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Partial Differential Equations

The role of eigenvalues and eigenvectors of the symmetrized gradient of velocity in the theory of the Navier–Stokes equations

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Abstract

In this Note, we formulate sufficient conditions for regularity of a so called suitable weak solution $(v; p)$ in a sub-domain D of the time–space cylinder Q_T by means of requirements on one of the eigenvalues or on the eigenvectors of the symmetrized gradient of velocity. ***To cite this article: J. Neustupa, P. Penel, C. R. Acad. Sci. Paris, Ser. I 336 (2003).***

Résumé

Rôle des valeurs propres et des vecteurs propres du gradient symétrisé des vitesses en théorie des équations de Navier–Stokes. Dans cette Note, on formule des conditions géométriques suffisantes pour la régularité intérieure des solutions faibles (« suitable weak ») des équations de Navier–Stokes dans un sous-domaine D du cylindre spatio–temporel Q_T : ces conditions suffisantes portent sur une des valeurs propres ou bien sur les composantes des vecteurs propres du gradient symétrisé. ***Pour citer cet article : J. Neustupa, P. Penel, C. R. Acad. Sci. Paris, Ser. I 336 (2003).***

Version française abrégée

Soit $\Omega \subset \mathbb{R}^3$ un ouvert régulier, et T un nombre réel positif. On considère le problème suivant :

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\nabla p + \nu \Delta \mathbf{v} \quad \text{in } Q_T, \tag{1}$$

$$\operatorname{div} \mathbf{v} = 0 \quad \text{in } Q_T, \tag{2}$$

$$\mathbf{v} = 0 \quad \text{on } \partial\Omega \times]0, T[, \tag{3}$$

$$\mathbf{v}|_{t=0} = \mathbf{v}_0. \tag{4}$$

Les équations de Navier–Stokes sont bien connues, $\mathbf{v} = (v_1, v_2, v_3)$ décrit la vitesse d'un écoulement, p la pression et ν est le coefficient de viscosité. Les notions de solutions faibles au sens de Hopf–Leray ou au sens de Caffarelli–

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Kohn–Nirenberg sont bien connues. On peut facilement trouver les définitions de ces solutions et des revues de leurs propriétés importantes (par exemple G.P. Galdi [2]). S'il existe un voisinage de (t,x) dans Q_T tel que $\mathbf{v}$ y soit essentiellement borné, on dit que (t,x) est un point régulier (un point singulier sinon) ; l'ensemble $S(v)$ qui est fermé dans Q_T désigne l'ensemble des points singuliers. Pour v solution faible au sens de Caffarelli–Kohn–Nirenberg on sait que la mesure de Hausdorff unidimensionnelle de $S(v)$ est zéro : Ainsi peut-on localiser le problème de conditions aux limites et initiales (1)–(4) dans $]t_0-\tau,t_0[\times C$ où $C=\prod_{i=1}^3]x_{0i}-r,x_{0i}+r[$ pour τ et r petits convenables, t_0 étant une possible époque de singularité de $\mathbf{v}$, et $(x_0,t_0)\in S(v)$.

En définitive on peut se ramener dans $]t_0-\tau,t_0[\times C$ à l'étude de la régularité de $\mathbf{u}$ solution de

$$\frac{\partial \mathbf{u}}{\partial t}+(\mathbf{u}\cdot\nabla)\mathbf{u}=\mathbf{h}-\nabla(\eta p)+\nu\,\Delta\mathbf{u}, \tag{5}$$

$$\operatorname{div}\mathbf{u}=0, \tag{6}$$

$$\mathbf{u}(\cdot,t_0-\tau)=\mathbf{u}_0, \tag{7}$$

$$\mathbf{u}=\mathbf{0}\quad \text{pour } (x,t)\in\partial C\times]t_0-\tau,t_0[\,; \tag{8}$$

où $\mathbf{u}=\eta\mathbf{v}-\mathbf{V}$ avec $\operatorname{div}\mathbf{V}=\operatorname{div}(\eta\mathbf{v})$ et η opérant une troncature adéquate de $\mathbf{v}$. La régularité de tous les termes intervenant dans $\mathbf{h}$ (entièrement calculé dans la partie anglaise) est assurée, les supports de $\nabla\eta$ et de $\mathbf{V}$ étant de la forme $\overline{C_2}-C_1$ où $C_1\subset C\subset C_2$; on notera que $\mathbf{v}$ et $\mathbf{u}$ coincident sur le plus petit cube C_1.

L'étude de (5)–(8) nous a permis d'établir un certain nombre de critères de régularité intérieure pour la solution $\mathbf{v}$ du problème (1)–(4). Pour contrôler $\|\nabla\mathbf{u}\|_{L^\infty(t_0-\tau,t_0;\,L^2(C))}$ il est classique d'intégrer sur C le produit de (5) par $\Delta\mathbf{u}(t,\cdot)$ ou d'opérer $\operatorname{curl}(\cdot)$ sur l'Éq. (5) avant d'intégrer sur C le produit par $\omega(\cdot,t)=\operatorname{curl}\mathbf{u}(\cdot,t)$. Notre objectif est ici de voir plus précisement quelle est la structure du terme non linéaire $\int_C(\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\Delta\mathbf{u}\,\mathrm{d}x$.

Observez que l'on a

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\int_C|\nabla\mathbf{u}|^2\,\mathrm{d}x+\nu\int_C|\Delta\mathbf{u}|^2\,\mathrm{d}x=-\int_C h\cdot\Delta\mathbf{u}\,\mathrm{d}x+\int_C(\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\Delta\mathbf{u}\,\mathrm{d}x,$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\int_C|\omega|^2\,\mathrm{d}x+\nu\int_C|\nabla\omega|^2\,\mathrm{d}x=\int_C(\omega\cdot\nabla)\mathbf{u}\cdot\omega\,\mathrm{d}x+\int_C\operatorname{curl}\mathbf{h}\cdot\omega\,\mathrm{d}x,$$

où $\int_C|\omega|^2\,\mathrm{d}x=\int_C|\nabla\mathbf{u}|^2\,\mathrm{d}x$ et $\int_C|\nabla\omega|^2\,\mathrm{d}x=\int_C|\Delta\mathbf{u}|^2\,\mathrm{d}x$ et avec les notations usuelles $\sigma(\mathbf{u})_{ij}=\frac{1}{2}(u_{i,j}+u_{j,i})$.

Il vient $\int_C(\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\Delta\mathbf{u}\,\mathrm{d}x=\int_C u_{i,j}u_{i,k}u_{j,k}\,\mathrm{d}x=-\int_C\sigma_{ij}\sigma_{ik}\sigma_{jk}\,\mathrm{d}x+\frac{1}{4}\int_C\sigma_{ij}\omega_i\omega_j\,\mathrm{d}x$ alors $\frac{\mathrm{d}}{\mathrm{d}t}\frac{3}{8}\int_{C_2}|\nabla\mathbf{u}|^2\,\mathrm{d}x+\frac{3\nu}{4}\int_{C_2}|\Delta\mathbf{u}|^2\,\mathrm{d}x=-\int_{C_2}\sigma_{ij}\sigma_{ik}\sigma_{jk}\,\mathrm{d}x-\frac{5}{4}\int_{C_2}\mathbf{h}\cdot\Delta\mathbf{u}\,\mathrm{d}x$. Les valeurs propres (réelles) et les vecteurs propres associés (orthogonaux) du tenseur $\sigma(\mathbf{u})$ sont des notions naturelles qui sont en relation locale avec l'écoulement $\mathbf{u}$: l'équation de continuité $\operatorname{div}\mathbf{u}=0$ s'écrit $(\lambda_1+\lambda_2+\lambda_3)(x,t)=0$ et $(\sigma_{ij}\sigma_{ik}\sigma_{jk})(x,t)=3(\lambda_1\lambda_2\lambda_3)(x,t)$ $=3\det\sigma(\mathbf{u}(x,t))$ est indépendant du système de coordonnées. Donc $\frac{\mathrm{d}}{\mathrm{d}t}\frac{3}{8}\int_C|\nabla\mathbf{u}|^2\,\mathrm{d}x+\frac{3\nu}{4}\int_C|\Delta\mathbf{u}|^2\,\mathrm{d}x+3\int_C\lambda_1\lambda_2\lambda_3\,\mathrm{d}x=\frac{5}{4}\int_{C_2}\mathbf{h}\cdot\Delta\mathbf{u}\,\mathrm{d}x$. Le premier résultat principal de la note est le suivant :

Théorème 1. *Soit $D\subset Q_T$ ouvert et $\mathbf{v}$ solution faible («suitable weak») du problème* (1)–(4). *Soient $\zeta_1\leqslant\zeta_2\leqslant\zeta_3$ les valeurs propres du tenseur $\frac{1}{2}(v_{i,j}+v_{j,i})$. On désigne par $(\zeta_2)_+$ la fonction* $\max\{\zeta_2;0\}$. *Si une des fonctions ζ_1, $(\zeta_2)_+$, ζ_3 appartient à $L^{r,s}_{\mathrm{loc}}(D)$ avec $2/r+3/s\leqslant 2$; $r\in[1,+\infty]$, $s\in]3/2,+\infty]$, alors $S(\mathbf{v})\cap D=\emptyset$, $(\mathbf{v};p)$ est régulier dans D.*

Idée de la démonstration : $\zeta_1\leqslant 0$, $\zeta_3\geqslant 0$ et $\zeta_1+\zeta_2+\zeta_3=0$, ζ_i et λ_i coincident dans le plus petit cube C_1, et dans $]t_0-\tau,t_0[\times(C-C_1)$ les λ_i sont bornés. La structure du terme non linéaire est expliquée par $\int_C\lambda_1\lambda_2\lambda_3\mathrm{d}x=\int_{C-C_1}\lambda_1\lambda_2\lambda_3\,\mathrm{d}x+\int_{C_1}\zeta_1(\zeta_2)_+\zeta_3\,\mathrm{d}x-\int_{C_1}\zeta_1(\zeta_2)_-\zeta_3\,\mathrm{d}x$. Il suffit de traiter $\int_{t_0-\tau}^{t_0}\int_{C_1}\zeta_1(\zeta_2)_+\zeta_3\,\mathrm{d}x\,\mathrm{d}t$ pour en déduire $\|\nabla\mathbf{u}\|$ et $\|\nabla\mathbf{v}\|$ dans $L^{\infty,2}\cap L^{2,6}$. On conclut à la régularité comme dans [1].

La contrainte de divergence nulle permet de bien gérer les signes des valeurs propres, cela est assez clair ci-dessus. Nous allons remarquer qu'en se fondant sur elle, d'autres possibilités de majorations a priori peuvent être envisagées, ainsi le second résultat principal de la note :

Théorème 2. *Soit $D \subset Q_T$ ouvert et* $\mathbf{v}$ *solution faible* («*suitable weak*») *du problème* (1)–(4). *Soient* $\mathbf{e}^i$ $(i = 1, 2, 3)$ *la base orthogonale des vecteurs propres du tenseur* $\frac{1}{2}(v_{i,j} + v_{j,i})$, *en tout point* (x, t) *de* D. *Si les fonctions* $\mathbf{e}^i$ *sont continues sur* D *et leurs dérivées partielles premières bornées dans* D, *alors* $S(\mathbf{v}) \cap D = \emptyset$, $(\mathbf{v}; p)$ *est régulier dans* D.

La démonstration est technique, localement dans un système de coordonnées choisi tel que les directions des axes de coordonnées soient en cohérence avec celles des vecteurs propres. On utilise essentiellement l'équation de continuité.

Après dérivation, l'équation de continuité conduit a trois équations d'ondes spatiales, du type $u_{1,11} - u_{1,22} - u_{1,33} = (a_k^{1,2} u_{k,k})_{,2} + (a_k^{1,3} u_{k,k})_{,3}$ et le fait remarquable est que l'on peut contrôler $\|\nabla u_1\|_{L^\infty(t_0-\tau,t_0;\, L^2(C_1))}$ par $\|\nabla u_2\|_{L^\infty(t_0-\tau,t_0;\, L^2(C_1))}$ et $\|\nabla u_3\|_{L^\infty(t_0-\tau,t_0;\, L^2(C_1))}$ avec des constantes plus petites que 1 pourvu que τ et r_1 soient suffisamment petits. On conclut comme pour le Théorème 1 en contrôlant $\|\nabla \mathbf{u}\|$ dans $L^{\infty,2} \cap L^{2,6}$ et en suivant [1].

1. Introduction

Let Ω be a domain in $\mathbb{R}^3$, T be a positive number and $Q_T = \Omega \times]0, T[$. We deal with the Navier–Stokes initial-boundary value problem

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\nabla p + \nu \Delta \mathbf{v} \quad \text{in } Q_T, \tag{1}$$

$$\operatorname{div} \mathbf{v} = 0 \quad \text{in } Q_T, \tag{2}$$

$$\mathbf{v} = 0 \quad \text{on } \partial\Omega \times]0, T[, \tag{3}$$

$$\mathbf{v}|_{t=0} = \mathbf{v}_0, \tag{4}$$

where $\mathbf{v} = (v_1, v_2, v_3)$ and p denote the velocity and the pressure and $\nu > 0$ is the viscosity coefficient.

The notions of a weak solution and a suitable weak solution to the problem (1)–(4) are well known. The readers can find the definitions and surveys of their important properties, e.g., in Galdi [2] and in Neustupa and Penel [3].

A point $(x, t) \in Q_T$ is called a regular point of a weak solution $\mathbf{v}$ of (1)–(4) if there exists a neighborhood U of (x, t) in Q_T such that $\mathbf{v}$ is essentially bounded in U. Points of Q_T which are not regular are called *singular*. We denote by $S(\mathbf{v})$ the set of all singular points of $\mathbf{v}$.

We suppose that $(\mathbf{v}; p)$ is a suitable weak solution of the problem (1)–(4). Then the singular set $S(\mathbf{v})$ has the 1-dimensional Hausdorff measure, equal to zero. (See Caffarelli, Kohn and Nirenberg [1].)

Suppose that D is a sub-domain of Q_T and D' is a domain in D such that $D' \subset \overline{D'} \subset D$. Since $S(\mathbf{v})$ is closed in Q_T, $S(\mathbf{v}) \cap \overline{D'}$ is a closed set. Denote by $\mathcal{T}(D')$, respectively G', the projection of D', respectively the projection of $S(\mathbf{v}) \cap \overline{D'}$, onto the time axis. Then $\mathcal{T}(D') = \bigcup_{\gamma \in \Gamma'}]a'_\gamma, b'_\gamma[\cup G'$ where the sets on the right-hand side are mutually disjoint.

Assume that t_0 is equal to b'_γ for some $\gamma \in \Gamma'$ and (x_0, t_0) is a singular point of $\mathbf{v}$ in D'. Our goal, as in [3], is first to localize the problem around (x_0, t_0) (Lemma 1.1), and then assuming some regularity conditions to show a contradiction with the possibility for (x_0, t_0) to be singular (Theorem 1 in Section 2 and Theorem 2 in Section 3).

Lemma 1.1. *There exist positive numbers* τ, r_1, r_2 *such that* $r_1 < r_2$ *and if we denote* $C_i =]x_{01} - r_i, x_{01} + r_i[\times]x_{02} - r_i, x_{02} + r_i[\times]x_{03} - r_i, x_{03} + r_i[$ $(i = 1, 2)$ *then*

(i) τ *is so small that* $a'_\gamma < b'_\gamma - \tau = t_0 - \tau$,

(ii) $\overline{C_2} \times [t_0 - \tau, t_0] \subset D'$,

(iii) $\{(\overline{C_2} - C_1) \times [t_0 - \tau, t_0]\} \cap S(\mathbf{v}) = \emptyset$,
(iv) $\mathbf{v}$ *and all its space derivatives are bounded on* $(\overline{C_2} - C_1) \times [t_0 - \tau, t_0]$,
(v) p, *respectively* $\partial\mathbf{v}/\partial t$, *has all space derivatives in* $L^\alpha(t_0 - \tau, t_0; L^\infty(C_2 - \overline{C_1}))$, *respectively in* $L^\alpha(t_0 - \tau, t_0; L^\infty(C_2 - \overline{C_1})^3)$, *for each* α *such that* $1 < \alpha < 2$.

(See [3] and [4] for the proof.) Numbers r_1, r_2 and τ given by Lemma 1.1, are not unique. On the other hand, there exist decreasing sequences $\{r_1^n\}$, $\{r_2^n\}$, $\{\tau^n\}$ of numbers with the properties of r_1, r_2 and τ stated in Lemma 1.1 which tend to zero.

Put $r_3 = (2r_1 + r_2)/3$, $r_4 = (r_1 + 2r_2)/3$ and $C_i =]x_{01} - r_i, x_{01} + r_i[\times]x_{02} - r_i, x_{02} + r_i[\times]x_{03} - r_i, x_{03} + r_i[$ for $i = 3, 4$. $L^{a,b}$ denotes the space $L^a(t_0 - \tau, t_0; L^b(C_2))$ and $|||\cdot|||_{a,b}$ denotes the norm in $L^{a,b}$. $|||\cdot|||_{(\infty,2)\cap(2,6)} = |||\cdot|||_{\infty,2} + |||\cdot|||_{2,6}$ and $\|\cdot\|_k$ denotes the norm in $L^k(C_2)$.

The problem (1)–(4) can be localized to C_2 in the spatial variables: let η be a C^∞ cut-off function on $\mathbb{R}^3$ such that $\operatorname{supp}(\eta) \subset C_4$, $\eta = 1$ on $\overline{C_3}$ and $0 \leqslant \eta \leqslant 1$ on $\overline{C_4} - C_3$. Since $\eta\mathbf{v}$ does not satisfy the equation of continuity, we put $\mathbf{u} = \eta\mathbf{v} - \mathbf{V}$ where $\mathbf{V}$ is an appropriate function such that $\operatorname{div}\mathbf{V} = \operatorname{div}(\eta\mathbf{v})$. Function $\mathbf{V}(\cdot, t)$ can be constructed in such a way that it has a compact support in $C_2 - \overline{C_1}$ and all its space derivatives are bounded on $\overline{C_2} \times [t_0 - \tau, t_0]$. Function $\mathbf{u}$ satisfies in a strong sense the equations

$$\frac{\partial\mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} = \mathbf{h} - \nabla(\eta\, p) + \nu\,\Delta\mathbf{u} \quad \text{in } C_2 \times]t_0 - \tau, t_0[, \tag{5}$$

$$\operatorname{div}\mathbf{u} = 0 \quad \text{in } C_2 \times]t_0 - \tau, t_0[, \tag{6}$$

$$\mathbf{u}(\cdot, t_0 - \tau) = \mathbf{u}_0 \quad \text{in } C_2, \tag{7}$$

$$\mathbf{u} = \mathbf{0} \quad \text{for } (x, t) \in \partial C_2 \times]t_0 - \tau, t_0[; \tag{8}$$

$$\begin{aligned}\mathbf{h} = {}& -\frac{\partial\mathbf{V}}{\partial t} - \mathbf{V}\cdot\nabla(\eta\,\mathbf{v}) - (\eta\,\mathbf{v})\cdot\nabla\mathbf{V} + \mathbf{V}\cdot\nabla\mathbf{V} + (\eta\,\mathbf{v}\cdot\nabla\eta)\,\mathbf{v} \\ & - \eta(1-\eta)\,\mathbf{v}\cdot\nabla\mathbf{v} - 2\nu\nabla\eta\cdot\nabla\mathbf{v} - \nu\,\mathbf{v}\,\Delta\eta + \nu\Delta\mathbf{V} + p\nabla\eta.\end{aligned}$$

In fact, $\mathbf{u}$ has all derivatives equal to zero on $\partial C_2 \times]t_0 - \tau, t_0[$. One can deduce that function $\mathbf{h}$ has all its space derivatives in $L^\alpha(t_0 - \tau, t_0; L^\infty(C_2)^3)$ for every $\alpha \in]1, 2[$. Moreover, $\mathbf{h}$ has a compact support in $(C_2 - \overline{C_1}) \times [t_0 - \tau, t_0]$. (See [3] and [4].) The components of $\mathbf{u}$ are denoted by u_1, u_2 and u_3. $u_{i,j}$ is the partial derivative of u_i with respect to x_j and $\omega = (\omega_1, \omega_2, \omega_3) = \operatorname{curl}\mathbf{u}$.

2. Regularity in dependence on eigenvalues of the rate of deformation tensor

Suppose that $t \in]t_0 - \tau, t_0[$. Multiplying Eq. (5) by $\Delta\mathbf{u}$ and integrating on C_2, we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\int_{C_2} |\nabla\mathbf{u}|^2\,\mathrm{d}x + \nu\int_{C_2} |\Delta\mathbf{u}|^2\,\mathrm{d}x &= -\int_{C_2} h\cdot\Delta\mathbf{u}\,\mathrm{d}x + \int_{C_2} (\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\Delta\mathbf{u}\,\mathrm{d}x \\ &= -\int_{C_2} h\cdot\Delta\mathbf{u}\,\mathrm{d}x - \int_{C_2} u_{i,j}\,u_{i,k}\,u_{j,k}\,\mathrm{d}x \leqslant c_1\big\|\mathbf{h}(\cdot,t)\big\|_1 - \int_{C_2} u_{i,j}u_{i,k}u_{j,k}\,\mathrm{d}x.\end{aligned} \tag{9}$$

(Function $\mathbf{h}$ has the support in $(\overline{C_2} - C_1) \times]t_0 - \tau, t_0[$ where functions $\mathbf{v}$ and $\mathbf{V}$ have all their space derivatives bounded. Since $\mathbf{u} = \eta\mathbf{v} - \mathbf{V}$, $\mathbf{u}$ has all its space derivatives bounded in $(\overline{C_2} - C_1) \times]t_0 - \tau, t_0[$, too. Hence the integral of $\mathbf{h}\,\Delta\mathbf{u}$ can be estimated by $c_1\|\mathbf{h}(\cdot, t)\|_1$.)

Let us denote $\sigma_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$. If we use the symmetry of (σ_{ij}) then we can show that the last integral on the right-hand side of (9) can be written in the form

$$-\int_{C_2} u_{i,j}u_{i,k}u_{j,k}\,\mathrm{d}x = -\int_{C_2} \sigma_{ij}\sigma_{ik}\sigma_{jk}\,\mathrm{d}x + \frac{1}{4}\int_{C_2} \sigma_{ij}\omega_i\omega_j\,\mathrm{d}x. \tag{10}$$

Applying operator curl to Eq. (5) and multiplying the equation by ω, we get the identity

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\int\limits_{C_2}|\omega|^2\,\mathrm{d}x+\nu\int\limits_{C_2}|\nabla\omega|^2\,\mathrm{d}x=\int\limits_{C_2}\sigma_{ij}\omega_i\omega_j\,\mathrm{d}x+\int\limits_{C_2}\operatorname{curl}\mathbf{h}\cdot\omega\,\mathrm{d}x$$

which enables to express the last integral on the right-hand side of (10). Thus, we can obtain:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\frac{3}{8}\int\limits_{C_2}|\nabla\mathbf{u}|^2\,\mathrm{d}x+\frac{3\nu}{4}\int\limits_{C_2}|\Delta\mathbf{u}|^2\,\mathrm{d}x&=-\int\limits_{C_2}\sigma_{ij}\sigma_{ik}\sigma_{jk}\,\mathrm{d}x-\int\limits_{C_2}h\cdot\Delta\mathbf{u}\,\mathrm{d}x-\frac{1}{4}\int\limits_{C_2}\operatorname{curl}\mathbf{h}\cdot\omega\,\mathrm{d}x\\&=-\int\limits_{C_2}\sigma_{ij}\sigma_{ik}\sigma_{jk}\,\mathrm{d}x-\frac{5}{4}\int\limits_{C_2}\mathbf{h}\cdot\Delta\mathbf{u}\,\mathrm{d}x\leqslant-\int\limits_{C_2}\sigma_{ij}\sigma_{ik}\sigma_{jk}\,\mathrm{d}x+c_1\|\mathbf{h}(\cdot,t)\|_1.\end{aligned}\tag{11}$$

Let x be a chosen point in C_2. Then the system of coordinates can be chosen so that the tensor (σ_{ij}) has a diagonal representation at point x: $(\sigma_{ij})(x,t)=\lambda_i$ if $i=j$ and $\sigma_{ij}(x,t)=0$ if $i\neq j$. $\lambda_1,\lambda_2,\lambda_3$ are the eigenvalues of $(\sigma_{ij})(x,t)$. We can suppose without the loss of generality that $\lambda_1\leqslant\lambda_2\leqslant\lambda_3$. The equation of continuity (6) implies that $\lambda_1+\lambda_2+\lambda_3=0$. Then

$$(\sigma_{ij}\sigma_{ik}\sigma_{jk})(x,t)=\lambda_1^3+\lambda_2^3+\lambda_3^3=3\lambda_1\lambda_2\lambda_3.\tag{12}$$

The product $\lambda_1\lambda_2\lambda_3$ is an invariant of the tensor $(\sigma_{ij})(x,t)$ and so it is independent of the choice of the system of coordinates. Hence (12) holds in all points $x\in C_2$ and we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{3}{8}\int\limits_{C_2}|\nabla\mathbf{u}|^2\,\mathrm{d}x+\frac{3\nu}{4}\int\limits_{C_2}|\Delta\mathbf{u}|^2\,\mathrm{d}x\leqslant-3\int\limits_{C_2}\lambda_1\lambda_2\lambda_3\,\mathrm{d}x+c_1\|\mathbf{h}(\cdot,t)\|_1.\tag{13}$$

Estimate (13) is the main tool which enables to prove the following theorem. (See [4] for more details.) However, the theorem is formulated in terms of solution $\mathbf{v}$ and not $\mathbf{u}$ $(=\eta\mathbf{v}-\mathbf{V})$. So we speak about the eigenvalues $\zeta_1,\ \zeta_2$ and ζ_3 of the symmetric tensor $\frac{1}{2}(v_{i,j}+v_{j,i})$ (so called "rate of deformation tensor") and not about the eigenvalues $\lambda_1,\ \lambda_2,\ \lambda_3$ of the tensor $(\sigma_{ij})=\frac{1}{2}(u_{i,j}+u_{j,i})$.

Theorem 2.1. *Suppose that D is an open sub-domain of Q_T, $(\mathbf{v};\,p)$ is a suitable weak solution of the problem* (1)–(4), *$\zeta_1\leqslant\zeta_2\leqslant\zeta_3$ are the eigenvalues of the tensor $\frac{1}{2}(v_{i,j}+v_{j,i})$ and one of the functions ζ_1, $(\zeta_2)_+$, ζ_3 belongs to $L^{r,s}_{\mathrm{loc}}(D)$ for some real numbers r, s such that $1\leqslant r\leqslant+\infty$, $\frac{3}{2}<s\leqslant+\infty$ and $2/r+3/s\leqslant 2$. ($(\zeta_2)_+$ denotes the positive part of ζ_2.) Then the solution $(\mathbf{v};\,p)$ is regular in D.*

The proof is based on inequality (13) which enables to show that the norm $|||\mathbf{u}|||_{(\infty,2)\cap(2,6)}$ is finite. A similar techniques as in [1] then leads to the conclusion that (x_0,t_0) is a regular point of the weak solution $(\mathbf{v};\,p)$.

It follows from Theorem 2.1 that the case when $\zeta_2\leqslant 0$ supports regularity, while the case when, i.e., $\zeta_2>0$ supports a hypothetical singularity of the solution $(\mathbf{v};\,p)$. Since the sign of ζ_2 is connected with the types of deformation of "infinitely small" volumes of the fluid, we can also interpret our result geometrically: *such deformations, when the "infinitely small" volumes of the fluid are compressed in two dimensions and stretched in one dimension support regularity, while the cases when the "infinitely small" volumes of the fluid are compressed in one dimension and stretched in two dimensions support the hypothetical "blow up"*.

One remark more: according to conventional ideas, the so-called stretching term is the real source term in the vorticity equation, $\omega\cdot\nabla\mathbf{v}$, generically three-dimensional (as it is well known this term is identically zero in the 2D-vorticity equation). Here we are not interesting in an explicit representation of this term, we only observe that the integral of its product by ω gives the second term at the r.h.s. of (10) as the important quantity, which is also proportional to $\int_{C_2}\lambda_1\lambda_2\lambda_3\,\mathrm{d}x$.

3. Regularity in dependence on eigenvectors of the rate of deformation tensor

Suppose that $\mathbf{e}^i=(e^i_1,e^i_2,e^i_3)$ $(i=1,2,3)$ are the eigenvectors of the rate of deformation tensor $\frac{1}{2}(v_{i,j}+v_{j,i})$ in D such that $\mathbf{e}^i\cdot\mathbf{e}^j=\delta_{ij}$ $(i,j=1,2,3)$ for a.a. $(x,t)\in D$ and

($*$) $\mathbf{e}^i$ $(i=1,2,3)$ *are continuous in* D *and their* 1*st order derivatives with respect to the space variables* $x_1,\ x_2,\ x_3$ *are essentially bounded in* D.

Such conditions concern the deformation directions. In the given form, the conditions are clearly too strong, it is not difficult to define weaker forms, rather it would be of particular interest to find their best weaker form: but to do that one has to face a correlative previous question on the competitive role of the vorticity and of the pressure (the Hessian of the pressure).

We are able to prove the following technical result, as a preliminary report on a progressing work. All the details of proofs will appear in [4].

Then $(v_{k,l}+v_{k,l})\,e^i_k\,e^j_l=0$ $(i,j=1,2,3;\ i\neq j)$ in a.a. points $(x,t)\in D$. We can assume without loss of generality that $\mathbf{e}^1(x_0,t_0)=(1,0,0)$, $\mathbf{e}^2(x_0,t_0)=(0,1,0)$ and $\mathbf{e}^3(x_0,t_0)=(0,0,1)$. Function $\mathbf{u}$ $(=\eta\mathbf{v}-\mathbf{V})$ coincides with $\mathbf{v}$ in C_1, hence

$$(u_{k,l}+u_{k,l})e^i_k e^j_l=0 \quad (i,j=1,2,3;\ i\neq j) \tag{14}$$

in $C_1\times\,]t_0-\tau,t_0[$. One can deduce from this system and from condition ($*$) that

$$u_{i,j}+u_{j,i}=a^{i,j}_k\,u_{k,k}=a^{i,j}_1\,u_{1,1}+a^{i,j}_2\,u_{2,2}+a^{i,j}_3\,u_{3,3} \quad (i,j=1,2,3;\ i\neq j), \tag{15}$$

where

$$\sum_{k=1}^{3}\bigl(\bigl|a^{1,2}_k\bigr|+\bigl|a^{1,3}_k\bigr|+\bigl|a^{2,3}_k\bigr|\bigr)\leqslant\varepsilon(r_1,\tau) \tag{16}$$

in $C_1\times\,]t_0-\tau,t_0[$. The right-hand side of (16) satisfies: $\varepsilon(r_1,\tau)\to 0$ as $r_1\to 0+$ and $\tau\to 0+$. (See [4] for more details.) Differentiating the equation of continuity (6) with respect to x_1 and substituting for $u_{2,21}$ and $u_{3,31}$ from (15), we obtain the spatial wave equation

$$u_{1,11}-u_{1,22}-u_{1,33}=\bigl(a^{1,2}_k u_{k,k}\bigr)_{,2}+\bigl(a^{1,3}_k u_{k,k}\bigr)_{,3}. \tag{17}$$

A deeper and relatively technical analysis of this equation (and also similar equations for u_2 and u_3) again enables to show the finiteness of the norm $|||\mathbf{u}|||_{(\infty,2)\cap(2,6)}$ which leads, analogously as in Section 2, to the regularity of the solution $(\mathbf{v};\,p)$ at the point (x_0,t_0). Thus, we obtain:

Theorem 3.1. *Suppose that* D *is an open sub-domain of* Q_T, $(\mathbf{v};\,p)$ *is a suitable weak solution of the problem* (1)–(4), $\mathbf{e}^i=(e^i_1,e^i_2,e^i_3)$ $(i=1,2,3)$ *are the eigenvectors of the rate of deformation tensor* $\frac{1}{2}(v_{i,j}+v_{j,i})$ *in* D *such that* $\mathbf{e}^i\cdot\mathbf{e}^j=\delta_{ij}$ $(i,j=1,2,3)$ *for a.a.* $(x,t)\in D$ *and* $\mathbf{e}^i$ $(i=1,2,3)$ *satisfy condition* ($*$). *Then the solution* $(\mathbf{v};\,p)$ *is regular in* D.

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