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The weak null condition for Einstein's equations

Hans Lindblad [a,1], Igor Rodnianski [b]

[a] *Department of Mathematics, University of California, San Diego, La Jolla, CA, USA*
[b] *Department of Mathematics, Princeton University, Princeton, NJ, USA*

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Presented by Yvonne Choquet-Bruhat

Abstract

We show that Einstein's equations of General Relativity expressed in wave coordinates satisfy a 'weak null condition'. In a forthcoming article we will use this to prove a global existence result for Einstein's equations in wave coordinates with small initial data. ***To cite this article: H. Lindblad, I. Rodnianski, C. R. Acad. Sci. Paris, Ser. I 336 (2003).***

Résumé

Condition nulle faible pour les équations d'Einstein. Nous montrons que les équations d'Einstein de la relativité générale exprimées en coordonnées des ondes satisfont une « condition de nullité faible ». Dans un futur article, nous utilisons ceci pour démontrer un résultat global d'existence pour des équations d'Einstein en coordonnées des ondes avec donnèes initiales petites. ***Pour citer cet article : H. Lindblad, I. Rodnianski, C. R. Acad. Sci. Paris, Ser. I 336 (2003).***

Version française abrégée

Les équations d'Einstein de la relativité générale exprimées en coordonnées des ondes forment un système d'équations d'ondes quasilinéaires. Pour qu'un tel système puisse avoir des solutions globales pour des petites données initiales, il doit y avoir une simplification des termes non-linéaires car sinon les solutions peuvent exploser. La « condition de nullité » de Klainerman [12] (voir aussi [5]) assure l'existence globale mais n'est pas satisfaite par les équations d'Einstein en coordonnées des ondes. En fait, Choquet-Bruhat [3] a montré que les coordonnées des ondes sont instables au sens où une deuxième itérée ne décroit pas aussi vite qu'une solution d'une équation d'ondes linéaire. Néanmoins, Christodoulou et Klainerman [6] ont démontré l'existence globale pour les équations d'Einstein de façon invariante par rapport aux coordonnées (cf. [7,13]).

On a alors pensé que les coordonnées des ondes pourraient exploser mais, dans une formulation invariante des coordonnées, la courbure reste bornée. Néanmoins, Choquet-Bruhat [4] a montré, indépendamment de toute

E-mail addresses: lindblad@math.ucsd.edu (H. Lindblad), irod@math.princeton.edu (I. Rodnianski).

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condition sur les coordonnées, qu'il n'y a pas de généralisation de la condition de nullité pour les équations d'Einstein. La condition nulle a été conçue pour déterminer les équations d'ondes non-linéaires pour lesquelles il existe de petites solutions globales dont la décroissance est analogue à celle des solutions d'ondes linéaires. Néanmoins, comme l'a remarqué Lindlad [15], il existe des équations d'ondes quasilinéaires qui ne satisfont pas la condition de nullité et pour lesquelles il existe de petites solutions globales qui décroissent plus lentement que les solutions d'ondes linéaires.

Nous introduisons la notion de « condition de nullité faible », basée sur l'idée que l'équation asymptotique correspondante a des solutions globales. Nous montrons que les équations d'Einstein en coordonnées d'ondes satisfont la « condition de nullité faible ». Dans [16], nous utilisons ceci pour démontrer un résultat global d'existence des solutions pour des équations d'Einstein en coordonnées d'ondes avec donnèes initiales petites.

1. Introduction

Einstein's equations of General Relativity expressed in wave coordinates are a system of quasilinear wave equations. In order for such a system to have global solutions for small initial data there must be some cancellation of the nonlinear terms or else solutions can blow up. The 'null condition' of Klainerman [12] (see also [5]) ensures global existence but it is not satisfied for Einstein's equations in wave coordinates. In fact Choquet-Bruhat [3] showed that the wave coordinates are unstable in the sense that a second iterate does not decay as fast as a solution of a linear wave equation. Nevertheless, Christodoulou and Klainerman [6] succeeded in proving global existence for Einstein's equations in a coordinate invariant way (cf. [7,13]).

This resulted in a belief that the wave coordinates might blow up but, in a coordinate invariant formulation, the curvature remains bounded. However, recently Choquet-Bruhat [4] showed that irrespective of coordinate condition no natural generalization of the null condition holds for Einstein's equations. The null condition was designed to detect nonlinear wave equations for which there are small global solutions that decay like solutions of linear wave equations. However, as was pointed out by Lindblad [15], there are quasilinear wave equations that do not satisfy the null condition for which there are small global solutions that decay slower than solutions of linear wave equations.

We introduce the notion of a 'weak null condition', based on the idea that the corresponding asymptotic equation has global solutions. We show that the Einstein equations in wave coordinates satisfy the weak null condition. In [16] we will use this to prove a global existence result for Einstein's equations in wave coordinates with small initial.

2. The weak null condition

Consider the Cauchy problem for a system of nonlinear wave equations in three space dimensions:

$$-\Box u_i = F_i(u, u', u''), \quad i = 1, \dots, N, \ u = (u_1, \dots, u_N), \tag{1}$$

where $-\Box = -\partial_t^2 + \sum_{j=1}^3 \partial_{x^j}^2$. We assume that F is a function of u and its derivatives of the form

$$F_i(u, u', u'') = a_{i\alpha\beta}^{jk} \partial^\alpha u_j \partial^\beta u_k + G_i(u, u', u''), \tag{2}$$

where $G_i(u, u', u'')$ vanishes to third order as $(u, u', u'') \to 0$ and $a_{i\,\alpha\beta}^{jk} = 0$ unless $|\alpha| \leqslant |\beta| \leqslant 2$ and $|\beta| \geqslant 1$. Here we used the summation convention over repeated indices. We assume that the initial data

$$u(0, x) = \varepsilon u_0(x) \in C^\infty, \qquad u_t(0, x) = \varepsilon u_1(x) \in C^\infty \tag{3}$$

is small and decays fast as $|x| \to \infty$. We are going to determine conditions on the nonlinearity such that Eq. (1) is compatible with the asymptotic expansion as $|x| \to \infty$ and $|x| \sim t$,

$$u(t,x) \sim \varepsilon U(q,s,\omega)/|x|, \quad \text{where } q = |x| - t,\ s = \varepsilon \ln|x|,\ \omega = x/|x|, \tag{4}$$

for all sufficiently small $\varepsilon > 0$. The linear and some nonlinear wave equations allow for such an expansion with U independent of s and the next term decaying like $\varepsilon/|x|^2$, see [8,9]. Substituting (4) into (1) and equating powers of order $\varepsilon^2/|x|^2$ we see that

$$2\partial_s \partial_q U_i = A_{i,mn}^{jk}(\omega)\big(\partial_q^m U_j\big)\big(\partial_q^n U_k\big), \qquad U|_{s=0} = F_0, \tag{5}$$

where

$$A_{i,mn}^{jk}(\omega) = \sum_{|\alpha|=m,\ |\beta|=n} a_{i,\alpha\beta}^{jk} \widehat{\omega}^\alpha \widehat{\omega}^\beta, \quad \text{where } \widehat{\omega} = (-1,\omega) \text{ and } \widehat{\omega}^\alpha = \widehat{\omega}_{\alpha_1} \cdots \widehat{\omega}_{\alpha_k}. \tag{6}$$

One can show that (1)–(3) has a solution as long as $\varepsilon \log t$ is bounded, provided that $\varepsilon > 0$ is sufficiently small and the solution of (5) exists up to that time, [11,8,9,14,15]. The only exception is the case $A_{i00}^{jk} \neq 0$, which has shorter life span. In cases where the solution of (5) blows up it has been shown that solutions of (1)–(3) also break down in some finite time $T_\varepsilon \leqslant e^{C/\varepsilon}$, [10,2]. John's example was

$$\Box u = u_t \Delta u \tag{7}$$

for which (5) is the Burger's equation $(2\partial_s - U_q \partial_q)U_q = 0$, which is known to blow up. The equation

$$\Box u = u_t^2 \tag{8}$$

is another example where solutions blow up, for which (5) is $\partial_s U_q = U_q^2$, that also blows up.

The *null condition* of [12] is equivalent to

$$A_{i\,mn}^{jk}(\omega) = 0 \quad \text{for all } (i,j,k,m,n),\ \omega \in \mathbb{S}^2. \tag{9}$$

The result of [5,12] asserts that (1)–(3) has global solutions for all sufficiently small initial data, provided that the null condition is satisfied. In this case the asymptotic equation (5) trivially can be solved globally. Moreover, similar to the linear case, its solutions approach a limit as $s \to \infty$ and the solutions of (1)–(3) decay like solutions of linear equations. A typical example of an equation satisfying the null condition is:

$$\Box u = u_t^2 - |\nabla_x u|^2. \tag{10}$$

There is however a more general class of nonlinearities for which solutions of (5) do not blow up:

Definition 2.1. We say that a system (1) satisfies the *weak null condition* if the solutions of the corresponding asymptotic system (5) exist for all s and if the solutions together with its derivatives grow at most exponentially in s for all initial data decaying sufficiently fast in q.

Under the weak null condition assumption solutions of (5) satisfy Eq. (1) up to terms of order $\varepsilon^2/|x|^{3-C\varepsilon}$, but need only decay like $\varepsilon/|x|^{1-C\varepsilon}$. An example satisfying the weak null condition is given by:

$$\Box u = u \Delta u. \tag{11}$$

In [15] it was proven that (11) have small global solutions in the spherically symmetric case and recently [1] established this result without the symmetry assumption. Eq. (11) appears to be similar to (7) but a closer look shows that the corresponding asymptotic equation:

$$(2\partial_s - U\partial_q)U_q = 0 \tag{12}$$

has global solutions growing exponentially in s, see [15]. The system:

$$\Box u = v_t^2, \qquad \Box v = 0 \tag{13}$$

is another example that satisfies the weak null condition. Eq. (13) appears to resemble (8). The system however decouples: v satisfies a linear homogeneous equation and, given v, we have a linear inhomogeneous equation for u, and global existence follows. The corresponding asymptotic system is:

$$\partial_s \partial_q U = (\partial_q V)^2, \qquad \partial_s \partial_q V = 0. \tag{14}$$

The solution of the second equation in (14) is independent of s: $V_q = V_q(q,\omega)$ and substituting this into the first equation we see that $U_q(s,q,\omega) = s V_q(q,\omega)^2$ so ∂u only decays like $|x|^{-1}\ln|x|$.

We show below that the Einstein vacuum equations in wave coordinates satisfy the weak null condition, i.e., that the corresponding asymptotic system (5) admits global solutions. In fact, each of the quadratic nonlinear terms in the Einstein equations is either of the type appearing in (10), (11) or (13).

3. The Einstein vacuum equations in wave coordinates and the quadratic approximation

We assume that $g_{\mu\nu}$, where $\mu,\nu = 0,\ldots,n$, is a Lorentzian metric with signature $(-1,1,\ldots,1)$. Let

$$\Gamma_{\mu\ \nu}^{\ \lambda} = \frac{1}{2} g^{\lambda\delta}(\partial_\mu g_{\delta\nu} + \partial_\nu g_{\delta\mu} - \partial_\delta g_{\mu\nu}) \tag{15}$$

be the Christoffel symbols, where $g^{\lambda\delta}$ is the inverse of $g_{\mu\nu}$. The Riemann curvature tensor is

$$R_{\mu\ \nu\delta}^{\ \lambda} = \partial_\delta \Gamma_{\mu\ \nu}^{\ \lambda} - \partial_\nu \Gamma_{\mu\ \delta}^{\ \lambda} + \Gamma_{\rho\ \delta}^{\ \lambda}\Gamma_{\mu\ \nu}^{\ \rho} - \Gamma_{\rho\ \nu}^{\ \lambda}\Gamma_{\mu\ \delta}^{\ \rho} \tag{16}$$

and the Ricci tensor $R_{\mu\nu} = g^{\alpha\beta} R_{\mu\alpha\nu\beta} = R_{\mu\ \nu\alpha}^{\ \alpha}$. The Einstein vacuum equations assert that

$$R_{\mu\nu} = 0. \tag{17}$$

We now recall that in wave coordinates the Christoffel symbols verify:

$$g^{\alpha\beta}\, \Gamma_{\alpha\ \beta}^{\ \lambda} = 0. \tag{18}$$

We denote by m the standard Minkowski metric $m_{00} = -1,\ m_{ii} = 1,\ i = 1,\ldots,n$ and $m_{\mu\nu} = 0$, if $\mu \neq \nu$.

Lemma 3.1. *Let g be a solution of the Einstein equations* (17) *and assume that the wave coordinate condition* (18) *holds. Then the symmetric* 2*-tensor $h_{\mu\nu} = g_{\mu\nu} - m_{\mu\nu}$ verifies the wave equation*:

$$\widetilde{\Box} h_{\mu\nu} = F_{\mu\nu}(h)(\partial h, \partial h), \quad \text{where } \widetilde{\Box} = g^{\alpha\beta}\partial_\alpha\partial_\beta. \tag{19}$$

Here $F_{\mu\nu}(h)(\partial h,\partial h) = M(\partial_\mu h, \partial_\nu h) + Q_{\mu\nu}(\partial h,\partial h) + G_{\mu\nu}(h)(\partial h,\partial h)$, where

$$M(\Pi,\Sigma) = \frac{1}{2} m^{\alpha\alpha'} m^{\beta\beta'} \Pi_{\alpha\beta}\, \Sigma_{\alpha'\beta'} - \frac{1}{4} m^{\alpha\alpha'} \Pi_{\alpha\alpha'}\, m^{\beta\beta'} \Sigma_{\beta\beta'}, \tag{20}$$

$$Q_{\mu\nu}(\partial h,\partial h) = \partial_\alpha h_{\beta\mu}\, m^{\alpha\alpha'} m^{\beta\beta'} \partial_{\alpha'} h_{\beta'\nu} + m^{\alpha\alpha'} m^{\beta\beta'} (\partial_\alpha h_{\beta\mu}\ \partial_{\beta'} h_{\alpha'\nu} - \partial_{\beta'} h_{\beta\mu}\ \partial_\alpha h_{\alpha'\nu}) \tag{21}$$

and $G_{\mu\nu}(h)(\partial h,\partial h)$ is a quadratic form in ∂h with coefficients depending smoothly on h such that $G_{\mu\nu}(0)(\partial h,\partial h) = 0$. Furthermore, the wave coordinate condition can be expressed in the form:

$$m^{\alpha\beta}\partial_\alpha h_{\beta\mu} = \frac{1}{2} m^{\alpha\beta}\partial_\mu h_{\alpha\beta} + G_\mu(h)(\partial h), \tag{22}$$

where $G_\mu(h)(\partial h)$ is a linear function in ∂h with coefficients depending smoothly on h and $G_\mu(0)(\partial h) = 0$. The terms $Q_{\mu\nu}$ are combinations of classical null forms.

4. The null frame and null forms

At each point $(t,x)=(t,r\omega)\in\mathbf{R}^{3+1}$, $\omega\in\mathbf{S}^2$ we can introduce the Minkowski null pair of vectors:

$$(L,\underline{L})=\left(L^\mu\partial_\mu,\underline{L}^\mu\partial_\mu\right),\quad \text{where } \left(L^\mu\right)=\left(1,\omega^i\right),\ \left(\underline{L}^\mu\right)=\left(1,-\omega^i\right). \tag{23}$$

At each point $(t,r\omega)\in\mathbf{R}^{3+1}$ we can complement the null pair $(L,\underline{L})$ by a pair of orthonormal vector fields (A,B) orthogonal to the vector $\omega\in\mathbf{S}^2$. We shall refer to the collection $\{L,\underline{L},A,B\}$ as a null frame.

Let Π be a 2-tensor and $\Pi_{XY}=\Pi_{\alpha\beta}X^\alpha Y^\beta$. We now express M in terms of the null frame $\{L,\underline{L},A,B\}$:

Lemma 4.1. *Let M be as in Lemma* 3.1. *If Π and Σ are symmetric* 2*-tensors, then*:

$$\begin{aligned}M(\Pi,\Sigma)\ =\ &\frac{1}{8}(\Pi_{LL}\Sigma_{\underline{LL}}+\Pi_{\underline{LL}}\Sigma_{LL})+\frac{1}{4}\delta^{AB}\delta^{A'B'}(2\Pi_{AA'}\Sigma_{BB'}-\Pi_{AB}\Sigma_{A'B'})\\ &-\frac{1}{4}\delta^{AB}(2\Pi_{AL}\Sigma_{B\underline{L}}+2\Pi_{A\underline{L}}\Sigma_{BL}-\Pi_{AB}\Sigma_{L\underline{L}}-\Pi_{L\underline{L}}\Sigma_{AB}).\end{aligned}$$

5. Asymptotic expansion of Einstein's equations in wave coordinates

Theorem 5.1. *Let h be a symmetric* 2*-tensor and let*

$$h_{\mu\nu}(t,x)\sim\varepsilon H_{\mu\nu}(s,q,\omega)/|x|,\quad \textit{where } q=|x|-t,\ s=\varepsilon\ln|x|,\ \omega=x/|x| \tag{24}$$

be an asymptotic ansatz. Then the asymptotic system for the the Einstein equations in wave coordinates (19)*, obtained by formally equating the terms with the coefficients $\varepsilon^2|x|^{-2}$, takes the following form*:

$$(2\partial_s-H_{LL}\partial_q)\partial_q H_{\mu\nu}=L_\mu L_\nu M(\partial_q H,\partial_q H),\quad \forall\mu,\nu=0,\dots,3, \tag{25}$$

where $H_{LL}=m^{\alpha\alpha'}m^{\beta\beta'}H_{\alpha'\beta'}L_\alpha L_\beta$. The asymptotic form of the wave coordinate condition (22) *is*:

$$2\partial_q H_{L\mu}=L_\mu\,\partial_q\operatorname{tr}H,\quad \forall\mu=0,\dots,3, \tag{26}$$

where $H_{L\mu}=m^{\alpha\alpha'}H_{\alpha'\mu}L_\alpha$ and $\operatorname{tr}H=m^{\alpha\beta}H_{\alpha\beta}$. The solution of the system (25)–(26) *exists globally and, thus, the Einstein vacuum equations* (19) *in wave coordinates satisfies the weak null condition. Moreover, the component $\partial_q H_{\underline{LL}}$ grows at most as s while the remaining components are uniformly bounded.*

The asymptotic form (25) follows by a direct calculation from (19). Observe that the null form $Q_{\mu\nu}(\partial h,\partial h)$ disappears after passage to the asymptotic system.

Next we note that (26) is preserved under the flow of (25). Contracting (25) with $L^\mu L^\nu$ we obtain:

$$(2\partial_s-H_{LL}\partial_q)\partial_q H_{LL}=0,$$

which can be solved globally. More generally, contracting (25) with the vector fields $\{L,A,B\}$ we obtain:

$$(2\partial_s-H_{LL}\partial_q)\partial_q H_{TU}=0,\quad \text{if } T\in\{L,A,B\} \text{ and } U\in\{L,\underline{L},A,B\}, \tag{27}$$

which can be solved globally now that H_{LL} has been determined. Note that the components $\partial_q H_{TU}$ are constant along the integral curves of the vector field $2\partial_s-H_{LL}\partial_q$. The remaining unknown component $H_{\underline{LL}}$ can be determined by contracting Eq. (25) with the vector field $\underline{L}$,

$$(2\partial_s-H_{LL}\partial_q)\partial_q H_{\underline{LL}}=4M(\partial_q H,\partial_q H). \tag{28}$$

By Lemma 4.1 the quantity $M(\partial_q H,\partial_q H)$ does not contain the term $(\partial_q H_{\underline{LL}})^2$. Thus, Eq. (28) can be solved globally and produces solutions growing exponentially in s. More precise information can be obtained from the asymptotic form of the wave coordinate condition (26). For contracting it with the null frame $\{L, A, B\}$ we obtain $\partial_q H_{LT} = 0$, if $T \in \{L, A, B\}$. Therefore:

$$M(\partial_q H, \partial_q H) = \frac{1}{4}\delta^{AB}\delta^{A'B'}(2\partial_q H_{AA'}\,\partial_q H_{BB'} - \partial_q H_{AB}\,\partial_q H_{A'B'}) + \frac{1}{2}\delta^{AB}\partial_q H_{AB}\,\partial_q H_{L\underline{L}}. \tag{29}$$

It follows from (27) that M is already determined and is, in fact, constant along the characteristics of the field $2\partial_s - H_{LL}\partial_q$. Therefore, integrating (28) we infer that $\partial_q H_{\underline{L}\underline{L}}$ grows at most like s.

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