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COMPTES RENDUS MATHEMATIQUE

Ordinary Differential Equations

Analytical classification of saddle-node vector fields

Classification analytique des champs de vecteurs nœud-cols

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Abstract

We describe the classification of germs of holomorphic saddle-node vector fields Z at $(0,0)\in\mathbb{C}^2$, up to analytical change of local coordinates. The classification of saddle-node foliations $\mathcal{F}_Z$, obtained by Martinet and Ramis, led to functional moduli. We reduce the conjugacy equation between two vector fields inducing the same foliation to a homological equation. We derive then a complete set of additional functional invariants for vector fields. ***To cite this article: L. Teyssier, C. R. Acad. Sci. Paris, Ser. I 336 (2003).***

Résumé

Nous décrivons la classification des germes de champs de vecteurs holomorphes de type nœud-col en $(0,0)\in\mathbb{C}^2$ à changement analytique de coordonnées locales près. La classification des feuilletages nœud-cols, obtenue par Martinet et Ramis, est décrite par des modules fonctionnels. Nous réduisons la conjugaison de deux champs induisant le même feuilletage à une équation homologique. Nous en déduisons un ensemble d'invariants fonctionnels qui complète les modules de Martinet–Ramis. ***Pour citer cet article : L. Teyssier, C. R. Acad. Sci. Paris, Ser. I 336 (2003).***

Version française abrégée

Nous considérons des germes de champs de vecteurs holomorphes en $(0,0)\in\mathbb{C}^2$ de type noeud-col, c'est-à-dire qu'ils peuvent s'écrire dans un système de coordonnées linéaires adéquat $Z(x,y)=\lambda y\frac{\partial}{\partial y}+\cdots$ avec $\lambda\neq 0$ (les « $\cdots$ » sont des termes d'ordre supérieur à 1). Nous supposons de plus que $(0,0)$ est un zéro isolé de Z. Deux champs de vecteurs seront dit analytiquement (resp. formellement) conjugués s'il existe un germe de difféomorphisme (resp. un changement de coordonnées formel) qui transforme un champ en le second.

Dans [2,11] ou [13] on montre que Z est formellement conjugué à $Z_0=P(x)X_0$ où P est un polynôme de degré au plus k satisfaisant $P(0)=\lambda$ et où X_0 est la forme normale formelle de Poincaré–Dulac [4] du feuilletage

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associée, donnée par $X_0 = x^{k+1}\frac{\partial}{\partial x} + y(1+\mu x^k)\frac{\partial}{\partial y}$ pour $k \in \mathbb{N}^*$ et $\mu \in \mathbb{C}$. Cette écriture est unique modulo l'action des rotations $x \mapsto \alpha x$, $\alpha^k = 1$.

Pour décrire l'espace des modules de la classification analytique des champs de vecteurs, nous introduisons l'espace fonctionnel $\mathcal{D}$ dont les éléments $(\varphi^n, \varphi^s, f)$ sont des triplets de transformations affines $\varphi^n(h) = h + b$, $b \in \mathbb{C}$, de séries convergentes $\varphi^s(h) = \sum_{n>0} a_n h^n$ et $f(h) = \sum_{n>0} f_n h^n$, avec $(a_n, f_n)_{n\in\mathbb{N}} \subset \mathbb{C}\times\mathbb{C}$ et $a_1 = \exp(2i\pi\mu/k)$. Dans [14] on associe à tout champ de vecteurs Z, formellement conjugué à Z_0, un $3k$-uple $m(Z) = (\varphi_j^n, \varphi_j^s, f_j)_{j\in\mathbb{Z}/k} \in \mathcal{D}^k$ déterminant la classe analytique de Z.

Théorème [14]. *Soient Z_1 et Z_2 deux champs de vecteurs formellement conjugués à Z_0 avec $m(Z_1) = (\varphi_j^n, \varphi_j^s, f_j)_j$ et $m(Z_2) = (\psi_j^n, \psi_j^s, g_j)_j$. Ces champs sont analytiquement conjugués si et seulement si il existe $c \in \mathbb{C}^*$ et $\theta \in \mathbb{Z}/k$ tels que $P(e^{2i\pi\theta/k}x) = P(x)$ et pour tout $j \in \mathbb{Z}/k$, $\varphi_{j+\theta}(ch) = c\psi_j(h)$ et $f_{j+\theta}(ch) = g_j(h)$. De plus l'application m est surjective et pour toute déformation analytique $\varepsilon \mapsto Z_\varepsilon$ de dimensoin finie, l'application $\varepsilon \mapsto m(Z_\varepsilon)$ est analytique.*

Cet invariant se scinde naturellement en l'invariant de Martinet–Ramis $(\varphi_j^n, \varphi_j^s)_j$, décrivant la classe analytique du feuilletage sous-jacents, ainsi qu'en une partie tangentielle $(f_j)_j$ classifiant le « temps » U.

Peu après avoir achevé ce travail j'ai appris que Yu.I. Meshcheryakova et V.M. Voronin avaient récemment annoncé un résultat similaire pour le cas $k = 1$ dans [12]. Leur démarche géométrique pour réaliser les invariants s'appuie sur le théorème de Newlander–Nirenberg, après avoir recollé des copies sectorielles du modèle formel Z_0. C'est aussi le point de vue choisi dans [6,7] pour les selles résonnantes. Notre approche se base sur le théorème de Ramis–Sibuya résultant en des constructions plus explicites. Nous montrons que conjuguer Z à UZ est équivalent à résoudre $Z \cdot F = 1/U - 1$, où Z agit par dérivation. Dans le cas des selles résonnantes l'équation $Z \cdot F = G$ a été étudiée dans [1] et [5], ce dernier étudiant aussi l'équation homologique associée à la forme normale X_0. Dans [13] nous considérons le cas général d'un champ de vecteurs noeud-col ; cette approche géométrique, semblable à [1], nous permet ici de dégager une représentation intégrale de l'invariant tangentiel $(f_j)_j$ qui, lorsqu'elle est explicitement calculable, nous fourni des formes normales.

1. Introduction

We consider germs of holomorphic vector fields at $(0,0) \in \mathbb{C}^2$. We say that $Z(x,y) = A(x,y)\frac{\partial}{\partial x} + B(x,y)\frac{\partial}{\partial y}$ has a singularity of saddle-node type at $(0,0)$ if it can be written in a suitable linear chart $Z(x,y) = \lambda y\frac{\partial}{\partial y} + \cdots$ with $\lambda \neq 0$ (the "$\cdots$" stand for higher order terms). In addition we assume that the singularity of Z at $(0,0)$ is isolated. Two of these germs are analytically (resp. formally) conjugated when there exists a germ of holomorphic (resp. a formal) change of coordinates ψ sending one onto the other. We will denote by $\psi^* Z$ the pullback by ψ of Z, i.e., $\psi^* Z = D\psi^{-1}(Z \circ \psi)$.

The germ of singular foliation induced by such a vector field Z will be denoted $\mathcal{F}_Z$. The leaves of the foliation are the integral curves of Z so that another vector field X induces the same foliation as Z if, and only if, there exists a non-vanishing germ of holomorphic function U such that $Z = UX$. More generally we will say that the foliations $\mathcal{F}_Z$ and $\mathcal{F}_X$ are analytically (resp. formally) conjugated when there exists a non-vanishing holomorphic function U (resp. formal power series) such that Z and UX are conjugated. In that case the vector fields Z and X will be said orbitally equivalent.

We wish to describe the classification of saddle-node vector fields with respect to changes of coordinates. This problem has been solved for foliations leading to important works (Poincaré and Dulac for the formal point of view, Martinet and Ramis [10] for the analytical one). In the case of resonant saddles, Grintchy and Voronin completed the classification for vector fields [7]. We want here to introduce another geometrical technique to tackle this classification problem.

Dulac [4] showed that Z is analytically orbitally equivalent to a pre-normal form (i.e., non-unique)

$$X = x^{k+1}\frac{\partial}{\partial x} + \big(y\big(1+\mu x^k\big) + x^{k+1}R(x,y)\big)\frac{\partial}{\partial y},$$

where $k \in \mathbb{N}$, $\mu \in \mathbb{C}$ and R is a germ of holomorphic function at $(0,0)$. Notice that, in these coordinates, $\{x=0\}$ is an invariant curve of $\mathcal{F}_Z$ and the foliation is transverse to all other vertical lines $\{x = cte\}$. In general there does not exist another analytic invariant curve passing through $(0,0)$.

In the sequel we will consider two types of change of coordinates: the fibered ones $\psi_N(x,y) = (x, f(x,y))$ and the tangential ones $\psi_T = \Phi_X^F$. The latter is obtained by replacing in the germ of a flow $\Phi_X^t(x,y)$, which is a convergent power series near $(x,y,t) = (0,0,0)$, the entry t with a power series or a function F depending on (x,y). Basically ψ_N modifies the foliation $\mathcal{F}_X$ but does not change the time: if two vector fields X_1 and X_2, written as above, are orbitally equivalent by ψ_N then ψ_N is a conjugacy. On the contrary ψ_T preserves $\mathcal{F}_X$ but modifies the time in the following way: $\psi_T^* X = (1/(1 + X \cdot F))X$ (here $X \cdot F$ denotes the derivative of F with respect to X). As a consequence we deduce that $U_1 X$ and $U_2 X$ are (formally, analytically) conjugated if, and only if, the following homological equation admits a (formal, analytical) solution:

$$X \cdot F = \frac{1}{U_1} - \frac{1}{U_2}. \tag{1}$$

This approach is similar to the work of Christopher, Mardešić and Rousseau for irrational saddles in [3]. They locate the obstruction for a vector field, which is orbitally linearizable, to be analytically linearizable in the obstruction to solve such an equation. Homological equations have also been studied by Elizarov [5] in the case of X_0, and by Berthier and Loray [1] in the case of resonant saddles.

From now on we fix a pair (k,μ) and will only consider vector fields $Z = UX$ written in Dulac's form as above.

2. Formal normal forms

To begin with, X is formally conjugated to the following Poincaré–Dulac's normal form [4]

$$X_0 = x^{k+1}\frac{\partial}{\partial x} + y\left(1 + \mu x^k\right)\frac{\partial}{\partial y}$$

by a fibered change of coordinates. Two such vector fields cannot be orbitally equivalent for different values of (k,μ).

Next, a formal computation ensures that the equation $X \cdot F = G$ possesses a formal solution if, and only if, the second term satisfies $G(x,0) = \mathrm{o}(x^k)$. It is then possible to deduce formal normal forms by writing Taylor's expansion $U(x,0) = P(x) + \mathrm{o}(x^k)$ and using Eq. (1).

Lemma 1 [2,11,13]. *The vector field $Z = UX$ is formally conjugated to $Z_0 = P(x)X_0$. Two of these vector fields PX_0 and QX_0 are conjugated if, and only if, $P(\alpha x) = Q(x)$ for some complex number α satisfying $\alpha^k = 1$.*

Notice that the formal change of coordinates can be written $\hat{\psi}_N \circ \hat{\psi}_T$ where $\hat{\psi}_N$ is Poincaré–Dulac's fibered conjugacy between PX_0 and PX, and $\hat{\psi}_T$ is a tangential conjugacy between PX and UX. We wish to describe how the previous formal approach can be applied to analytical classification.

3. Sectorial structure of the foliation

Definition. Take $\theta \in [0, 2\pi[$, $0 < \beta \leqslant \pi$ and $r > 0$. We define the fibered open sector $V(r,\theta,\beta) = \{x : |\arg x - \theta| < \beta \text{ and } 0 < |x| < r\} \times \{y\colon |y| < r\}$. We will say that a biholomorphism $\psi : V \to W$ is a sectorial diffeomorphism over $V(r,\theta,\beta)$ if (1) For all $\varepsilon > 0$ small enough there exists $r' > 0$ such that $V(r',\theta,\beta-\varepsilon) \subset V$ and $V(r',\theta,\beta-\varepsilon) \subset W$. (2) There exists $r'' > 0$ such that ψ extends continuously along $\{0\} \times \{|y| < r''\}$ to some diffeomorphism.

We wish now to present the sectorial structure of the foliation induced by X. It is sectorially equivalent to its formal normal form X_0 over each one of the $2k$ fibered open sectors defined for $j \in \mathbb{Z}/k$ and some $r > 0$ by

$$V_j^{ns} = V\left(r, (4j+1)\frac{\pi}{2k}, \frac{\pi}{k}\right) \quad \text{and} \quad V_j^{sn} = V\left(r, (4j-1)\frac{\pi}{2k}, \frac{\pi}{k}\right)$$

as is recalled in the following lemma:

Lemma (Hukuhara, Kimura and Matuda [8]). *On each V_j^{sn} (resp. V_j^{ns}) there exists a sectorial diffeomorphism ψ_N conjugating X_0 to X, that is $\psi_N^* X_0 = X$.*

The leaves of the foliation induced by X_0 are parametrized by a fixed determination of

$$y = hx^\mu \mathrm{e}^{-x^{-k}/k} \tag{2}$$

for $h \in \mathbb{C}$. In the saddle parts $V_j^s = V_j^{ns} \cap V_{j+1}^{sn}$, corresponding to $\mathrm{Re}(x^k) < 0$, the modulus of $y(x)$ tends to infinity as x goes radially to 0, except when $h = 0$. On the contrary, in the node parts $V_j^n = V_j^{sn} \cap V_j^{ns}$, for all value of $h \in \mathbb{C}$ the leaf tends radially to $(0,0)$. The space of leaves Ω_j^s of Z_0 over V_j^s, consisting of those h such that the leaf given by (2) intersects V_j^s, is a bounded simply-connected domain of $\mathbb{C}$ containing 0. On the contrary over V_j^n the space of leaves Ω_j^n is the whole line (so that $\Omega_j^{sn} = \Omega_j^{ns} = \mathbb{C}$ over V_j^{sn} and V_j^{ns} as well). The same description holds for the foliation $\mathcal{F}_Z$ and we will use the notations Ω_j in this case as well.

4. Homological equation

For $G \in \mathbb{C}\{x, y\}$ consider the equation

$$Z \cdot F = G. \tag{3}$$

In restriction to a leaf of $\mathcal{F}_Z$ the vector field Z can be sectorially straighten to $\frac{\mathrm{d}}{\mathrm{d}t}$ so that the above equation becomes $f' = g$: we obtain solutions by integrating $g(t)\,\mathrm{d}t$. The 1-form corresponding to $\mathrm{d}t$ can be chosen as the time form $\tau_Z = x^{-k-1} \frac{\mathrm{d}x}{U(x,y)}$. By integrating $G\tau_Z$ along a path included in a leaf of $\mathcal{F}_Z$ starting from $(0,0)$, we will obtain sectorial solutions F_j on each sector $V_j = V_j^{sn} \cup V_j^{ns}$. The integrating paths are given by:

Lemma 2 [13]. *For $r > 0$ small enough and for each $p \in V_j$ there exists a path tangent to Z*

$$\gamma_j(p) : [0,1] \to V_j \cup \big\{(0,0)\big\}, \quad \gamma_j(0) = (0,0) \quad \text{and} \quad \gamma_j(1) = p.$$

This path is unique up to tangential homotopy with fixed extremities in $V_j \cup \{(0,0)\}$.

Given $p \in V_j^s$, we denote by $\gamma_j^\infty(p)$ the concatenation of $\gamma_j(p)$ and $\gamma_{j+1}(p)$ with the negative orientation: $\gamma_j^\infty(p) : [0,1] \to V_j \cup V_{j+1} \cup \{(0,0)\}$ is an "asymptotic cycle" tangent to Z with basepoint $(0,0)$. We can now state the result regarding the homological equation.

Theorem 1 [13]. (1) *Given $p \in V_j$, the integral $F_j(p) = \int_{\gamma_j(p)} G\tau_Z$ is convergent if, and only if, $G(x,0) = \mathrm{o}(x^k)$. If this is the case it is the unique bounded holomorphic solution to* (3) *on V_j up to addition by a constant.*

(2) *The formal solution F with $F(0,0) = 0$ converges if, and only if, the integrals $\int_{\gamma_j^\infty} G\tau_Z$ are zero for all asymptotic cycles γ_j^∞. The sum of F is obtained by glueing the different F_j.*

Definition. Consider $p \in V_j^s$ and the asymptotic cycle $\gamma(p) = \gamma_j^\infty(p)$ given by Lemma 2. The value of $T(p) = \int_{\gamma(p)} G\tau_Z$ only depends on the tangential homotopical class of $\gamma(p)$ with fixed extremity. Thus $p \mapsto T(p)$ is constant on each of the sectorial leaves of the foliation. This defines a holomorphic function on the space of leaves over V_j^s which we call the period of G with respect to Z and which we write $\mathcal{T}_Z^j(G) : h \in \Omega_j^s \to T(p) \in \mathbb{C}$.

5. Sectorial normalization

Lemma 3 [14]. *Let G be a holomorphic function and consider the sectorial solution $F = F_j$ to the homological equation* (3) *as in Theorem* 1. *Then the map $\psi_T(x,y) = \Phi_Z^{F(x,y)}(x,y)$ is a sectorial diffeomorphism over V_j which satisfies $\psi_T^* Z = (1/(1 + Z \cdot F))Z$.*

Applying this lemma to $G = 1/U - 1/P$ the diffeomorphism ψ_T conjugates PX to UX. We compose it with Hukuhara–Kimura–Matuda's fibered conjugacy ψ_N between X_0 and X, to obtain a counterpart for vector fields of their result:

Corollary 1 [14]. *On each one of the sectors V_j^{sn} (resp. V_j^{ns}) the vector field Z is analytically conjugated to its formal normal form Z_0 by a sectorial diffeomorphism ψ_j^{sn} (resp. ψ_j^{ns}) which factorizes into $\psi_N \circ \psi_T$.*

6. Analytical classification

The analytical class of Z is determined by how we glue together the $2k$ sectorial parts of Z_0. The gluing mappings g_j^n, g_j^s defined by $\psi_j^{ns} \circ g_j^s = \psi_{j+1}^{sn}$ and $\psi_j^{sn} \circ g_j^n = \psi_j^{ns}$ are sectorial symmetries of Z_0: they are sectorial diffeomorphisms satisfying in addition $g^* Z_0 = Z_0$. The next proposition describes the structure of the group $\mathcal{G}(Z_0, V_0)$ of germs of sectorial symmetries of Z_0 over V_0, when V_0 is one of the sectors V_j^s, V_j^n, V_j^{sn} or V_j^{ns}, in order to give the analytical classification of vector fields. We write $\mathrm{Aff}(\mathbb{C})$ as the group of affine maps of the line, $\mathbb{C}\{h\}$ as the ring of germs of holomorphic functions at $0 \in \mathbb{C}$ (convergent power series in $h \in \mathbb{C}$), and $\mathrm{Diff}(\mathbb{C}, 0)$ as the group of germs of biholomorphisms φ fixing $0 \in \mathbb{C}$, i.e., $\mathrm{Diff}(\mathbb{C}, 0) = \{\varphi \in \mathbb{C}\{h\}: \varphi(0) = 0 \text{ and } \varphi'(0) \neq 0\}$.

Proposition 1 [14]. *The group $\mathcal{G}(Z_0, V_j^s)$ is isomorphic to the semi-direct product* $\mathrm{Diff}(\mathbb{C}, 0) \ltimes \mathbb{C}\{h\}$ *where* $(\varphi, f) \ltimes (\psi, g) = (\varphi \circ \psi, g + f \circ \psi)$. *The group $\mathcal{G}(Z_0, V_j^n)$ is isomorphic to the product* $\mathrm{Aff}(\mathbb{C}) \times \mathbb{C}$. *The groups $\mathcal{G}(Z_0, V_j^{sn})$ and $\mathcal{G}(Z_0, V_j^{ns})$ are isomorphic to $\mathbb{C}^* \times \mathbb{C}$.*

Let us describe the isomorphisms involved, for instance for $\mathcal{G}(Z, V_j^s)$. Eq. (2) provides a first integral $H(x, y): V_j^s \to \Omega_j^s$ which, for all fixed x, is a linear isomorphism $H_x: y \mapsto h$ between a vertical open disc $\{x = cte,\ |y| < r\}$ and a disc $\Omega_x \subset \mathbb{C}$ containing 0. Consider now a sectorial symmetry g over V_j^s. Since the map sends the foliation onto itself it induces a diffeomorphism $\varphi \in \mathrm{Diff}(\Omega_x, 0)$. The map φ induces *via* H_x a fibered sectorial symmetry $(x, y) \mapsto (x, H_x^* \varphi)$ we call g_N. The composition $g_T = g_N^{-1} \circ g$ is still a symmetry of Z_0, this time sending every leaf of the foliation onto itself. We derive easily that $g_T = \Phi_{Z_0}^F$ for a function F constant on the leaves of the foliation: F factorizes into some function $f \in \mathcal{O}(\Omega_x)$. We then identify g to the pair (φ, f).

Applying these isomorphisms to the gluing mappings $(g_j^s, g_j^n)_j$ we identify g_j^s with $(\varphi_j^s, f_j^s) \in \mathrm{Diff}(\mathbb{C}, 0) \times \mathbb{C}\{h\}$ and g_j^n with $(\varphi_j^n, t_j) \in \mathrm{Aff}(\mathbb{C}) \times \mathbb{C}$. Beside the gluing mappings are unique up to a choice of a symmetry over each V_j^{sn} and V_j^{ns}. After a convenient choice we can assume that $(\varphi_j^s)'(0) = \exp(2\mathrm{i}\pi\mu/k)$, $(\varphi_j^n)'(0) = 1$ and $f_j^s(0) = t_j = 0$. This particular choice is unique up to a simultaneous change of coordinates $c \mapsto ch$ in all the spaces of leaves Ω_j^{sn} and Ω_j^{ns}, corresponding to the global symmetry $(x, y) \mapsto (x, cy)$ identified to $(c, 0)$. We define $m(Z) = (\varphi_j^s, \varphi_j^n, f_j)_j$. In this setting Martinet–Ramis' invariant corresponds to $(\varphi_j^n, \varphi_j^s)_j$.

Theorem 2 [14]. (1) *Let Z_1 and Z_2 be two vector fields analytically conjugated to Z_0 with invariants $m(Z_1) = (\varphi_j^n, \varphi_j^s, f_j)_j$ and $m(Z_2) = (\psi_j^n, \psi_j^s, g_j)_j$. Those vector fields are analytically conjugated if, and only if, there exists $c \in \mathbb{C}^*$ and $\theta \in \mathbb{Z}/k$ such that $P(\mathrm{e}^{2\mathrm{i}\pi\theta/k} x) = P(x)$, and for all $j \in \mathbb{Z}/k$ we have $\varphi_{j+\theta}(ch) = c\psi_j(h)$ and $f_{j+\theta}(ch) = g_j(h)$.*

(2) *Reciprocally let $m_0 = (\varphi_j^n, \varphi_j^s, f_j)_j$ be given satisfying $\varphi_j^n(h) = h + a_j$, $\varphi_j^s(0) = f_j(0) = 0$ and $(\varphi_j^s)'(0) = \exp(2\mathrm{i}\pi\mu/k)$. Then there exists a vector field Z, formally conjugated to Z_0, whose invariant $m(Z)$ equals the given m_0. In other words the map m is surjective.*

(3) *For all finite-dimensional analytical deformation $\varepsilon \mapsto Z_\varepsilon$ the map $\varepsilon \mapsto m(Z_\varepsilon)$ is analytic as well.*

After I finished this work I learned that Meshcheryakova and Voronin had announced a similar result in [12] for the case $k = 1$. Their proof is different, in particular for the realization of invariants, i.e., point (2) of the previous theorem. This is established by gluing 2 $(= 2k)$ sectorial parts of the model Z_0, obtaining an integrable almost-complex structure which, by Newlander–Niremberg's theorem, is induced by a germ of holomorphic vector field Z. Our proof uses the realization part of Martinet–Ramis' classification to obtain a foliation $\mathcal{F}_X$ with prescribed invariant $(\varphi_j^n, \varphi_j^s)_j$ as above. We then identify the invariant $(f_j)_j$ with the functional obstruction to solve the homological equation $X \cdot F = 1/U - 1/P$ (i.e., the period of the second term).

Corollary 2 [14]. *Let* $(f_j)_j$ *be the tangential invariant of* $Z = UX$. *Then* $f_j = \mathcal{T}_X^j(1/U - 1/P)$.

The map m is surjective if, and only if, the linear map $G \mapsto (\mathcal{T}_X^j(G))_j \in \{f \in \mathbb{C}\{h\} : f(0) = 0\}^k$ is. The latter statement is proved using Ramis–Sibuya's theorem [9, p. 176].

7. Normal forms

When $X = X_0$ it is possible to compute the periods $\mathcal{T}_X^j(x^n y^m)$ in terms of the Γ-function (see [5,10,13]). In that case we are able to find "explicit" analytical normal forms. For $\beta \leqslant 0$ we set

$$\mathcal{G}_\beta = \left\{ yx^{k+1} \sum_{m \geqslant 0} y^m x^{[-m\beta]} P_m(x) \in \mathbb{C}\{x, y\} : P_m(x) = \sum_{n=0}^{k-1} p_{m,n} x^n \right\}.$$

Let us define $\beta(\mu) = \min\{0, \mathrm{Re}(\mu)\}$ and $\mathcal{G} = \mathcal{G}_{\beta(\mu)}$.

Corollary 3. *Each* Z *analytically orbitally equivalent to* X_0 *is analytically conjugated to some* $Z_{P,G} = \frac{P}{1+PG} X_0$ *with* $G \in \mathcal{G}$ *and* $P(x) = \lambda + \cdots + a_k x^k$ *with* $\lambda \neq 0$. *This form is unique up to linear change of coordinates* $(x, y) \mapsto (\alpha x, cy)$ *with* $\alpha^k = 1$ *and* $c \neq 0$.

In [6] similar normal forms are found for vector fields associated to normalizable resonant saddle foliations, whose formal invariants are $p/q = 1$, $k = 1$ and $\mu = 0$. The method we introduce, together with the computations of [5], can provide normal forms for all parameters $(p/q, k, \mu) \in \mathbb{Q}_- \times \mathbb{N}^* \times \mathbb{C}$.

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