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Dynamical Systems

Recurrence and genericity

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Abstract

We prove a C^1-connecting lemma for pseudo-orbits of diffeomorphisms on compact manifolds. We explore some consequences for C^1-generic diffeomorphisms. For instance, C^1-generic conservative diffeomorphisms are transitive. ***To cite this article: C. Bonatti, S. Crovisier, C. R. Acad. Sci. Paris, Ser. I 336 (2003).***

Résumé

Récurrence et généricité. Nous montrons un lemme de connexion C^1 pour les pseudo-orbites des difféomorphismes des variétés compactes. Nous explorons alors les conséquences pour les difféomorphismes C^1-génériques. Par exemple, les difféomorphismes conservatifs C^1-génériques sont transitifs. ***Pour citer cet article : C. Bonatti, S. Crovisier, C. R. Acad. Sci. Paris, Ser. I 336 (2003).***

Version française abrégée

Soit M une variété riemannienne compacte connexe munie d'une forme volume ω. Notons $\mathrm{Diff}^1(M)$ l'ensemble des difféomorphismes de classe C^1 de M muni de la topologie C^1 et $\mathrm{Diff}^1_\omega(M) \subset \mathrm{Diff}^1(M)$ le sous-ensemble de ceux qui préservent le volume ω. Rappelons qu'une partie d'un espace métrique complet est dite *résiduelle* si elle contient une intersection dénombrable d'ouverts denses. Une propriété est dite *générique* si elle est vérifiée sur un ensemble résiduel.

Soit $f \in \mathrm{Diff}^1(M)$ un difféomorphisme de M. Pour tout $\varepsilon > 0$, une *ε-pseudo-orbite* de f est une suite (finie ou infinie) de points (x_i) vérifiant $d(x_{i+1}, f(x_i)) < \varepsilon$ pour tout i. On définit les relations binaires suivantes pour les paires (x, y) de points de M :

- On note $x \dashv y$ si, pour tout $\varepsilon > 0$, il existe une ε-pseudo-orbite $x_0 = x, \dots, x_n = y$ avec $n \geqslant 1$. La relation $\dashv$ est transitive. L'ensemble récurrent par chaînes $\mathcal{R}(f)$ est l'ensemble des points x de M tels que $x \dashv x$.
- On note $x \prec y$ si, pour tous voisinages U, V de x et y respectivement, il existe $n \geqslant 1$ tel que $f^n(U)$ rencontre V. La relation $x \prec y$ n'est pas *a priori* transitive, cependant Arnaud montre dans [1] que la relation $\prec$ est transitive pour les difféomorphismes génériques. L'ensemble non-errant $\Omega(f)$ est l'ensemble des points x de M tels que $x \prec x$.

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Notre résultat principal est un lemme de connexion pour les pseudo-orbites généralisant celui de Hayashi (voir [1, 5,11]) :

Théorème 0.1. *Soit f un difféomorphisme d'une variété compacte M dont toutes les orbites périodiques sont hyperboliques (si f est un difféomorphisme conservatif de surface, on autorise aussi les orbites périodiques elliptiques irrationnelles, i.e., la différentielle est conjuguée à une rotation irrationnelle). Soit $\mathcal{U}$ un C^1-voisinage de f dans $\mathrm{Diff}^1(M)$ (ou dans $\mathrm{Diff}^1_\omega(M)$, si f préserve une forme volume ω).*

Alors, pour toute paire (x, y) de points de M telle que $x \dashv y$, il existe un difféomorphisme g dans $\mathcal{U}$ et un entier $n > 0$ tel que $g^n(x) = y$.

En conséquence directe, on obtient :

Théorème 0.2. *Il existe une partie résiduelle $\mathcal{G}$ de $\mathrm{Diff}^1(M)$ (ou de $\mathrm{Diff}^1_\omega(M)$) telle que pour tout difféomorphisme f de $\mathcal{G}$ et tout couple (x, y) de points de M on a : $x \dashv y \Leftrightarrow x \prec y$.*

Voici quelques conséquences de ces résultats :

Corollaire 0.3. *Il existe une partie résiduelle $\mathcal{G}$ de $\mathrm{Diff}^1(M)$ telle que pour tout difféomorphisme f de $\mathcal{G}$, on a $\mathcal{R}(f) = \Omega(f)$; de plus, si $f \in \mathcal{G}$ vérifie $\Omega(f) = M$ alors il est transitif et M est une unique classe homocline.*

On en déduit de nombreuses propriétés des systèmes génériques qui nous permettent par exemple de donner une réponse positive, en topologie C^1, à une conjecture de Hurley [7] ; Hurley appelle un *quasi-attracteur* une intersection d'une famille d'attracteurs topologiques (non-transitifs). Nous montrons :

Corollaire 0.4. *Il existe une partie résiduelle $\mathcal{G}$ de $\mathrm{Diff}^1(M)$ telle que pour tout $f \in \mathcal{G}$, l'union des bassins des quasi-attracteurs récurrents par chaînes est une partie résiduelle de M.*

Dans le cas conservatif, l'ensemble non-errant coïncide toujours avec la variété M. On obtient donc :

Théorème 0.5. *Il existe une partie résiduelle $\mathcal{G}_\omega$ dans l'ensemble $\mathrm{Diff}^1_\omega(M)$ des difféomorphismes préservant ω pour laquelle tout difféomorphisme $f \in \mathcal{G}_\omega$ est transitif. De plus M est une unique classe homocline.*

Arbieto et Matheus nous ont signalé que nos résultats permettent de répondre à une question posée par Herman dans [6] :

Théorème 0.6. *Soit f un difféomorphisme de classe C^∞ d'une variété compacte connexe préservant une forme volume ω. Alors, f vérifie une des deux propriétés suivantes* :

(1) *ou bien f possède une décomposition dominée,*
(2) *ou bien f n'a pas d'exposants stables : f est approché en topologie C^1 par une suite de difféomorphismes g de classe C^∞ possédant une orbite p de période n telle que $Dg^n(p) = \mathrm{Id}$.*

1. The connecting lemma for pseudo-orbits

Let M be a compact connected Riemannian manifold endowed with a volume form ω, $\mathrm{Diff}^1(M)$ the space of C^1-diffeomorphisms of M endowed with the C^1-topology and $\mathrm{Diff}^1_\omega(M) \subset \mathrm{Diff}^1(M)$ the subspace of ω-preserving diffeomorphisms. Recall that a subset is *residual* if it contains a countable intersection of dense open sets. A property is *generic* if it holds on a residual subset. A property is *locally generic* if it holds on a residual subset of some non-empty open subset.

For $f \in \mathrm{Diff}^1(M)$ and $\varepsilon > 0$, a *ε-pseudo-orbit* of f is a (finite or infinite) sequence of points (x_i) such that $d(x_{i+1}, f(x_i)) < \varepsilon$. We define two binary relations for pairs (x, y) of points in M:

– We write $x \dashv y$ if there exists a ε-pseudo-orbit $x_0 = x, \ldots, x_k = y$ with $k \geqslant 1$ for any $\varepsilon > 0$. This relation is transitive. The chain recurrent set $\mathcal{R}(f)$ is the set of points $x \in M$ such that $x \dashv x$.
– We write $x \prec y$ if, for any neighborhoods U and V of x and y, there is $n \geqslant 1$ such that $f^n(U)$ intersects V. This relation is not in general transitive, however Arnaud [1] shows that $\prec$ is transitive for generic diffeomorphisms. The nonwandering set $\Omega(f)$ is the set of points $x \in M$ such that $x \prec x$.

Our main result is a connecting lemma for pseudo-orbits, generalizing Hayashi's connecting lemma in [1,5,11]:

Theorem 1.1. *Let f be a diffeomorphism of M whose periodic orbits are hyperbolic (if f is a conservative diffeomorphism of a compact surface, we allow also irrational elliptic periodic orbits, i.e., the derivative is conjugate to an irrational rotation). Let $\mathcal{U}$ be a C^1-neighborhood of f in $\mathrm{Diff}^1(M)$ (or in $\mathrm{Diff}^1_\omega(M)$, if f preserves ω).*

Then, for any pair (x, y) of points in M verifying $x \dashv y$, there is a diffeomorphism $g \in \mathcal{U}$ and an integer $n > 0$ such that $g^n(x) = y$.

As a direct consequence, we get:

Theorem 1.2. *There is a residual subset $\mathcal{G} \subset \mathrm{Diff}^1(M)$ (or $\mathcal{G} \subset \mathrm{Diff}^1_\omega(M)$) such that for any diffeomorphism $f \in \mathcal{G}$ and any pair (x, y) of points in M we have: $x \dashv y \Leftrightarrow x \prec y$.*

We present some consequences of these results.

2. Dynamics of generic diffeomorphisms

Corollary 2.1. *There is a residual subset $\mathcal{G} \subset \mathrm{Diff}^1(M)$ such that for any diffeomorphism $f \in \mathcal{G}$, we have $\mathcal{R}(f) = \Omega(f)$; moreover, if $f \in \mathcal{G}$ verifies $\Omega(f) = M$ then it is transitive and M is a unique homoclinic class.*[1]

The relation $\vdash\!\dashv$ defined by $(x \vdash\!\dashv y) \Leftrightarrow (x \dashv y$ and $y \dashv x)$ induces on $\mathcal{R}(f)$ an equivalence relation, whose classes are called *chain recurrent classes*. A f-invariant compact set Λ is *weakly transitive* if $x \prec y$ for any $x, y \in \Lambda$. It is a *maximal weakly transitive* set if it is maximal for $\subset$ among the weakly transitive sets. From Zorn's lemma, any weakly transitive set is contained in some maximal weakly transitive set. If f is generic, so that the relation $\prec$ is transitive, the maximal weakly transitive sets are the equivalence classes of the symetrized relation $\succ$ and $\prec$ induced on $\Omega(f)$. For f generic we get:

Corollary 2.2. *There is a residual subset $\mathcal{G} \subset \mathrm{Diff}^1(M)$ such that for any $f \in \mathcal{G}$ the chain recurrence classes are the maximal weakly transitive sets.*

Let us recall (see [4]) that for any homeomorphism h of some metric compact space X, there exists a continuous function $\phi : X \to \mathbb{R}$, called *Lyapunov fuction for h*, that increases along the orbits of $X \setminus \mathcal{R}(h)$, is constant on each chain recurrence class and injective on the set of these classes. Moreover the compact subset $\phi(\mathcal{R}(h))$ of $\mathbb{R}$ is totally discontinuous. From our results above, it follows now that the nonwandering set of a generic diffeomorphism splits in maximal weakly transitive sets separated by a filtration given by some Lyapunov function.

[1] The *homoclinic class* of an hyperbolic periodic orbit is the closure of the transverse intersections of its stable and unstable manifolds.

We would expect that, for any generic diffeomorphism, any generic point belongs to the basin of some transitive topological attractor. A weaker version of this problem has been proposed by Hurley in [7]. He defines a *quasi-attractor* as any intersection of (non-transitive) topological attractors. He conjectures:

Conjecture 1 (Hurley). *For any $r \geqslant 1$, there exists some residual subset $\mathcal{G} \subset \mathrm{Diff}^r(M)$ such that for any $f \in \mathcal{G}$, the union of the basins of all the chain recurrent quasi-attractors is a residual subset of M.*

We present here a positive answer to Hurley's conjecture in the C^1-topology:

Corollary 2.3. *There is a residual subset $\mathcal{G} \subset \mathrm{Diff}^1(M)$ such that for any $f \in \mathcal{G}$, the union of the basins of all the chain recurrent quasi-attractors is a residual subset of M; more precisely, the set of points whose ω-limit set is a Lyapunov stable[2] chain recurrence class is a residual subset of M.*

This is a consequence of partial results in [1,8], of Theorem 1.2 and of the following proposition:

Proposition 2.4. *There is a residual subset $\mathcal{G} \in \mathrm{Diff}^1(M)$ such that for any $f \in \mathcal{G}$, a Lyapunov stable chain recurrence class is a quasi-attractor.*

The homoclinic classes play an important role: they are transitive; for a generic diffeomorphism f they are also dense in the nonwandering set (from Pugh's closing lemma [9,10], the periodic points are dense in $\Omega(f)$); for a generic diffeomorphism, [1,3] show that any homoclinic class is a maximal weakly transitive set, hence by Corollary 2.2 a chain recurrence class. A chain recurrence class of a generic diffeomorphism which is not an homoclinic class does not contain any periodic orbit and will be called an *aperiodic class* (see [2] for examples of locally generic diffeomorphisms with aperiodic classes).

Corollary 2.5. *There is a residual subset $\mathcal{G} \subset \mathrm{Diff}^1(M)$ such that for any $f \in \mathcal{G}$,*

(1) *any component of $\Omega(f) = \mathcal{R}(f)$ with non-empty interior is periodic and its orbit is an homoclinic class,*
(2) *any chain recurrence class which is isolated in $\mathcal{R}(f)$ is an homoclinic class. For instance, this is the case of the topological attractors or repellors.*

An invariant compact set Λ is *robustly transitive* if there exists an isolating compact neighborhood U of Λ (i.e., Λ is the maximal invariant set in U) such that for any C^1-close diffeomorphism g, the maximal invariant set Λ_g of g in U is transitive. We would expect that for a generic diffeomorphism the isolated homoclinic classes are robustly transitive. We have a partial result:

Corollary 2.6. *There is a residual subset $\mathcal{G} \subset \mathrm{Diff}^1(M)$ such that, for any $f \in \mathcal{G}$, any homoclinic class Λ which is isolated in $\mathcal{R}(f)$ is robustly chain recurrent: for any isolating compact neighborhood U of Λ and any C^1-close diffeomorphism de g, the maximal invariant set of g in U is chain recurrent.*

3. Conservative dynamics

In the conservative world, the nonwandering set is the whole manifold M. Hence we obtain:

Theorem 3.1. *There is a residual subset $\mathcal{G}_\omega \subset \mathrm{Diff}^1_\omega(M)$ such that any diffeomorphism $f \in \mathcal{G}_\omega$ is transitive. Moreover M is a unique homoclinic class.*

[2] An invariant compact set is *Lyapunov stable* if it has arbitrarily small neighborhoods U that are positively invariant: $f(U) \subset U$.

Arbieto et Matheus noticed that our results solve a question of Herman in [6]:

Theorem 3.2. *Let f be any C^∞-diffeomorphism that preserves ω. Then f satisfies one of the following properties*:

(1) *either f has a dominated splitting,*[3]
(2) *or f has no stable exponents*: *f can be approached in the C^1-topology by a sequence of C^∞-diffeomorphisms g that have a periodic point of period n such that $Dg^n(p) = \mathrm{Id}$.*

4. Open problems on dynamics of C^1-generic (non-conservatif) diffeomorphisms

Problem. Let us consider a generic C^1-diffeomorphism f.

(1) Does every homoclinic class of f with non-empty interior coincide with the whole manifold M?
(2) Is every Lyapunov stable homoclinic class of f a topological attractor?
(3) Is every robustly chain recurrent homoclinic class of f robustly transitive?
(4) Is every maximal weakly transitive set of f transitive?
(5) Are the unions of the stable and unstable manifolds of periodic orbits of f dense in M?
(6) Does every generic point of M belong to the basin of some weakly transitive topological attractor?

5. Sketch of the proof of Theorem 1.1

From the proof of Hayashi's Connecting Lemma given by Arnaud [1], we extract a slightly stronger statement that we explain now.

We consider the standard cube in $\mathbb{R}^d$ endowed with some reference tiling by smaller cubes (see Fig. 4 in [1]). A tiled cube of $\mathbb{R}^d$ is obtained from the standard one by homotheties and translations. For any chart $\varphi : U \to \mathbb{R}^d$ (where $d = \dim(M)$), we call *tiled cube* of the chart (U, φ) the preimage by φ of any tiled cube contained in $\varphi(U)$. Given a tiled cube $\mathcal{C}$ of a chart and an integer $N \geqslant 1$, we say that a sequence $x_0, \ldots, x_n$ preserves the tiling during N iterations if:

- the points x_0 and x_n are not contained in $\mathcal{C} \cup f(\mathcal{C}) \cup \cdots \cup f^N(\mathcal{C})$,
- if x_i (with $i \in \{0, \ldots, n-1\}$) belongs to the tiled cube $\mathcal{C}$ then $f^{-1}(x_{i+1})$ belongs to the same tile as x_i,
- if x_i belongs to some iterate $f^j(\mathcal{C})$ (with $j \in \{1, \ldots, N-1\}$) of the cube then $x_{i+1} = f(x_i)$.

Let $\mathcal{U}$ be a C^1-neighborhood of f. A tiled cube $\mathcal{C}$ of a chart is a *perturbation box of length $N \geqslant 1$ for $(f, \mathcal{U})$* if for any sequence $x_0, \ldots, x_n$ that preserves the tiling of $\mathcal{C}$ during N iterations there exists a perturbation $g \in \mathcal{U}$ that coincides with f outside $\mathcal{C} \cup f(\mathcal{C}) \cup \cdots \cup f^{N-1}(\mathcal{C})$ (so that $g^j(\mathcal{C}) = f^j(\mathcal{C})$ for $j \in \{1, \ldots, N\}$) and there exists a sequence $y_0 = x_0, y_1, \ldots, y_m = x_n$ joining x_0 to x_n that has no jump in $\mathcal{C} \cup f(\mathcal{C}) \cup \cdots \cup f^N(\mathcal{C})$: if y_i belongs to some iterate $f^j(\mathcal{C})$ (with $j \in \{0, \ldots, N-1\}$) then $y_{i+1} = g(y_i)$. Moreover the segments of the sequence that do not intersect $f(\mathcal{C}) \cup \cdots \cup f^{N-1}(\mathcal{C})$ are segments of the initial sequence (x_i).

The Connecting Lemma. *Given any C^1-neighborhood $\mathcal{U}$ of f, there is an integer N and, for any point $x \in M$, a chart centered at x such that any tiled cube $\mathcal{C}$ of the chart, which is disjoint from its iterates $f^i(\mathcal{C})$, $i \in \{1, \ldots, N\}$, is a perturbation box of length N for $(f, \mathcal{U})$.*

[3] A f-invariant set K has a dominated splitting if the tangent bundle $TM|_K$ to M over K splits as a direct sum $TM(x) = E(x) \oplus F(x)$, $x \in K$ of two proper subbundles that are Df-invariant, continuous and such that for some integer ℓ, any point $x \in K$, and any pair of unitary vectors $u \in E(x)$ and $v \in F(x)$, we have: $\|Df^\ell(u)\| \leqslant \frac{1}{2}\|Df^\ell(v)\|$.

We want to use the Connecting Lemma in order to delete all the jumps of a pseudo-orbit. The difficulty is that the jumps of a pseudo-orbit have no reason to respect the tiles of some perturbation box. Our idea consists in constructing finitely many disjoint perturbation boxes whose union meets every orbit of the initial dynamics. If every orbit spends a uniformly bounded time between the visits of the interior of tiles of these boxes, then the same holds for ε-pseudo-orbits with small ε. Furthermore, choosing ε even smaller, one can modify the pseudo-orbit in the following way. Each time the pseudo-orbit visits the interior of some tile, one defines a new jump by adding at this time all the following jumps of the pseudo-orbit until the next visit of a tile. Since the number of jumps we grouped together was uniformly bounded, the resulting jump is small and respects the tile. In that way, we get a pseudo-orbit that preserves the tiling of the boxes (during N iterations) and has no jumps outside the boxes. Boxes after boxes, the Connnecting Lemma erases all the jumps of the pseudo-orbit, giving the perturbation of Theorem 1.1.

In order to build these perturbation boxes, we have to solve some difficulty: the union of such boxes gives an open set which is disjoint from its N first iterates and meets all the orbits of M. The first step of the construction is to build an open set U with these properties. Such an open set will be called a *topological tower*[4] of order N.

Let us sketch the construction of the topological tower in the case f has no periodic orbit: we choose an integer $K \gg N$; as there are no periodic orbit, one can cover the manifold by a finite family of open sets (V_i), each of them disjoint from its K first iterates. In order to build U, we first keep V_0 and rename it U_0. One removes from V_1 the points that fall in U_0 after less than K iterates. The other points of V_1 are covered by a family of open sets (W_j) and a "Colouring Lemma" gives integers $k_j \leqslant K$ (with $|k_i - k_j| > N$ if $W_i \cap W_j \neq \emptyset$ when $i \neq j$) such that the open set $U_1 = U_0 \cup \bigcup_j f^{k_j}(W_j)$ is disjoint from its N first iterates and meets every orbit passing through $V_0 \cup V_1$. In the same way, one removes from V_2 the points that fall in U_1 after less than K iterates and uses the same process to get an open set U_2 which is disjoint from its N first iterates and meets all the orbits passing through $V_0 \cup V_1 \cup V_2$. The announced open set U is obtained by incorporating in U_i the orbits of V_{i+1} by the same construction.

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[4] Clearly the periodic orbits of period less than N are an obstruction for the existence of a topological tower but this definition can be adapted.