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COMPTES RENDUS MATHEMATIQUE

Partial Differential Equations

Anisotropic regularity results for Laplace and Maxwell operators in a polyhedron

Régularité anisotrope pour le Laplacien et l'opérateur de Maxwell dans un polyèdre

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Abstract

As representatives of a larger class of elliptic boundary value problems of mathematical physics, we study the Dirichlet problem for the Laplace operator and the electric boundary problem for the Maxwell operator. We state regularity results in two families of weighted Sobolev spaces: A classical isotropic family, and a new anisotropic family, where the hypoellipticity along an edge of a polyhedral domain is taken into account. ***To cite this article: A. Buffa et al., C. R. Acad. Sci. Paris, Ser. I 336 (2003).***

Résumé

Nous choisissons d'étudier le problème de Dirichlet pour le Laplacien et le problème de Maxwell électrique, comme représentants de classes plus larges de problèmes intéressant la modélisation de phénomènes physiques stationnaires. Nous énonçons des résultats de régularité dans deux familles d'espaces de Sobolev à poids : l'une, classique, isotrope, et l'autre, nouvelle, anisotrope, où l'on tient compte de l'hypoellipticité le long des arêtes d'un domaine polyédral. ***Pour citer cet article : A. Buffa et al., C. R. Acad. Sci. Paris, Ser. I 336 (2003).***

Version française abrégée

Nous considérons dans cette Note le problème de Dirichlet pour le Laplacien et le problème de Maxwell électrique sur un polyèdre $\Omega \subset \mathbb{R}^3$. Les solutions de ces problèmes ont des singularités auprès des arêtes et coins du domaine. Notre but est de décrire le plus précisément possible la régularité des solutions lorsque le second membre est très régulier. Comme les conditions aux limites homogènes impliquent suffisamment d'annulation aux arêtes et coins, les espaces à poids $\mathrm{K}^m_\beta(\Omega)$ du type de ceux de Kondrat'ev semblent bien adaptés.

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Pourtant, du fait de l'hypoellipticité *le long des arêtes*, les solutions peuvent avoir un supplément de régularité quand on les dérive le long des arêtes. C'est pourquoi nous introduisons des espaces anisotropes à poids $\mathrm{M}^m_\beta(\Omega)$ et donnons des résultats de régularité dans le cadre de ces espaces. L'existence de tels résultats est le fondement des méthodes d'approximation par éléments finis avec maillages anisotropes.

Soit $\mathbf{e}$ et $\mathbf{c}$ les arêtes et coins du domaine Ω. Soit $\mathcal{V}^{\mathbf{c}}_{\mathbf{e}}$, $\mathcal{V}^0_{\mathbf{e}}$, $\mathcal{V}^0_{\mathbf{c}}$ et $\mathcal{V}^0$ des sous-régions de Ω soumises respectivement à l'influence

- conjointe de l'arête $\mathbf{e}$ et du coin $\mathbf{c}$ (applicable si $\mathbf{c}$ est une extrémité de $\mathbf{e}$),
- de la seule arête $\mathbf{e}$, • du seul coin $\mathbf{c}$, • d'aucun coin ni arête.

Avec $r_{\mathbf{e}}$ et $r_{\mathbf{c}}$ les fonctions distance à $\mathbf{e}$ et $\mathbf{c}$, et $\beta_{\mathbf{e}}$, $\beta_{\mathbf{c}}$ des exposants associés à $\mathbf{e}$ et $\mathbf{c}$, les poids dans les régions $\mathcal{V}^{\mathbf{c}}_{\mathbf{e}}$, $\mathcal{V}^0_{\mathbf{e}}$, $\mathcal{V}^0_{\mathbf{c}}$ et $\mathcal{V}^0$ prennent respectivement la forme

$$r_{\mathbf{c}}^{\beta_{\mathbf{c}}}\left(\frac{r_{\mathbf{e}}}{r_{\mathbf{c}}}\right)^{\beta_{\mathbf{e}}}, \quad r_{\mathbf{e}}^{\beta_{\mathbf{e}}}, \quad r_{\mathbf{c}}^{\beta_{\mathbf{c}}} \quad \text{et} \quad 1,$$

ce qui définit les premiers espaces $\mathrm{K}^0_\beta(\Omega)=\mathrm{M}^0_\beta(\Omega)$ des deux familles. La famille $\mathrm{K}^m_\beta(\Omega)$ est obtenue en décalant régulièrement le poids en fonction de l'ordre de dérivation : $r_{\mathbf{e}}^{\beta_{\mathbf{e}}+|\boldsymbol{\alpha}|}$, $r_{\mathbf{c}}^{\beta_{\mathbf{c}}+|\boldsymbol{\alpha}|}$, etc., pour la dérivée d'ordre $\boldsymbol{\alpha}$. Par contre, pour les espaces $\mathrm{M}^m_\beta(\Omega)$, le poids *ne dépend pas de l'ordre de dérivation de long de l'arête*. Ainsi, pour $\boldsymbol{\alpha}=(\boldsymbol{\alpha}_\perp,\alpha_3)$ avec α_3 indiquant la direction de $\mathbf{e}$, le poids $r_{\mathbf{e}}^{\beta_{\mathbf{e}}+|\boldsymbol{\alpha}|}$ est remplacé par $r_{\mathbf{e}}^{\beta_{\mathbf{e}}+|\boldsymbol{\alpha}_\perp|}$. Les définitions précises sont données en (3) et (5).

Laplacien

Soit pour $f\in\mathrm{H}^{-1}(\Omega)$, $u\in\mathrm{H}^1_0(\Omega)$ la solution du problème de Dirichlet $\Delta u=f$. Pour tout multi-exposant $\beta=\{\beta_{\mathbf{e}},\beta_{\mathbf{c}}\}$ avec $\beta_{\mathbf{e}}$, $\beta_{\mathbf{c}}\geqslant 0$, l'espace $\mathrm{K}^0_{1-\beta}(\Omega)$ est contenu dans $\mathrm{H}^{-1}(\Omega)$. Si β est assez petit pour satisfaire la condition (4), alors $f\in\mathrm{K}^0_{1-\beta}(\Omega)$ implique $u\in\mathrm{K}^2_{-1-\beta}(\Omega)$.

Le résultat général de régularité isotrope est que si $f\in\mathrm{K}^m_{1-\beta}(\Omega)$ alors $u\in\mathrm{K}^{m+2}_{-1-\beta}(\Omega)$ pour le même ensemble de multi-exposants β satisfaisant la condition (4). Quant au résultat de régularité anisotrope, il énonce que si $f\in\mathrm{M}^m_{1-\beta}(\Omega)$, alors $u\in\mathrm{M}^m_{-1-\beta}(\Omega)$, Théorème 3.3.

Maxwell

La difficulté spécifique du problème de Maxwell est que son espace variationnel naturel $\mathbf{X}_N$ qui est l'espace des champs $\mathbf{u}$ dans L^2, avec divergence et rotationnel dans L^2 et composante tangentielle 0 sur $\partial\Omega$, n'est pas contenu dans $\mathrm{H}^1(\Omega)^3$ dès que Ω est non-convexe. C'est pourquoi l'on doit déjà prendre quelques précautions avec le second membre de l'équation pour s'assurer qu'il soit dans $\mathbf{X}'_N$, voir (8).

D'autre part, tout résultat un peu précis sur la régularité de la solution $\mathbf{u}$ passe par une décomposition en potentiel singulier ∇q plus champ moins singulier $\mathbf{u}_0$, sur le même modèle que ce que l'on a déjà pour l'espace variationnel $\mathbf{X}_N$ lui-même [3]. Nous obtenons que si le second membre $\mathbf{f}\in\mathrm{M}^m_{1-\gamma}(\Omega)^3$ pour un multi-exposant γ, $0\leqslant\gamma\leqslant 1$, convenable, alors il existe un autre multi-exposant β, $0<\beta\leqslant\gamma$, tel que

$$\mathbf{u}=\mathbf{u}_0+\nabla q, \quad \text{avec } \mathbf{u}_0\in\mathrm{M}^m_{-1-\beta}(\Omega)^3 \text{ et } q\in\mathrm{M}^{m+1}_{-1-\beta}(\Omega)\cap\mathrm{H}^1_0(\Omega).$$

1. Introduction

Many stationary phenomena are modelled by elliptic boundary value problems, which in general possess extra coercivity properties. In this Note, we choose two of them. (i) The Dirichlet problem for the Laplace operator, whose analysis can serve as model for systems like linear elasticity with clamped boundary conditions. (ii) The Maxwell operator with electric boundary conditions (zero tangential component on the boundary), which has its own peculiarities and is interesting by itself.

The corners and edges of a three-dimensional domain Ω stop the regularity of solutions, even if the right-hand side is very smooth. The description of the optimal regularity can be done in the standard scale of Sobolev spaces [9], or in classical weighted spaces of Kondrat'ev type [11,13]. The results in such scales are, in some sense, not optimal when the right-hand side is very smooth: In this case, the underlying coerciveness of the operator yields extra regularity along the edges, and, for piecewise smooth domains containing only edges (but no corners) improved results can be obtained in anisotropic scales of Sobolev spaces [12,6].

We want to combine such improved regularity results with the presence of corners. For the Laplace operator, we obtain a kind of shift theorem. For the Maxwell operator, the main singularities having nonsquare-integrable gradients in general, an optimal result will be obtained via the splitting of solutions into the gradient of a very singular potential, and a less singular part. Such results are essential for the method of anisotropic finite elements.

2. Subregions and distance functions

Let Ω be a three-dimensional polyhedral domain with plane faces. We denote by $\mathcal{E}$ and $\mathcal{C}$ the set of edges and corners of Ω. Moreover for every $\mathbf{c} \in \mathcal{C}$, we denote by $\mathcal{E}_{\mathbf{c}}$ the set of edges $\mathbf{e}$ such that $\mathbf{c} \subset \bar{\mathbf{e}}$, and for every $\mathbf{c}$, by $\mathcal{C}_{\mathbf{e}}$ the set of the two corners which are the endpoints of $\mathbf{e}$.

For any $\mathbf{c} \in \mathcal{C}$ and $\mathbf{e} \in \mathcal{E}$, we define the distance functions on Ω:

$$r_{\mathbf{e}}(\mathbf{x}) = \operatorname{dist}(\mathbf{x}, \mathbf{e}), \quad r_{\mathbf{c}}(\mathbf{x}) = \operatorname{dist}(\mathbf{x}, \mathbf{c}), \quad \mathbf{x} \in \Omega. \tag{1}$$

We are going to identify subregions of the domain Ω governed by a corner $\mathbf{c}$, an edge $\mathbf{e}$, or both, cf. [10,8]. Let $\mathbf{c} \in \mathcal{C}$ and $B_\varepsilon(\mathbf{c})$ be a ball centered in $\mathbf{c}$ and with radius ε which is not intersecting any other corner of Ω. We denote by $G_{\mathbf{c}} \subset \mathbb{S}^2$ the spherical polygonal domain corresponding to $\varepsilon^{-1}(\partial B_\varepsilon(\mathbf{c}) \cap \Omega)$. There is a bijection between the vertices $\mathbf{x}_{\mathbf{e}}$ of $G_{\mathbf{c}}$ and the edges $\mathbf{e}$ in $\mathcal{E}_{\mathbf{c}}$. For every $\mathbf{e} \in \mathcal{E}_{\mathbf{c}}$, let $\mathcal{V}(\mathbf{x}_{\mathbf{e}})$ be a neighborhood of $\mathbf{x}_{\mathbf{e}}$ in $G_{\mathbf{c}}$ such that $\mathcal{V}(\mathbf{x}_{\mathbf{e}})$ does not contain any other vertex of $G_{\mathbf{c}}$.

We introduce spherical coordinates $(r_{\mathbf{c}}, \vartheta_{\mathbf{c}})$, $\vartheta \in \mathbb{S}^2$ associated with the corner $\mathbf{c}$. This allows to define:

$$\begin{aligned} \mathcal{V}^{\mathbf{c}}_{\mathbf{e}} &= \bigl\{(r_{\mathbf{c}}, \vartheta_{\mathbf{c}}),\ r_{\mathbf{c}} < \varepsilon,\ \vartheta_{\mathbf{c}} \in \mathcal{V}(\mathbf{x}_{\mathbf{e}})\bigr\}, \\ \mathcal{V}^{0}_{\mathbf{c}} &= \left\{(r_{\mathbf{c}}, \vartheta_{\mathbf{c}}),\ r_{\mathbf{c}} < \varepsilon,\ \vartheta_{\mathbf{c}} \in G_{\mathbf{c}} \Big\backslash \left(\bigcup_{\mathbf{e} \in \mathcal{E}_{\mathbf{c}}} \overline{\mathcal{V}}(\mathbf{x}_{\mathbf{e}}) \right) \right\}. \end{aligned} \tag{2}$$

Besides the neighborhoods $\mathcal{V}^{\mathbf{c}}_{\mathbf{e}}$ and $\mathcal{V}^{0}_{\mathbf{e}}$, we introduce $\mathcal{V}^{0}_{\mathbf{e}}$ such that $\overline{\mathcal{V}^{0}_{\mathbf{e}}}$ does not contain any other edge than $\mathbf{e}$, nor any corner and such that $\bar{\mathbf{e}}$ is contained in $\overline{\mathcal{V}^{0}_{\mathbf{e}}} \cup (\bigcup_{\mathbf{c} \in \mathcal{C}_{\mathbf{e}}} \overline{\mathcal{V}^{\mathbf{c}}_{\mathbf{e}}})$. And finally let $\mathcal{V}^0$ such that $\overline{\mathcal{V}^0}$ contains no edge and no corner and such that

$$\Omega = \mathcal{V}^0 \cup \bigcup_{\mathbf{e} \in \mathcal{E}} \mathcal{V}^{0}_{\mathbf{e}} \cup \bigcup_{\mathbf{c} \in \mathcal{C}} \left(\mathcal{V}^{0}_{\mathbf{c}} \cup \bigcup_{\mathbf{e} \in \mathcal{E}_{\mathbf{c}}} \mathcal{V}^{\mathbf{c}}_{\mathbf{e}} \right).$$

Finally, in $\mathcal{V}^{\mathbf{c}}_{\mathbf{e}}$ we also make use of the angular distance $\rho_{\mathbf{e},\mathbf{c}}$ defined as $\rho_{\mathbf{e},\mathbf{c}} = r_{\mathbf{e}}/r_{\mathbf{c}}$.

3. The Laplace operator

3.1. Isotropic spaces and isotropic regularity

For every $\mathbf{e} \in \mathcal{E}$ and $\mathbf{c} \in \mathcal{C}$, let $\beta_{\mathbf{e}}$ and $\beta_{\mathbf{c}}$ be real numbers. We denote by β the collection: $\{\beta_{\mathbf{e}}\}_{\mathbf{e} \in \mathcal{E}} \cup \{\beta_{\mathbf{c}}\}_{\mathbf{c} \in \mathcal{C}}$ and we call β *multi-exponent*. We say that $\beta < c$, for a constant c, when each component of β is smaller than c. For any fixed multi-exponent β, we introduce the scale of isotropic weighted Sobolev spaces $\mathrm{K}^0_\beta(\Omega) \supset \mathrm{K}^1_\beta(\Omega) \supset \cdots \supset \mathrm{K}^m_\beta(\Omega) \cdots$, cf. [11,13]:

$$\mathrm{K}^m_\beta(\Omega) := \big\{u \in \mathrm{L}^2_{\mathrm{loc}}(\Omega)\colon \forall|\boldsymbol{\alpha}| \leqslant m,\ \partial^{\boldsymbol{\alpha}} u \in \mathrm{L}^2\big(\mathcal{V}^0\big),\ \forall \mathbf{c} \in \mathcal{C},\ r_{\mathbf{c}}^{\beta_{\mathbf{c}}+|\boldsymbol{\alpha}|}\partial^{\boldsymbol{\alpha}} u \in \mathrm{L}^2\big(\mathcal{V}^0_{\mathbf{c}}\big),$$
$$\forall \mathbf{e} \in \mathcal{E},\ r_{\mathbf{e}}^{\beta_{\mathbf{e}}+|\boldsymbol{\alpha}|}\partial^{\boldsymbol{\alpha}} u \in \mathrm{L}^2\big(\mathcal{V}^0_{\mathbf{e}}\big) \text{ and } r_{\mathbf{c}}^{\beta_{\mathbf{c}}+|\boldsymbol{\alpha}|}\rho_{\mathbf{e},\mathbf{c}}^{\beta_{\mathbf{e}}+|\boldsymbol{\alpha}|}\partial^{\boldsymbol{\alpha}} u \in \mathrm{L}^2\big(\mathcal{V}^{\mathbf{c}}_{\mathbf{e}}\big)\big\}. \tag{3}$$

We call this space "isotropic" because the transverse and longitudinal derivatives along the edge are treated in the same way, in contrast with the "anisotropic" weighted Sobolev spaces introduced later. As in the case of cones, see [11], these spaces are well suited for the functional analysis of elliptic boundary value problems in polyhedra, see [13].

Concerning the Dirichlet problem for the Laplace operator, a precise statement needs the introduction, for each corner $\mathbf{c}$, of the smallest eigenvalue $\mu_{\mathbf{c}}$ of the Laplace–Beltrami operator with Dirichlet boundary condition on $G_{\mathbf{c}}$, defining the real number $\lambda_{\mathbf{c}} = -\frac{1}{2} + (\mu_{\mathbf{c}} + \frac{1}{4})^{1/2}$. We also need the opening angle $\omega_{\mathbf{e}}$ of the edge $\mathbf{e}$. For the following theorem we refer to [13,9,8].

Theorem 3.1. *The Laplace operator Δ is an isomorphism from $\mathrm{K}^2_{-1-\beta}(\Omega) \cap \mathrm{H}^1_0(\Omega)$ to $\mathrm{K}^0_{1-\beta}(\Omega)$ for every $\beta = (\beta_{\mathbf{e}}, \beta_{\mathbf{c}})$ verifying*:

$$\forall \mathbf{e} \in \mathcal{E}, \quad 0 \leqslant \beta_{\mathbf{e}} < \frac{\pi}{\omega_{\mathbf{e}}} \quad \text{and} \quad \forall \mathbf{c} \in \mathcal{C}, \quad 0 \leqslant \beta_{\mathbf{c}} < \lambda_{\mathbf{c}} + \frac{1}{2}. \tag{4}$$

Note that if $\beta \leqslant 1$, the space $\mathrm{K}^0_{1-\beta}$ of right-hand sides is contained in H^{-1} and contains L^2, whereas $\mathrm{K}^2_{-1-\beta}$ is contained in H^1. If Ω is not convex, some $\beta_{\mathbf{e}}$ has to be < 1 and then $\mathrm{K}^2_{-1-\beta} \not\subset \mathrm{H}^2$.

The Dirichlet boundary conditions are important here, because they allow the variational space H^1_0 to be a subspace of K^1_{-1}. In the case of Neumann boundary conditions, either the weight associated with $\boldsymbol{\alpha} = 0$ has to be relaxed, or the solution has to be split in a part belonging to a space K as above and another part taking into account the non-zero traces along the edges.

The following statement, still very classical, see [11,13], provides a general shift theorem for the Laplace operator (with homogeneous Dirichlet boundary condition) in isotropic weighted Sobolev spaces:

Theorem 3.2. *Let β be a multi-exponent verifying* (4)*, then for any $m \in \mathbb{N}$*

$$\Delta : \mathrm{K}^{m+2}_{-1-\beta}(\Omega) \cap \mathrm{H}^1_0(\Omega) \to \mathrm{K}^m_{1-\beta}(\Omega) \quad \textit{is an isomorphism}.$$

The proof of this theorem relies on Theorem 3.1 combined with local elliptic estimates on a "dyadic" covering of Ω by countable families of similar domains, adapted to edges or corners.

3.2. Anisotropic spaces and anisotropic regularity

Although this result is optimal in a certain sense, it does not describe the extra regularity which can be obtained by differentiating *along the edges* (see [12,6]). Moreover, some important applications, like the analysis of approximation by anisotropic finite elements [1,5,10], take advantage of this improved behaviour of the solution. More precisely, in $\mathcal{V}^0_{\mathbf{e}}$ and $\mathcal{V}^{\mathbf{c}}_{\mathbf{e}}$ for a fixed $\mathbf{e} \in \mathcal{E}$, we want sharper statements about the differentiability in the direction of the edge. For that, we introduce anisotropic spaces.

To this aim, in the regions $\mathcal{V}^0_{\mathbf{e}}$ and $\mathcal{V}^{\mathbf{c}}_{\mathbf{e}}$ we choose a local system of coordinates in which the direction of the edge is the third one $(0,0,1)$. The subindex $\perp$ will always denote the directions transverse to the edge: for example let $\boldsymbol{\alpha}$ be a derivation multi-index, then $\boldsymbol{\alpha} = (\boldsymbol{\alpha}_\perp, \alpha_3)$ means $\boldsymbol{\alpha}_\perp$ derivatives in the transverse direction and α_3 in the longitudinal one. We then define the scale of anisotropic weighted Sobolev spaces $\mathrm{M}^0_\beta(\Omega) \supset \mathrm{M}^1_\beta(\Omega) \supset \cdots \supset \mathrm{M}^m_\beta(\Omega) \cdots$

$$\mathrm{M}^m_\beta(\Omega) := \big\{u \in \mathrm{L}^2(\Omega)\colon \forall|\boldsymbol{\alpha}| \leqslant m,\ \partial^{\boldsymbol{\alpha}} u \in \mathrm{L}^2\big(\mathcal{V}^0\big),\ \forall \mathbf{c} \in \mathcal{C},\ r_{\mathbf{c}}^{\beta_{\mathbf{c}}+|\boldsymbol{\alpha}|}\partial^{\boldsymbol{\alpha}} u \in \mathrm{L}^2\big(\mathcal{V}^0_{\mathbf{c}}\big),$$
$$\forall \mathbf{e} \in \mathcal{E},\ r_{\mathbf{e}}^{\beta_{\mathbf{e}}+|\boldsymbol{\alpha}_\perp|}\partial^{\boldsymbol{\alpha}} u \in \mathrm{L}^2\big(\mathcal{V}^0_{\mathbf{e}}\big) \text{ and } r_{\mathbf{c}}^{|\boldsymbol{\alpha}|+\beta_{\mathbf{c}}}\rho_{\mathbf{e},\mathbf{c}}^{\beta_{\mathbf{e}}+|\boldsymbol{\alpha}_\perp|}\partial^{\boldsymbol{\alpha}} u \in \mathrm{L}^2\big(\mathcal{V}^{\mathbf{c}}_{\mathbf{e}}\big)\big\}. \tag{5}$$

Note te presence of $|\boldsymbol{\alpha}_\perp|$ instead of $|\boldsymbol{\alpha}|$ in the last two conditions. This means that the weight is independent of the number of derivatives in the longitudinal direction. Our main result concerning the Laplacian is:

Theorem 3.3. *Let β be a multi-exponent satisfying* (4). *For every $m \in \mathbb{N}$, if $u \in \mathrm{H}_0^1(\Omega)$ and $\Delta u \in \mathrm{M}_{1-\beta}^m(\Omega)$, then $u \in \mathrm{M}_{-1-\beta}^m(\Omega)$ with the estimate $\|u\|_{\mathrm{M}_{-1-\beta}^m(\Omega)} \leqslant C\|\Delta u\|_{\mathrm{M}_{1-\beta}^m(\Omega)}$.*

The proof uses a local estimate along the edge based on the coercivity of the operator, combined with the technique of differential quotients.

Note that if $\beta \leqslant 1$, any right-hand side in H^m also belongs to $\mathrm{M}_{1-\beta}^m$. Thus, if the right-hand side is smooth up to the boundary, the statement yields that the solution belongs to $\mathrm{M}_{-1-\beta}^m$ for any $m \in \mathbb{N}$.

4. The Maxwell operator

The Maxwell operator acts on 3-component fields. We denote by bold letters the spaces of complex vector fields, like $\mathbf{L}^2(\Omega)$, $\mathbf{H}^1(\Omega)$, $\mathbf{K}_\beta^m(\Omega)$, $\mathbf{M}_\beta^m(\Omega)$ for $\mathrm{L}^2(\Omega)^3$, $\mathrm{H}^1(\Omega)^3$, $\mathrm{K}_\beta^m(\Omega)^3$, $\mathrm{M}_\beta^m(\Omega)^3$ respectively. The space associated with Maxwell electric boundary conditions is:

$$\mathbf{X}_N = \{\mathbf{u} \in \mathbf{L}^2(\Omega):\ \operatorname{div}\mathbf{u} \in \mathrm{L}^2(\Omega),\ \mathbf{curl}\,\mathbf{u} \in \mathbf{L}^2(\Omega),\ \mathbf{u}\times\mathbf{n} = 0 \text{ on } \partial\Omega\}.$$

We call Maxwell operator the operator A associated with the bilinear form

$$a(\mathbf{u},\mathbf{v}) = \int_\Omega (\mathbf{curl}\,\mathbf{u}\cdot\mathbf{curl}\,\mathbf{v} + \operatorname{div}\mathbf{u}\,\operatorname{div}\mathbf{v})\,\mathrm{d}\mathbf{x}, \quad \mathbf{u},\mathbf{v} \in \mathbf{X}_N.$$

The coercivity of a gives that A induces an isomorphism between $\mathbf{X}_N$ and $\mathbf{X}_N'$ and that the equation $A\mathbf{u} = \mathbf{f}$, with $\mathbf{f} \in \mathbf{X}_N'$, corresponds to the boundary value problem (see, e.g., [4]):

$$-\boldsymbol{\Delta}\mathbf{u} = \mathbf{f} \quad \text{on } \Omega \quad \text{and} \quad \mathbf{u}\times\mathbf{n} = 0, \quad \operatorname{div}\mathbf{u} = 0 \quad \text{on } \partial\Omega. \tag{6}$$

The difficulty with this problem is the lack of regularity of the variational space $\mathbf{X}_N$: if Ω is not convex, $\mathbf{X}_N$ can be decomposed as the sum of its subspace $\mathbf{X}_N \cap \mathbf{H}^1(\Omega)$ and of the space $\nabla(D^{\mathrm{Dir}}(\Delta))$ where $D^{\mathrm{Dir}}(\Delta)$ is the domain of the Dirichlet problem for the Laplacian $\{q \in \mathrm{H}_0^1(\Omega):\ \Delta q \in \mathrm{L}^2(\Omega)\}$, see [3,2].

The boundary conditions allow to prove that $\mathbf{X}_N \cap \mathbf{H}^1(\Omega)$ is contained in $\mathbf{K}_{-1}^1(\Omega)$. From Theorem 3.1, we see that $D^{\mathrm{Dir}}(\Delta)$ is contained in $\mathrm{K}_{-1-\beta}^2(\Omega)$ for all $\beta \leqslant 1$ satisfying (4), therefore we have

$$\forall\mathbf{u} \in \mathbf{X}_N, \quad \mathbf{u} = \mathbf{u}_0 + \nabla q, \quad \mathbf{u}_0 \in \mathbf{K}_{-1}^1(\Omega),\ q \in \mathrm{K}_{-1-\beta}^2(\Omega)\cap\mathrm{H}_0^1(\Omega)\ \forall\beta \leqslant 1 \text{ s.t. (4)}. \tag{7}$$

As a consequence, $\nabla(D^{\mathrm{Dir}}(\Delta))$ and also $\mathbf{X}_N$, is a subset of $\mathbf{K}_{-\beta}^1(\Omega)$ for all $\beta \leqslant 1$ satisfying (4). Thus for such β we have the embedding $\mathbf{K}_\beta^0(\Omega) \subset \mathbf{X}_N'$. Introducing another multi-exponent $\gamma = \{\gamma_{\mathbf{e}}, \gamma_{\mathbf{c}}\}$, we have obtained that $\mathbf{K}_{1-\gamma}^0(\Omega) = \mathbf{M}_{1-\gamma}^0(\Omega)$ is contained in $\mathbf{X}_N'$ if $\gamma \geqslant 0$ and

$$\forall\mathbf{e} \in \mathcal{E}, \quad 1-\gamma_{\mathbf{e}} < \frac{\pi}{\omega_{\mathbf{e}}} \quad \text{and} \quad \forall\mathbf{c} \in \mathcal{C}, \quad 1-\gamma_{\mathbf{c}} < \lambda_{\mathbf{c}} + \frac{1}{2}. \tag{8}$$

The main result of our Note can be expressed as the weighted anisotropic elliptic regularity of the decomposition (7):

Theorem 4.1. *Let γ be a multi-exponent satisfying*: $0 < \gamma \leqslant 1$ *and* (8). *Let $\mathbf{u} \in \mathbf{X}_N$ be the solution of problem* (6) *with right-hand side $\mathbf{f} \in \mathbf{M}_{1-\gamma}^m(\Omega)$. Then, there exists a multi-exponent $\beta = \beta(\gamma,\Omega)$, $0 < \beta \leqslant \gamma$, such that*

$$\mathbf{u} = \mathbf{u}_0 + \nabla q, \quad \mathbf{u}_0 \in \mathbf{M}_{-1-\beta}^m(\Omega),\ q \in \mathrm{M}_{-1-\beta}^{m+1}(\Omega)\cap\mathrm{H}_0^1(\Omega), \tag{9}$$

Moreover, we have the estimate $\|\mathbf{u}_0\|_{\mathbf{M}^m_{-1-\beta}(\Omega)} + \|q\|_{\mathrm{M}^{m+1}_{-1-\beta}(\Omega)} \leqslant C\|\mathbf{f}\|_{\mathbf{M}^m_{1-\gamma}(\Omega)}$.

The proof of this theorem is more involved than that of Theorem 3.3, because the arguments of scaling of local elliptic estimates and of differential quotients have to be combined with a full edge-corner decomposition of the solution in regular and singular parts, along the lines of [7]: roughly, the solution $\mathbf{u}$ can be split into

$$\mathbf{u} = \mathbf{u}_{\mathrm{reg}} + \sum_{\mathbf{c}\in\mathcal{C}} \mathbf{u}_{\mathbf{c}} + \sum_{\mathbf{e}\in\mathcal{E}} \mathbf{u}_{\mathbf{e}}, \quad \mathbf{u}_{\mathrm{reg}} \in \mathbf{M}^m_{-1-\gamma}(\Omega). \tag{10}$$

Each term $\mathbf{u}_{\mathbf{c}}$ and $\mathbf{u}_{\mathbf{e}}$ is itself made of several contributions, the most singular ones are of "Type 1" in the terminology of [7], and consequently can be written as gradients. The gradients are gathered in one part to make q, whereas the remainder $\mathbf{u}_{\mathrm{reg}}$ plus less singular parts in $\mathbf{u}_{\mathbf{c}}$ and $\mathbf{u}_{\mathbf{e}}$ are gathered in $\mathbf{u}_0$.

As a corollary of Theorem 4.1, we find a global anisotropic weighted regularity for $\mathbf{u}$ and its **curl**. In fact, examining the structure of the singular gradient part ∇q we find that $\mathbf{u}$ has locally a more regular component, that is, its longitudinal component along the edge. This is the reason for the introduction of the new space

$$\widetilde{\mathbf{M}}^m_\beta(\Omega) := \big\{\mathbf{u} \in \mathbf{M}^m_\beta(\Omega)\colon \forall \mathbf{e} \in \mathcal{E} \quad \text{and with } u_{\mathbf{e},3} \text{ the component of } \mathbf{u} \text{ along } \mathbf{e}, \\ u_{\mathbf{e},3} \in \mathrm{M}^m_{\beta_{\mathbf{e}}-1,\beta_{\mathbf{c}}}\big(\mathcal{V}^0_{\mathbf{e}}\big) \text{ and } u_{\mathbf{e},3} \in \mathrm{M}^m_{\beta_{\mathbf{e}}-1,\beta_{\mathbf{c}}}\big(\mathcal{V}^{\mathbf{c}}_{\mathbf{e}}\big)\big\} \tag{11}$$

whereas any component u_j of $\mathbf{u} \in \mathbf{M}^{m,p}_\beta(\Omega)$ belongs to $\mathrm{M}^m_{\beta_{\mathbf{e}},\beta_{\mathbf{c}}}(\mathcal{V}^0_{\mathbf{e}})$ and $\mathrm{M}^m_{\beta_{\mathbf{e}},\beta_{\mathbf{c}}}(\mathcal{V}^{\mathbf{c}}_{\mathbf{e}})$ only.

Corollary 4.2. *Under the assumptions of Theorem* 4.1, $\mathbf{curl}\,\mathbf{u} \in \mathbf{M}^{m-1}_{-\beta}(\Omega)$ *and* $\mathbf{u} \in \widetilde{\mathbf{M}}^m_{-\beta}(\Omega)$.

Proof. We use the decomposition (9). We have $\mathbf{curl}\,\mathbf{u} = \mathbf{curl}\,\mathbf{u}_0$ and since any partial derivative of order 1 is continuous from $\mathrm{M}^m_{-1-\beta}(\Omega)$ into $\mathrm{M}^{m-1}_{-\beta}(\Omega)$, we obtain the regularity of the **curl**. Concerning $\mathbf{u}$ itself, we look at the two terms in (9) separately. For the gradient part, it is enough to note that $\nabla : \mathrm{M}^{m+1}_{-1-\beta}(\Omega) \to \widetilde{\mathbf{M}}^m_{-\beta}(\Omega)$ is continuous. On the other hand, $\mathbf{M}^m_{-1-\beta}(\Omega) \subset \widetilde{\mathbf{M}}^m_{-\beta}(\Omega)$. Whence $\mathbf{u} \in \widetilde{\mathbf{M}}^m_{-\beta}(\Omega)$. □

References

[1] T. Apel, S. Nicaise, The finite element method with anisotropic mesh grading for elliptic problems in domains with corners and edges, Math. Methods Appl. Sci. 21 (1998) 519–549.
[2] F. Assous, P. Ciarlet Jr., P.-A. Raviart, E. Sonnendrücker, Characterization of the singular part of the solution of Maxwell's equations in a polyhedral domain, Math. Methods Appl. Sci. 22 (1999) 485–499.
[3] M. Birman, M. Solomyak, L^2-theory of the Maxwell operator in arbitrary domains, Russian Math. Surveys 42 (6) (1987) 75–96.
[4] A.-S. Bonnet-Ben Dhia, C. Hazard, S. Lohrengel, A singular field method for the solution of Maxwell's equations in polyhedral domains, SIAM J. Appl. Math. 59 (2000) 2028–2044.
[5] A. Buffa, M. Costabel, and M. Dauge, Algebraic convergence for anisotropic edge elements in polyhedral domains, Technical report, IMATI-CNR, 2003, in preparation.
[6] M. Costabel, M. Dauge, General edge asymptotics of solutions of second order elliptic boundary value problems I & II, Proc. Roy. Soc. Edinburgh Sect. A 123 (1993) 109–184.
[7] M. Costabel, M. Dauge, Singularities of electromagnetic fields in polyhedral domains, Arch. Rational Mech. Anal. 151 (2000) 221–276.
[8] M. Costabel, M. Dauge, Weighted regularization of Maxwell equations in polyhedral domains, Numer. Math. 93 (2002) 239–277.
[9] M. Dauge, Elliptic Boundary Value Problems on Corner Domains, in: Lecture Notes in Math., Springer-Verlag, Berlin, 1988.
[10] B.Q. Guo, The h-p version of the finite element method for solving boundary value problems in polyhedral domains, in: M. Costabel, M. Dauge, S. Nicaise (Eds.), Boundary Value Problems and Integral Equations in Nonsmooth Domains, Luminy, 1993, Dekker, New York, 1995, pp. 101–120.
[11] V.A. Kondrat'ev, Boundary-value problems for elliptic equations in domains with conical or angular points, Trans. Moscow Math. Soc. 16 (1967) 227–313.
[12] V.G. Maz'ya, J. Rossmann, Über die Asymptotik der Lösungen elliptischer Randwertaufgaben in der Umgebung von Kanten, Math. Nachr. 138 (1988) 27–53.
[13] S.A. Nazarov, B.A. Plamenevsky, Elliptic Problems in Domains with Piecewise Smooth Boundaries, in: de Gruyter Exp. Math., Vol. 13, de Gruyter, Berlin, 1994.