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Discriminant of a generic projection of a minimal normal surface singularity

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Abstract

Let $(S,0)$ be a rational complex surface singularity with reduced fundamental cycle, also known as a *minimal* singularity. Using a fundamental result of M. Spivakovsky, we explain how to describe the equisingularity type of the discriminant curve for a generic projection of $(S,0)$ onto $(\mathbb{C}^2,0)$ from the resolution of $(S,0)$. ***To cite this article: R. Bondil, C. R. Acad. Sci. Paris, Ser. I 337 (2003).***

Résumé

Discriminant d'une projection générique d'une singularité minimale de surface normale. Soit $(S,0)$ une singularité rationnelle de surface complexe à cycle fondamental réduit, appelée aussi singularité *minimale*. En utilisant un résultat fondamental de M. Spivakovsky, on montre comment le type d'équisingularité de la courbe plane discriminant d'une projection générique de $(S,0)$ sur $(\mathbb{C}^2,0)$ est déterminé par la résolution de $(S,0)$. ***Pour citer cet article : R. Bondil, C. R. Acad. Sci. Paris, Ser. I 337 (2003).***

Version française abrégée

Soit $(S,0)$ une singularité *minimale* de surface complexe normale, c'est-à-dire une singularité rationnelle à cycle fondamental réduit (cf. [10], II.2, en dimension quelconque voir [7] § 3.4 et [2] § 5).

On plonge $(S,0)$ dans un $(\mathbb{C}^N,0)$ et on considère une projection $p_D:(S,0)\to(\mathbb{C}^2,0)$, restriction à S d'une projection linéaire de noyau un $(N-2)$-plan D. Lorsque p_D est finie, on note $C_1(D)$ l'adhérence du lieu critique de $(p_D)_{|S\setminus\{0\}}$, appelée courbe polaire pour la direction D.

Pour D générique, on montre dans cette Note que le type d'équisingularité de la courbe plane $(\Delta_{p_D},0)\subset(\mathbb{C}^2,0)$, image de $C_1(D)$ par p_D et appelée *discriminant de la projection* p_D, ne dépend que du graphe dual de résolution de $(S,0)$ (cf. la remarque p. 6) et on décrit ce discriminant générique, que l'on notera $\Delta_{S,0}$.

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On se fonde sur un résultat de M. Spivakovsky (Théorème 2.3) décrivant la transformée stricte de $C_1(D)$ sur la résolution minimale X de la singularité minimale $(S,0)$, pour D dans un ouvert dense de la Grassmannienne $G(N-2,N)$ des $(N-2)$ plans de $\mathbb{C}^N$.

Les résultats clefs de cette Note sont démontrés au paragraphe 3 :

Soit $(S,0)$ une singularité *minimale* de surface normale plongée dans $(\mathbb{C}^N,0)$ et $\pi: X\to(S,0)$ sa résolution minimale, que l'on peut écrire comme une composition $\pi=\pi_1\circ\cdots\circ\pi_r$ d'éclatements de points. Alors π est une résolution *minimale* des singularités de la courbe polaire générique.

En outre la courbe polaire générique $C_1(D)$ a toutes ses branches de multiplicité 1 ou 2, et en particulier planes.

Enfin, la projection de $C_1(D)$ sur le discriminant Δ_{p_D} préserve le contact entre les branches, c'est-à-dire qu'elles sont séparées au bout du même nombre d'éclatements de points.

Avec ces résultats, et le théorème de Spivakovsky, il est facile d'obtenir pour chaque graphe de résolution Γ d'une singularité minimale, la classe d'équisingularité du discriminant générique $\Delta_{S,0}$ comme cela est expliqué au paragraphe 4, où l'on donne aussi des exemples.

0. Introduction

This Note is organized as follows: in Section 1 we recall the definitions of polar curves, discriminants and a remarkable property of transversality of Briançon–Henry (Theorem 1.2). For minimal surface singularities, a theorem of Spivakovsky describes the behavior of the generic polar curve (cf. Section 2). We use this theorem in Section 3 to prove lemmas relating on the one hand the resolution of the generic polar curve to the resolution of a minimal surface singularity, and on the other hand, polar curves and discriminants. Gathering these results, our main theorem in Section 4 provides a combinatorial way to describe the discriminant.

1. Polar curves and discriminants

Let $(S,0)$ be a normal complex surface singularity $(S,0)$, embedded in $(\mathbb{C}^N,0)$: for any $(N-2)$-dimensional vector subspace D of $\mathbb{C}^N$, we consider a linear projection $\mathbb{C}^N\to\mathbb{C}^2$ with kernel D and denote by $p_D:(S,0)\to(\mathbb{C}^2,0)$, the restriction of this projection to $(S,0)$.

Restricting ourselves to the D such that p_D is finite, and considering a small representative S of the germ $(S,0)$, we define, as in [9], (2.2.2), the *polar curve* $C_1(D)$ of the germ $(S,0)$ for the direction D, as the closure in S of the critical locus of the restriction of p_D to $S\setminus\{0\}$. As explained in loc. cit., it makes sense to say that for an open dense subset of the Grassmann manifold $G(N-2,N)$ of $(N-2)$-planes in $\mathbb{C}^N$, the space curves $C_1(D)$ are *equisingular* in terms of strong simultaneous resolutions.

Then we define the *discriminant* Δ_{p_D} as (the germ at 0 of) the reduced analytic curve of $(\mathbb{C}^2,0)$ image of $C_1(D)$ by the finite morphism p_D.

Again, one may show that, for a generic choice of D, the discriminants obtained are *equisingular germs of plane curves*, but we will need a much more precise result, that demands another definition (cf. [3], IV.3):

Definition 1.1. Let $(X,0)\subset(\mathbb{C}^N,0)$ be a germ of reduced curve. Then a linear projection $p:\mathbb{C}^N\to\mathbb{C}^2$ will be said to be *generic* with respect to $(X,0)$ if the kernel of p does not contain any limit of bisecants to X (cf. [3] for an explicit description of the cone $C_5(X,0)$ formed by the limits at 0 of bisecants to X).

We now state the following transversality result (proved for curves on surfaces of $\mathbb{C}^3$ in [4], Theorem 3.12 and in general as the "lemme-clé" in [11], V, (1.2.2)):

Theorem 1.2. *Let $p_D:(S,0)\to(\mathbb{C}^2,0)$ be as above, and $C_1(D)\subset(S,0)\subset(\mathbb{C}^N,0)$ be the corresponding polar curve. Then there is an open dense subset U of $G(N-2,N)$ such that for $D\in U$ the restriction of p to $C_1(D)$ is generic in the sense of Definition* 1.1.

Definition 1.3. For all D in the open subset U of Theorem 1.2, the discriminant Δ_{p_D} are equisingular in the sense of the well-known equisingularity theory for germs of plane curves (cf., e.g., the account at the beginning of [3]): we will call this equisingularity class the *generic discriminant* of $(S, 0)$.

2. Polar curves for minimal singularities of surface after Spivakovsky

We first recall how one may define a minimal singularity in the case of normal surfaces (cf. [10], II.2):

Definition 2.1. A normal surface singularity $(S, 0)$ is said to be *minimal* if it is rational with reduced fundamental cycle (see [1] for these latter notions).

Let $\pi : (X, E) \to (S, 0)$ be the minimal resolution of the singularity $(S, 0)$, where $E = \pi^{-1}(0)$ is the exceptional divisor, with components L_i. A *cycle* will be by definition a divisor with support on E, i.e., a linear combination $\sum a_i L_i$ with $a_i \in \mathbb{Z}$ (or $a_i \in \mathbb{Q}$ for a $\mathbb{Q}$-cycle).

Considering the dual graph Γ associated to the exceptional divisor E (cf. [10], I.1) in which each component L_x gives a vertex x and two vertices are connected if, and only if, the corresponding components intersect, the minimal singularities have the following easy characterization (cf. loc. cit. II.2):

Lemma 2.2. *Let Γ be as above the dual graph associated to the minimal resolution of a normal surface singularity $(S, 0)$. For each vertex $x \in \Gamma$, one defines its weight $w(x) := -L_x^2$ (self-intersection of the corresponding component L_x) and its valence $\gamma(x)$, which is the number of vertices connected to x. Then $(S, 0)$ is minimal if, and only if, Γ is a tree and for all $x \in \Gamma$, $L_x \simeq \mathbb{P}^1_{\mathbb{C}}$ and $w(x) \geqslant \gamma(x)$.*

To two vertices $x, y \in \Gamma$ (which is a tree), we associate the shortest chain in Γ connecting them, which we denote by $[x, y]$. The *distance* $d(x, y)$ is by definition the number of edges on $[x, y]$.

In [10], III.5, Spivakovsky further introduces the following number s_x associated to each vertex $x \in \Gamma$. If $Z.L_x < 0$ (where . denotes the intersection number and $Z = \sum_{x \in \Gamma} L_x$ the reduced fundamental cycle), i.e., if $w(x) > \gamma(x)$ put $s_x := 1$ (and x is said to be non-Tyurina). Otherwise (i.e., if $w(x) = \gamma(x)$), x is said to be a Tyurina vertex; then denote Δ the Tyurina component of Γ containing x (i.e., the maximal connected subgraph of Γ containing only Tyurina vertices), and put $s_x := d(x, \Gamma \setminus \Delta) + 1$.

Let x, y be two adjacent vertices: the edge (x, y) in Γ is called a *central arc* if $s_x = s_y$. A vertex x is called a *central vertex* if there are at least two vertices y adjacent to x such that $s_y = s_x - 1$ (cf. [10]).

Eventually, we define the following $\mathbb{Q}$-cycle Z_Ω on the minimal resolution X of $(S, 0)$ by:

$$Z_\Omega = \sum_{x \in \Gamma} s_x L_x - Z_K, \tag{1}$$

where Γ is the dual graph of the resolution, and Z_K is the numerically canonical $\mathbb{Q}$-cycle.[1]

One may now quote the important Theorem 5.4 in [10] in the following way[2] (in loc. cit. it is used to compare the minimal resolution with the resolution of the Nash transform):

Theorem 2.3. *Let $(S, 0)$ be a* minimal *normal surface singularity. There is an open dense subset U' of the open set U of Theorem* 1.2*, such that for all $D \in U'$ the strict transform $C_1'(D)$ of $C_1(D)$ on X:*

(a) *is a multi-germ of smooth curves intersecting each component L_x of E transversally in exactly $-Z_\Omega.L_x$ points*;

[1] Uniquely defined by the condition that for all $x \in \Gamma$, $Z_K.L_x = -2 - L_x^2$ since the intersection product on E is negative-definite.

[2] See also the account in [8], (7.4); just beware that one term is missing in the formula giving $m_x := -Z_\Omega.L_x$ there.

(b) *goes through the point of intersection of* L_x *and* L_y *if and only if* $s_x = s_y$ (*point corresponding to a* central arc *of the graph*). *Furthermore, the curves* $C_1'(D)$, *with* $D \in U'$ *do not share other common points* (*base points*) *and these base points are simple, i.e., the curves* $C_1'(D)$ *are separated when one blows up these points once.*

3. From polar curves to discriminants, key lemmas

Lemma 3.1. *Let* $(S,0)$ *be a* minimal *normal surface singularity, embedded in* $\mathbb{C}^N$ *and* $\pi : X \to (S,0)$ *its minimal resolution, which is a composition* $\pi_1 \circ \cdots \circ \pi_r$ *of point blow-ups. We claim that this composition of blow-ups is also the minimal resolution of the generic polar curve* $C_1(D)$ *for* $D \in U'$ *as in Theorem* 2.3.

Proof. The fact that π is a composition of point blow-ups is general for rational surface singularities (for a non-cohomological proof in the case of minimal singularities, see [2], 5.9). Conclusion (a) in Theorem 2.3 certainly gives that π is a resolution of $C_1(D)$. We prove that this resolution is minimal: among the exceptional components in X obtained by the last point blow-up, there is a component L_x corresponding either to a central vertex of Γ or to the boundary of a central arc.

If L_x corresponds to a central vertex, one computes from (1) and Theorem 2.3(a), the number of branches of $C_1'(D)$ intersecting L_x, i.e., $-Z_\Omega . L_x = -(\sum s_y + (s_x+1)L_x^2 + 2)$. By the definition of a central vertex (before Theorem 2.3), this must be at least two, which proves that these branches are not separated before L_x is obtained.

If L_x is the boundary of a central arc, let L_y be the other boundary: then both L_x and L_y appear as exceptional components of the last blow-up $\pi_r : X \to S_{r-1}$ at 0_{r-1}. Now, the strict transform of $C_1(D)$ at the point 0_{r-1} can not be smooth. Indeed, by an argument in [6], 1.1, if it were smooth, then its strict transform $C_1'(D)$ on X, smooth surface, would go through a smooth point of the exceptional divisor. □

Lemma 3.2. *For* $D \in U'$ *as in Theorem* 2.3, *the polar curve* $C_1(D)$ *on* $(S,0)$ *has only smooth branches and branches of multiplicity two, the latter being exactly those for which the strict transform goes through a central arc as in* (b) *of Theorem* 2.3.

Proof. Let the notation be the same as in Theorem 2.3. Since $(S,0)$ is a minimal singularity, the cycle on X defined by the maximal ideal is reduced.

Hence by the projection formula for intersections, for any branch C of the generic polar curve $C_1(D)$, the multiplicity of C at 0 is the intersection multiplicity of its strict transform C' with the reduced exceptional divisor.

Then the description of the strict transform in Theorem 2.3 gives the conclusion. □

Corollary 3.3. *Take the chain of point blow-ups over* $(\mathbb{C}^N,0)$ *that gives the minimal resolution of* $(C_1(D),0)$ *for* $D \in U'$. *Then, performing over* $(\mathbb{C}^2,0)$ *the "same" succession of blow-ups* (*this makes sense because of footnote* 3), *we get the minimal resolution of the plane curve* $\Delta_{p_D} = p_D(C_1(D))$.

Proof. Since, by Lemma 3.2, the multiplicity of the branches of $C_1(D)$ is at most two, these branches are plane curves and so are equisingular to their *generic* projection by p_D (here we use Theorem 1.2): so we have dealt with the branches. Further, by another result of Teissier's (see [11], Chapter I, (6.2.1) and remark p. 354) a generic projection is bi-Lipschitz, which implies that it preserves the contact between branches.[3] □

[3] Indeed, the contact between two branches $\gamma_1(t)$ and $\gamma_2(t)$ which are both of multiplicity one or two, that we define as the number of blow-ups to separate them, may be read from the order in t of the difference $\gamma_1(t) - \gamma_2(t)$, which is a bi-lipschitz invariant. Since we blow-up always in the "same chart" these blow-ups actually dominate the blow-ups of the plane, as claimed in the corollary.

4. Statement of the main result

Gathering the results from Lemma 3.1 to Corollary 3.3, we obtain:

Theorem 4.1. *Let $(S,0)$ be a minimal normal surface singularity, embedded in $\mathbb{C}^N$ and $\pi: X \to (S,0)$ its minimal resolution, which is a composition $\pi_1 \circ \cdots \circ \pi_r$ of point blow-ups in $\mathbb{C}^N$. Let $\Delta_{S,0}$ be the generic discriminant of $(S,0)$ (cf. Definition 1.3). Then, performing over $(\mathbb{C}^2,0)$ the "same" succession of blow-ups (cf. footnote 3), we get the minimal resolution of the plane curve $\Delta_{S,0}$.*

This result, together with Theorem 2.3 gives an easy way to get a combinatorial description of $(\Delta_{S,0},0)$:

Notation 4.2. (i) We denote by Δ_{A_n} the generic discriminant of the A_n surface singularity, which is the equisingularity class of the plane curve defined by $x^2 + y^{n+1} = 0$.

(ii) We denote by δ_n the generic discriminant of the singularity which is a cone over a rational normal curve of degree n in $\mathbb{P}^n_{\mathbb{C}}$: it is defined by $2n-2$ distinct lines through the origin.

The assertion in (ii) follows from the fact that $C_1(D)$ is the cone over the critical locus of the projection from the rational normal curve onto a line, which has degree $2n-2$ by the Hurwitz formula.

We need to introduce several subsets of a dual graph Γ: we denote by $\Gamma_{NT} = \{x_1,\ldots,x_n\}$ the set of Non-Tyurina vertices in Γ, which are here the $x \in \Gamma$ such that $w(x) > \gamma(x)$ (notation as in Lemma 2.2).

We denote by $\mathcal{C}_v$ and $\mathcal{C}_a$ the set of central vertices and central arcs respectively in Γ (cf. def. before Theorem 2.3).

Corollary 4.3. *From Theorem 2.3 we know that the components of the strict transform $C_1(D)'$ of $C_1(D)$ on the resolution X of $(S,0)$ go through components corresponding to elements of $\Gamma_{NT} \cup \mathcal{C}_a \cup \mathcal{C}_v$, and we also know the number of branches of $C_1(D)'$ on each of these components, and their multiplicity by Lemma 3.2.*

From Theorem 4.1 we know the contact between the corresponding branches of $C_1(D)$ (or $\Delta_{S,0}$): the contact between two branches whose strict transforms lie respectively on a component L_x and a component L_y equals $1+N$, where N is the number of blow-ups necessary so that L_x and L_y are no longer in the same Tyurina component of the corresponding $(S_N,0_N)$ singularity, with the further requirement that if, say, the first branch actually goes through a central arc $L_x \cap L_{x'}$, the number N corresponds to the number of blow-ups so that both *x and x' are no longer in the same Tyurina component as y.*

From this, we get a precise description of $\Delta_{S,0}$: each $x_i \in \Gamma_{NT}$ contributes with a $\delta_{x_i} := \delta_{w(x_i)-\gamma(x_i)}$ (cf. 4.2(ii)), i.e., $2(w(x_i)-\gamma(x_i))-2$ lines, and the contact between these δ_{x_i} and other branches of the discriminant is one. For the contribution of the central elements, we first compute the number of branches on each components with Theorem 2.3 and one easily shows (using 3.2) that they contribute as Δ_{A_n}-curves and we use Theorem 4.1 to determine the n and the contacts as in the following examples:

Example 1. Consider $(S,0)$ with the graph Γ as on Fig. 1, where the $\bullet$ denote Tyurina vertices (with $w(x) = \gamma(x)$), and $\Gamma_{NT} = \{x_1,\ldots,x_4\}$ with the weights indicated on the graph. Remark that as a general rule $\delta_{x_i} = \emptyset$ when $w(x_i) = \gamma(x_i)+1$, hence here only x_1 actually gives a δ_{x_1} equal to four lines.

(i) In the *first Tyurina component* (bounded by x_1, x_2, x_4) there is a central vertex and a central arc, which respectively give a Δ_{A_5} and a Δ_{A_4} curve.

After two blow-ups the boundaries of the central arc and the central vertex are in distinct Tyurina components, hence the contact between the Δ_{A_5} and Δ_{A_4} is three.

(ii) In the *second Tyurina component* (bounded by x_2, x_3), there is a central vertex: this gives a Δ_{A_3} which has contact 1 with the others Δ_{A_i} obtained.

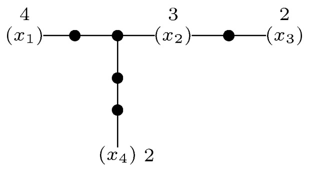

Fig. 1. Graph Γ as in Example 1.

Fig. 1. Graphe de Γ, comme dans l'Exemple 1.

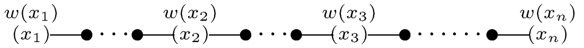

Fig. 2. Graph Γ as in Example 2.

Fig. 2. Graphe de Γ, comme dans l'Exemple 2.

Hence, using coordinates, we may give as representative of the equisingularity class of $\Delta_{S,0}$: $(x^4+y^4)(x^2+y^6)\cdot(x^2+y^5)(y^2+x^4)=0$.

Example 2. If $(S,0)$ is a cyclic-quotient singularity, i.e., has a graph Γ as on Fig. 2, we may order $\Gamma_{NT}=\{x_1<x_2<\cdots<x_n\}$ and each central element x (central vertex or central arc) lies in a unique $[x_i,x_{i+1}]$ and is easily seen to contribute to $\Delta_{S,0}$ by a $\Delta_x:=\Delta_{A_{l([x_i,x_{i+1}])}}$, where $l[x_i,x_{i+1}]$ is the number of vertices on the chain $[x_i,x_{i+1}]$; the contact between each Δ_x is one (i.e., their tangent cones have no common components). Here δ_{x_i} is $2w(x_i)-4$ lines for $i=1$ and $i=n$, and $2w(x_i)-6$ for $1<i<n$, all this lines being distinct. So, with Δ_{A_n} as in Notation 4.2(i): $\Delta_{S,0}=\delta_{x_1}\cup\Delta_{A_{l[x_1,x_2]}}\cup\delta_{x_2}\cup\cdots\cup\Delta_{A_{l[x_{n-1},x_n]}}\cup\delta_{x_n}$, with contact one between all the curves in the "$\cup$".

Remark 1. In particular, the equisingularity type of $(\Delta_{S,0},0)$ depends only on the resolution graph of $(S,0)$, i.e., of the topological type of $(S,0)$, a fact which is known to be wrong for other normal surface singularities as shown in [5].

Remark 2 (Added on proofs). The contribution of the components of the tangent cone ($x_i\in\Gamma_{NT}$) as δ_{x_i} in the generic discriminant of $(S,0)$ may be seen directly (i.e., without using Theorem 2.3) from the deformation on $(S,0)$ on its tangent cone. We hope to come back to this in a future paper.

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