_{\Sigma_t} u \mu_\sigma, \quad (u - \bar{u}) \cdot u' \equiv 2(\gamma - \bar{\gamma})\gamma' + \frac{1}{2} e^{-4\gamma} (\omega - \bar{\omega})\omega'.$$

The Cauchy–Schwarz inequality on (Σ, g) , $g \equiv e^{2\lambda}\sigma$ and the Poincaré inequality imply that:

$$\left| \int_{\Sigma_t} (\gamma - \bar{\gamma})\gamma' \mu_g \right| \leq e^{\lambda_M} \Lambda_\sigma^{-1/2} \|D\gamma\| \|\gamma'\|_g \quad (33)$$

with Λ_σ the first positive eigenvalue of the Laplacian Δ_σ on functions with mean value zero. An analogous procedure gives, with γ_m and γ_M the lower and upper bounds of γ :

$$\left| \int_{\Sigma_t} e^{-4\gamma} (\omega - \bar{\omega})\omega' \mu_g \right| \leq e^{2(\gamma_M - \gamma_m)} e^{\lambda_M} \Lambda_\sigma^{-1/2} \|e^{-2\gamma} D\omega\| \|e^{-2\gamma} \omega'\|_g. \quad (34)$$

We estimate $\gamma_M - \gamma_m$ using Sobolev embeddings, Poincaré inequality, the relation between $\widehat{\Delta}$ and Δ , and Lemma 4.1. We find that:

$$0 \leq \gamma_M - \gamma_m \leq 2\|\gamma - \bar{\gamma}\|_{L^\infty} \leq C_\sigma \|\gamma - \bar{\gamma}\|_{H_2} \leq C_{\sigma,\lambda,E} \{\varepsilon + \varepsilon_1\}.$$

We have obtained that

$$\left| \tau \int_{\Sigma_t} (u - \bar{u}) \cdot u' \mu_g \right| \leq \sqrt{2} \Lambda_\sigma^{-1/2} \|Du\| \|u'\|_g + C_{E,\lambda,\sigma} \varepsilon^2 (\varepsilon + \varepsilon_1). \quad (35)$$

Therefore, with $\|Du\| = x_1$, $\|u'\|_g = x_0$, there exists a number $\ell > 0$ such that

$$E_\alpha(t) \geq \frac{1}{2} (\|h\|_g^2 + x_0^2 + x_1^2) - \alpha\sqrt{2} \Lambda_\sigma^{-1/2} x_0 x_1 - C_{E,\lambda,\sigma} \varepsilon^2 (\varepsilon + \varepsilon_1) \geq \ell E(t), \quad (36)$$

if $\alpha < 2^{-1/2} \Lambda_\sigma^{1/2}$ and

$$\varepsilon + \varepsilon_1 \leq \eta_3, \quad \text{with } 0 < \eta_3 < \frac{1 - \alpha\sqrt{2} \Lambda_\sigma^{-1/2}}{C_{\sigma,E,\lambda}}. \quad (37)$$

If u satisfies the wave map equation, straightforward computations lead to the differential inequality

$$\frac{dE_\alpha}{dt} - k\tau E_\alpha \leq |\tau A|, \quad |\tau A| \leq |\tau| C_{E,\lambda,\sigma} (\varepsilon + \varepsilon_1)^3, \quad (38)$$

when the number k is such that

$$k \leq 4\alpha, \quad k \leq 2(1 - 2\alpha), \quad 2\alpha^2 \Lambda^{-1} (1 - k)^2 - 4 \left(2\alpha - \frac{k}{2} \right) \left(1 - 2\alpha - \frac{k}{2} \right) \leq 0. \tag{39}$$

- (i) If $\Lambda > \frac{1}{8}$ the best choice is $\alpha = \frac{1}{4}, k = 1$;
- (ii) $\Lambda \leq \frac{1}{8}$. Then α and k are such that (then also $1 - \alpha\sqrt{2}\Lambda\sigma^{-1/2} > 0$):

$$0 < \alpha < \frac{4}{8 + \Lambda^{-1}} \leq \frac{1}{4}, \quad 0 < k < 1 - \frac{1 - 4\alpha}{(1 - 2\Lambda^{-1}\alpha^2)^{1/2}}. \tag{40}$$

We define a *corrected second energy* $E_\alpha^{(1)}$ by the formula

$$E_\alpha^{(1)}(t) = E^{(1)}(t) + \alpha\tau E_c^{(1)}(t), \quad E_c^{(1)}(t) \equiv \int_{\Sigma_t} \hat{\Delta}_g u \cdot u' \mu_g. \tag{41}$$

One proves that if α satisfies (40) there exists a number $\eta_4 > 0$ and $L > 0$ such that

$$E_\alpha + \tau^{-2} E_\alpha^{(1)} \geq L(\varepsilon^2 + \varepsilon_1^2) \quad \text{if } \varepsilon + \varepsilon_1 \leq \eta_4 \tag{42}$$

and, through long computations and estimates, if u satisfies the wave map equation,

$$\frac{dE_\alpha^{(1)}}{dt} - (2 + k)\tau E_\alpha^{(1)} \leq |\tau|^3 B, \quad |B| \leq C_{E,\lambda,\sigma}(\varepsilon + \varepsilon_1)^3.$$

7. Decays and global existence

Hypothesis H_E^η . In addition to the hypotheses H_σ and H_E we suppose that ε^2 and ε_1^2 satisfy the inequalities (37), (42) the hypothesis H_λ is satisfied. We denote by $M_i > 0$ a bound of the form $C_{\sigma,E}$, by Λ a lower bound of Λ_σ . We choose α such that

$$\alpha = \frac{1}{4} \quad \text{if } \Lambda > \frac{1}{8}, \quad \alpha < \frac{4}{8 + \Lambda^{-1}} \leq \frac{1}{4} \quad \text{if } \Lambda \leq \frac{1}{8}. \tag{43}$$

Lemma 7.1. *Under the hypotheses H_σ and H_E^η there exists a number $k > 0$ such that the total energy $E_{\text{tot}}(t) \equiv \phi(t) \equiv \varepsilon^2 + \varepsilon_1^2$ satisfies an estimate of the form*

$$t^k \phi(t) \leq M_2 \phi(t_0) \tag{44}$$

if it is small enough initially.

Proof. We have shown that the *total corrected energy* $\psi \equiv E_\alpha + \tau^{-2} E_\alpha^{(1)}$ satisfies:

$$\frac{d\psi}{dt} \leq -\frac{k}{t} (\psi - M_1 \psi^{3/2}). \tag{45}$$

We suppose that $\psi_0 \equiv \psi(t_0)$ satisfies $\psi_0^{1/2} < M_1^{-1}$. Then ψ starts and continues to decrease, therefore $(\psi - M_1 \psi^{3/2}) > 0$. The announced decay estimate is obtained by integration of (45) and the use of (42). \square

Lemma 7.2. *Under the hypothesis H_E^ε and H_σ there exists a number M_3 such that:*

$$|Q(t) - Q(t_0)| \leq M_3 \phi(t_0)^{1/2}.$$

Proof. This uses the inequalities (25) and (44). \square

The proof of the following theorem is as in [7] a consequence of these two lemmas, and a continuity argument. Completeness results from a theorem of [4].

Theorem 7.3. *Let initial data for the Einstein equations with $U(1)$ isometry group satisfy the hypothesis of the local existence Theorem 2.1. Then there exists a number $\eta_0 > 0$ such that if $\phi(t_0) \equiv E(t_0) + \tau_0^{-2} E^{(1)}(t_0) < \eta_0$ these Einstein equations have a future timelike and null complete solution admitting these initial data.*

The asymptotic behaviour of the solution is given by (3).

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