Partial Differential Equations

Some non-existence results for critical equations on step-two stratified groups

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Abstract

We discuss a non-existence result for a semilinear sub-elliptic Dirichlet problem with critical growth on half-spaces of stratified groups of step two. Our result improves a recent theorem (N. Garofalo, D. Vassilev, Math. Ann. 318 (3) (2000) 453–516). ***To cite this article: A. Bonfiglioli, F. Uguzzoni, C. R. Acad. Sci. Paris, Ser. I 336 (2003).***
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Résumé

Quelques résultats de non-existence pour des équations critiques sur les groupes stratifiés de pas deux. Nous discutons un résultat de non-existence pour un problème de Dirichlet sous-elliptique semi-linéaire à croissance critique sur les demi-espaces de groupes stratifiés de pas deux. Notre résultat améliore un théorème récent en (N. Garofalo, D. Vassilev, Math. Ann. 318 (3) (2000) 453–516). ***Pour citer cet article : A. Bonfiglioli, F. Uguzzoni, C. R. Acad. Sci. Paris, Ser. I 336 (2003).***
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Version française abrégée

Dans cette Note nous annonçons le résultat suivant.

Théorème 0.1. *Soit $\mathbb{G}$ un groupe de type Heisenberg et soit Ω un demi-espace quelconque de $\mathbb{G}$. Alors il n'existe pas de solutions du problème de Dirichlet*

$$\begin{cases} -\Delta_{\mathbb{G}} u = u^{(Q+2)/(Q-2)}, \\ u \in S_0^1(\Omega), \quad u > 0. \end{cases} \tag{1}$$

De plus, si $\mathbb{G}$ est un groupe stratifié de pas deux quelconque et Ω est un demi-espace de $\mathbb{G}$ avec frontière parallèle au centre de $\mathbb{G}$, alors, dans ce cas aussi, le problème de Dirichlet (1) *n'a pas des solutions.*

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Ci-dessus, nous avons désigné par $\Delta_{\mathbb{G}}$ le sous-Laplacien canonique sur $\mathbb{G}$, avec Q la dimension homogène de $\mathbb{G}$ et avec $S_0^1(\Omega)$ l'espace de Sobolev approprié. Notre théorème améliore un résultat récent [5] où on a démontré la non-existence de solutions de (1) pour la sous-classe des groupes de type Heisenberg composée des groupes de type Iwasawa et pour les demi-espaces Ω avec frontière non parallèle au centre de $\mathbb{G}$.

1. Introduction

The aim of this Note is to present some nonlinear Liouville theorems on step-two stratified groups and in particular on Heisenberg-type groups (see Theorem 2.1 below). The study of nonlinear non-existence theorems within the degenerate-elliptic context has achieved a growing interest in recent years. Indeed, this sort of results plays a crucial rôle in applying blow-up techniques in order to obtain existence results for nonlinear subelliptic equations. Particularly relevant is the study of critical semilinear equations on stratified groups in connection with the Yamabe problem and the Webster scalar curvature problem on CR manifolds (see, e.g., [7]).

A considerable amount of the related literature is devoted to the Heisenberg group $\mathbb{H}^k$, the simplest case of a non-Abelian stratified group. For instance, in the papers [1,2], sub-critical semilinear equations on $\mathbb{H}^k$ are investigated: a priori estimates and existence theorems are obtained via non-existence results of "Gidas and Spruck type". A different approach, of variational nature, is followed in [9,12,3]. In [3] existence of solutions has been obtained by using the non-existence results in [9,12] on the half-spaces of $\mathbb{H}^k$ and by the P.-L. Lions concentration-compactness principle.

The variational setting seems to be appropriate in order to obtain existence results also in the wider case of general stratified groups $\mathbb{G}$ (whose relevance is highlighted by the celebrated paper of Rothschild and Stein [10]), provided suitable nonlinear Liouville theorems are established. More precisely, in using variational techniques, one is led to characterize the energy levels of the variational solutions of the following semilinear Dirichlet problem with critical growth

$$\begin{cases} -\Delta_{\mathbb{G}}u = u^{(Q+2)/(Q-2)}, \\ u \in S_0^1(\Omega), \quad u > 0, \end{cases} \tag{2}$$

when Ω is a half-space of $\mathbb{G}$ or the whole space. Here, $\Delta_{\mathbb{G}}$ is a sub-Laplacian on $\mathbb{G}$, Q denotes the homogeneous dimension of $\mathbb{G}$ and $S_0^1(\Omega)$ is the appropriate subelliptic Sobolev space (all the notation and definitions can be found below).

The classical analogue of problem (2), when $\mathbb{G}$ is the Euclidean group $\mathbb{R}^N$ and $\Delta_{\mathbb{G}}$ is the ordinary Laplace operator, has been intensively studied starting from the early 1980s (see, e.g., the monograph [11] and the therein references). However, in comparison to the classical setting, uniqueness and non-existence results for (2) present new and significant difficulties, even in the "simplest" case of the Heisenberg group $\mathbb{G} = \mathbb{H}^k$ [9,12]. These difficulties are mainly due to the lack of good a priori estimates for the Lie derivatives of the solutions along the directions of higher commutators. The case of general stratified groups presents further complications and, at the authors' knowledge, only very partial results have been given so far. This is true even in the case of the so-called H-type groups, despite they share with the Heisenberg groups $\mathbb{H}^k$ common features. H-type groups were introduced by Kaplan [8] and form a remarkable class of step-two stratified groups widely studied in the latest literature. In the same paper [8, Eq. (17)], Kaplan exhibited an explicit (cilindrically-symmetric) solution to (2) when $\mathbb{G}$ is a H-type group and $\Omega = \mathbb{G}$. Moreover, in the recent paper [6], Garofalo and Vassilev have established a uniqueness result for cylindrically-symmetric solutions to (2) when $\mathbb{G}$ is a H-type group of Iwasawa-type and Ω is the whole $\mathbb{G}$. Furthermore, in [5] Garofalo and Vassilev deal with the non-existence problem on half-spaces. The techniques in [5] are based on the use of a Kelvin-type transform: this forces the authors to work only in the case of the Iwasawa-type groups, a particular sub-class of H-type groups where the Kelvin transform possesses several useful properties. Besides, in [5] only certain classes of half-spaces are covered.

2. Result

We announce here the following result which exhausts the problem for *all* H-type groups and for *all* half-spaces. It also provides a partial answer for the problem on general step-two stratified groups.

Theorem 2.1. *Let $\mathbb{G}$ be a H-type group and let Ω be any half-space of $\mathbb{G}$. Then the Dirichlet problem* (2) *has no solution. Moreover, if $\mathbb{G}$ is a general step-two stratified group and Ω is any half-space of $\mathbb{G}$ whose boundary is parallel to the center of $\mathbb{G}$, then* (*also in this case*) *the Dirichlet problem* (2) *has no solution.*

We explicitly remark that not every non-characteristic half-space of a step-two stratified group $\mathbb{G}$ has the form in the second assertion of Theorem 2.1, if $\mathbb{G}$ is not a H-type group.

Though some of our techniques in approaching Theorem 2.1 are inspired by the ideas contained in the papers [9,12], we stress that the case of general H-type groups present several new difficulties. Broadly speaking, these complications are mainly due to the structure of the second layer in the stratification of the Lie algebra of $\mathbb{G}$, which (when the group is not $\mathbb{H}^k$) always has dimension strictly larger than one. In particular, the different geometry of $\mathbb{G}$ makes it harder to construct explicit barrier functions. Moreover one has to face with the more general features of the group structure at many different levels.

3. Proof of Theorem 2.1

We hereafter trace the line of the proof of the above theorem. First we need to fix some notation. N-dimensional stratified groups of step *two* and m generators are characterized by being (canonically isomorphic to) $\mathbb{G} = (\mathbb{R}^N, \circ)$ with the following Lie group law ($N = m + n$, $x \in \mathbb{R}^m$, $t \in \mathbb{R}^n$)

$$(x, t) \circ (\xi, \tau) = \begin{pmatrix} x_j + \xi_j, & j = 1, \dots, m \\ t_j + \tau_j + \frac{1}{2}\langle x, U^{(j)}\xi\rangle, & j = 1, \dots, n \end{pmatrix}, \tag{3}$$

where the $U^{(j)}$'s are $m \times m$ linearly independent skew-symmetric matrices. The canonical sub-Laplacian on $\mathbb{G}$ is the second order degenerate-elliptic operator $\Delta_{\mathbb{G}} = \sum_{j=1}^m X_j^2$, where X_j is the left-invariant vector field that agrees at the origin with $\partial/\partial x_j$. The Lie algebra $\mathfrak{g}$ of $\mathbb{G}$ admits the stratification $\mathfrak{g} = \mathfrak{G}_1 \oplus \mathfrak{G}_2$, where $\mathfrak{G}_1 = \mathrm{span}\{X_1, \dots, X_m\}$ and $\mathfrak{G}_2 = \mathrm{span}\{(\partial/\partial t_1), \dots, (\partial/\partial t_n)\}$. We denote by $Q = m + 2n$ the homogeneous dimension of $\mathbb{G}$ and by $\nabla_{\mathbb{G}} = (X_1, \dots, X_m)$ the subelliptic gradient operator related to the sub-Laplacian $\Delta_{\mathbb{G}}$. If $\Omega \subseteq \mathbb{G}$ is a smooth open set, we recall that the characteristic set of Ω is

$$\{z \in \partial\Omega \mid X_j(z) \in \mathrm{T}_z(\partial\Omega), \ j = 1, \dots, m\},$$

$\mathrm{T}_z(\partial\Omega)$ being the tangent space to $\partial\Omega$ at the point z. Stratified groups possess the following remarkable property: there exists a homogeneous norm d on $\mathbb{G}$ such that

$$\Gamma(z, \zeta) = d^{2-Q}(\zeta^{-1} \circ z), \quad z, \zeta \in \mathbb{G}, \tag{4}$$

is a fundamental solution for $\Delta_{\mathbb{G}}$. In the sequel, we shall denote by $B_d(z, r)$ the d-ball with radius $r > 0$ and center $z \in \mathbb{G}$.

The definition of H-type group due to Kaplan [8] is equivalent to the following one: H-type groups are the subclass of step-two stratified groups $(\mathbb{G}, \circ)$ as in (3), with the additional requirements that the $U^{(j)}$'s are *orthogonal* and satisfy

$$U^{(r)}U^{(s)} + U^{(s)}U^{(r)} = 0, \quad \text{for every } r, s \in \{1, \dots, n\} \text{ with } r \neq s.$$

The above characterization allows to write an explicit formula for the sub-Laplacian $\Delta_{\mathbb{G}}$:

$$\Delta_{\mathbb{G}} = \sum_{j=1}^m \left(\frac{\partial}{\partial x_j}\right)^2 + \frac{1}{4}|x|^2 \sum_{s=1}^n \left(\frac{\partial}{\partial t_s}\right)^2 + \sum_{s=1}^n \sum_{i,j=1}^m x_i U_{i,j}^{(s)} \frac{\partial^2}{\partial x_j \partial t_s}. \tag{5}$$

We finally fix the notation for the Dirichlet problem (2). We set $2^\star = 2Q/(Q-2)$. The exponent $2^\star - 1 = (Q+2)/(Q-2)$ is a critical exponent for the semilinear Dirichlet problem (2). We shall denote by $S^1(\Omega)$ the Sobolev space of the functions $u \in L^{2^\star}(\Omega)$ such that $\nabla_\mathbb{G} u \in L^2(\Omega)$. The norm in $S^1(\Omega)$ is given by $\|u\|_{S^1(\Omega)} = \|u\|_{2^\star} + \|\nabla_\mathbb{G} u\|_2$. We denote by $S^1_0(\Omega)$ the closure of $C^\infty_0(\Omega)$ with respect to this norm. A solution to the Dirichlet problem (2) is, by definition, a function $u \in S^1_0(\Omega)$, $u > 0$, such that $\int_\Omega \langle \nabla_\mathbb{G} u, \nabla_\mathbb{G} \varphi \rangle = \int_\Omega u^{2^\star - 1} \varphi$, for every $\varphi \in S^1_0(\Omega)$.

Throughout the sequel, $\mathbb{G}$ will always denote a stratified group of step two, Π will denote an arbitrary half-space of $\mathbb{G}$ and u will always denote a fixed solution to the boundary value problem (2) on $\Omega = \Pi$. The core of the proof of Theorem 2.1 consists in finding suitable asymptotic estimates for the second layer derivatives of u, which will allow us to apply some general Pohozaev-type identities. We point out that Pohozaev-type identities have been proved in the previous papers [4] in the setting of the Heisenberg group and in [5] for general stratified groups. Here we apply [5, Theorem 3.1] to vector fields of the following type:

$$Z^{z_0}(z) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \big((tz_0) \circ z\big) \quad \text{(for a fixed } z_0 \in \mathbb{G}). \tag{6}$$

It is easy to prove that $\mathrm{div}(Z^{z_0}) = 0$ and that Z^{z_0} commutes with $\nabla_\mathbb{G}$. We then have the following Pohozaev-type identity:

$$2 \int_{\partial\Omega} \sum_{j=1}^m X_j \varphi \langle X_j, \nu \rangle Z^{z_0} \varphi \, \mathrm{d}\sigma - \int_{\partial\Omega} \langle Z^{z_0}, \nu \rangle |\nabla_\mathbb{G} \varphi|^2 \, \mathrm{d}\sigma = 2 \int_\Omega \Delta_\mathbb{G} \varphi \, Z^{z_0} \varphi. \tag{7}$$

Arguments analogous to the ones given in [4,9], will prove our non-existence Theorem 2.1, provided suitable estimates of u and its Lie derivatives are established. In the sequel we trace the line of how these estimates are proved.

The first task is to prove L^p summability properties of u, namely that $u \in L^p(\Omega)$ for every $p \in (2^\star/2, \infty]$. We stress that the proof of the global L^p summability of u for p lower than $2^\star$ requires significant modifications of the standard arguments. The following task is to prove asymptotic estimates for u and its Lie derivatives at infinity, in terms of the fundamental solution Γ of $\Delta_\mathbb{G}$. If u is set to be zero outside Π and $f = u^{2^\star - 1}$, we introduce the function $w(z) = \int_\mathbb{G} \Gamma(z, \zeta) f(\zeta) \, \mathrm{d}\zeta$ so that we can write $u = w + v$, where v is the $\Delta_\mathbb{G}$-*harmonic* part of u. We have $0 \leqslant u \leqslant w$ in Π. In order to estimate the decay of u, we show that $w = \mathcal{O}(\Gamma)$, as infinity. This can be proved as a consequence of [13, Theorem 1.1] jointly with the cited L^p properties of u. We then get the needed asymptotic estimate $u \leqslant M \min\{1, \Gamma\}$. We now turn to the estimate of the Lie derivatives of $u = w + v$. Since $\mathbb{G}$ is step-two, it is not difficult to find estimates of w using direct representation formulas for the Lie derivatives of the convolution defining w, up to the second layer of stratification. The heart of the matter is then to obtain analogous estimates for the second layer derivatives of v. We now have to distinguish between two classes of half-spaces Π, whose different geometric structures require ad hoc approaches:

$$\begin{aligned} \Pi^{(1)} &= \big\{(x, t) \in \mathbb{G} \mid \langle a, x \rangle > 0\big\} \quad \text{(for a fixed } a \in \mathbb{R}^m), \\ \Pi^{(2)} &= \big\{(x, t) \in \mathbb{G} \mid, \langle b, t \rangle > 0\big\} \quad \text{(for a fixed } b \in \mathbb{R}^n). \end{aligned} \tag{8}$$

We explicitly remark that if $\mathbb{G}$ is a H-type group, then these two types of half-spaces characterize (up to a group left-translation) respectively all non-characteristic and all characteristic half-spaces. On the contrary, if $\mathbb{G}$ is a general step-two stratified group, the class of non-characteristic half-spaces in $\mathbb{G}$ contains all the $\Pi^{(1)}$'s, but may also be larger.

First of all we treat the case of a non-characteristic half-space $\Pi^{(1)}$. The main argument is based on the representation of the derivatives of v as the limit of a sequence of integral means modelled on the geometry of $\Pi^{(1)}$. More precisely, let $Z \in \mathfrak{G}_2$. Since v is a classical solution of $\Delta_\mathbb{G} v = 0$ in $\Pi^{(1)}$, $v = w$ in $\partial \Pi^{(1)}$, then Zv is a classical solution of the Dirichlet problem $\Delta_\mathbb{G}(Zv) = 0$ in $\Pi^{(1)}$, $Zv = Zw$ in $\partial \Pi^{(1)}$. We used here the facts that the operators $\Delta_\mathbb{G}$ and Z commute and that $\partial \Pi^{(1)}$ is invariant with respect to the Euclidean translations along the

x-coordinates. Let now $r(x) > 0$ be a continuous function *only depending on x* such that $B_d((x, t), r(x)) \Subset \Pi^{(1)}$ for every $(x, t) \in \Pi^{(1)}$. For every $g \in L^1_{\mathrm{loc}}(\Pi^{(1)})$ we define

$$Tg : \Pi^{(1)} \to \mathbb{R}, \quad (Tg)(x, t) = (M_{r(x)} g)(x, t),$$

where M_r is defined by (here $m_Q > 0$ is suitable constant and $K = |\nabla_{\mathbb{G}} d|^2$)

$$M_r(g)(z) = \frac{m_Q}{r^Q} \int\limits_{B_d(z, r)} K(z^{-1} \circ \zeta) g(\zeta) \, \mathrm{d}\zeta.$$

T is a linear operator with the following properties. (i) T is increasing and maps $L^1_{\mathrm{loc}}(\Pi^{(1)})$ into $C(\Pi^{(1)})$. (ii) If $g \in C^2(\Pi^{(1)})$ and $\Delta_{\mathbb{G}} g = 0$ then $Tg = g$. (iii) If $g \in C^2(\Pi^{(1)})$ and $\Delta_{\mathbb{G}} g \leqslant 0$ then $Tg \leqslant g$ and $(T^k g)_{k \in \mathbb{N}}$ is a non-increasing sequence. (iv) T commutes with second-layer Lie derivatives. By means of these properties of T, it is possible to show that $T^k u$ is a non-increasing sequence pointwise convergent to zero in $\Pi^{(1)}$ as $k \to \infty$. As a consequence, we are able to derive the result we needed:

$$|Zv| = \mathcal{O}(\Gamma), \quad \text{at infinity in } \Pi^{(1)}. \tag{9}$$

Indeed, from the above properties, it is $Tv = v$ hence $T^k w = T^k v + T^k u = v + T^k u \searrow v$; moreover $T^k(Zw) = Z(T^k w)$ whence (for every non-negative test function Φ) $\int \Phi T^k(Zw) = -\int (Z\Phi) T^k w \overset{k \to \infty}{\longrightarrow} \int \Phi Zv$. From the estimates of w and the $\Delta_{\mathbb{G}}$-harmonicity of Γ, it also follows $|T^k(Zw)| \leqslant T^k(c\,\Gamma) = c\,\Gamma$ in $\Pi^{(1)}$, whence $|\int_{\Pi^{(1)}} \Phi Zv| \leqslant M \int_{\Pi^{(1)}} \Phi \Gamma$ which implies (9). We explicitly remark that everything we have discussed so far is valid on a general step-two stratified group $\mathbb{G}$.

We are left with the investigation of the half-spaces $\Pi^{(2)}$ in (8). To this end, we now restrict ourselves to the case when $\mathbb{G}$ is a H-type group. Hence the $\Pi^{(2)}$'s exhaust (up to a left-translation) all characteristic half-spaces. A rather elaborated argument is exploited in obtaining the needed estimates of second layer Lie derivatives. This argument is based on the delicate construction of explicit barrier functions (modelled on the geometry of $\Pi^{(2)}$ and on the properties of H-type groups) and also relies on the knowledge of the explicit expression of $\Delta_{\mathbb{G}}$ in (5). We briefly summarize our approach. We first find an estimate of u near the boundary of Π which allows to obtain an estimate of the normal derivative Zu at $\partial \Pi$. Then, exploiting the fact that $\Delta_{\mathbb{G}}$ and Z commute, we are able to extend such estimate inside Π. In this way we get asymptotic behavior for Zu both at infinity and near the characteristic set $\{|x| = \langle b, t \rangle = 0\}$ (where Zu may fail to be smooth up to the boundary). We do not give details here. We only highlight that the lack of compactness of the characteristic set adds remarkable complications in this construction, in comparison to the case of the Heisenberg group $\mathbb{H}^k$.

Acknowledgements

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