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COMPTES RENDUS MATHEMATIQUE

Group Theory

A Wilson group of non-uniformly exponential growth

Un groupe de Wilson de croissance exponentielle non-uniforme

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Abstract

This Note constructs a finitely generated group W whose word-growth is exponential, but for which the infimum of the growth rates over all finite generating sets is 1 – in other words, of non-uniformly exponential growth.

This answers a question by Mikhael Gromov (Structures métriques pour les variétés riemanniennes, in: J. Lafontaine, P. Pansu (Eds.), CEDIC, Paris, 1981).

The construction also yields a group of intermediate growth V that locally resembles W in that (by changing the generating set of W) there are isomorphic balls of arbitrarily large radius in V and W's Cayley graphs. ***To cite this article: L. Bartholdi, C. R. Acad. Sci. Paris, Ser. I 336 (2003).***

Résumé

Cette Note construit un groupe W de type fini dont la croissance des boules est exponentielle, mais pour laquelle l'infimum des taux de croissance vaut 1 – en d'autres termes, W est de croissance exponentielle non-uniforme.

Ceci répond à une question de Mikhael Gromov (Structures métriques pour les variétés riemanniennes, in : J. Lafontaine, P. Pansu (Eds.), CEDIC, Paris, 1981).

Cette construction donne aussi un groupe de croissance intermédiaire V ressemblant localement à W dans le sens que (en changeant le système générateur de W) des boules de rayon arbitrairement grand coïncident dans les graphes de Cayley de V et W. ***Pour citer cet article : L. Bartholdi, C. R. Acad. Sci. Paris, Ser. I 336 (2003).***

Version française abrégée

Considérons le groupe $A = \mathsf{PSL}(3,2)$ agissant sur le plan projectif à sept points P. Ce groupe est engendré par trois involutions $x = (\mathbb{1},\mathbb{5})(\mathbb{3},\mathbb{7})$, $y = (\mathbb{2},\mathbb{3})(\mathbb{6},\mathbb{7})$, $z = (\mathbb{4},\mathbb{6})(\mathbb{5},\mathbb{7})$. On fait agir à droite deux copies $A, \bar{A}$ de A sur l'ensemble P^* des suites finies d'éléments de P de la façon suivante :

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$$(p_1p_2\dots p_n)a = (p_1a)p_2\dots p_n,$$
$$(p_1\dots p_mp_{m+1}\dots p_n)\bar a = \begin{cases} p_1\dots p_m(p_{m+1}a)p_{m+2}\dots p_n, & \text{si } p_1=\dots=p_{m-1}=\mathbb{1},\ p_m=\mathbb{2},\\ p_1\dots p_mp_{m+1}\dots p_n, & \text{sinon.}\end{cases}$$

On note W le groupe de permutations de P^* engendré par A et $\bar A$.

Par ailleurs, on considère les involutions $\tilde x$, $\tilde y$, $\tilde z$ de P^* définies par

$$(\mathbb{4}\dots\mathbb{4}p_m\dots p_n)\tilde x = \mathbb{4}\dots\mathbb{4}(p_mx)p_{m+1}\dots p_n, \qquad (\mathbb{1}\dots\mathbb{1}p_m\dots p_n)\tilde y = \mathbb{1}\dots\mathbb{1}(p_my)p_{m+1}\dots p_n,$$
$$(\mathbb{2}\dots\mathbb{2}p_m\dots p_n)\tilde z = \mathbb{2}\dots\mathbb{2}(p_mz)p_{m+1}\dots p_n,$$

où m est choisi maximal, de sorte que $p_m \neq p_{m-1}$. Elles engendrent le groupe $V = \langle \tilde x, \tilde y, \tilde z\rangle$.

Les résultats principaux de cette note sont les suivants :

- W est un groupe à croissance exponentielle non-uniforme.
- V est un groupe à croissance intermédiaire.
- Pour tout N il existe un système générateur $\{x_N, y_N, z_N\}$ de W tel que les boules de centre 1 et de rayon N dans V et W coïncident, si l'on identifie $\tilde x, \tilde y, \tilde z$ à x_N, y_N, z_N respectivement.

La construction de W est essentiellement semblable à celle annoncée par John Wilson [8].

1. Introduction

The purpose of this Note is to construct in an as short and elementary way as possible a group of non-uniformly exponential growth, i.e., a group of exponential growth with a family of generating sets for which the growth rate tends to 1. The "limit" of these generating sets generates a group of intermediate growth.

This construction is an adaptation of [1], which describes a family of groups of intermediate growth. I completed it after John Wilson announced that he had produced a group of non-uniformly exponential growth; my method is similar to his, and indeed only claims to be somewhat shorter and more explicit than that of his recent preprint [8]. The first appearance of these groups seems to be in [6].

The reader is directed to [5] for a survey on uniform growth of groups.

2. A group of non-uniformly exponential growth

First consider the group $A = \mathsf{PSL}(3,2)$ acting on the seven-points projective plane P over $\mathbb{F}_2$. The group A is generated by three reflections x, y, z in P, as in Fig. 1. We have $x = (\mathbb{1},\mathbb{5})(\mathbb{3},\mathbb{7})$, $y = (\mathbb{2},\mathbb{3})(\mathbb{6},\mathbb{7})$, $z = (\mathbb{4},\mathbb{6})(\mathbb{5},\mathbb{7})$.

We let two copies A, $\bar A$ of A act on the right on the set P^* of finite sequences in P by

$$(p_1p_2\dots p_n)a = (p_1a)p_2\dots p_n,$$
$$(p_1\dots p_mp_{m+1}\dots p_n)\bar a = \begin{cases} p_1\dots p_m(p_{m+1}a)p_{m+2}\dots p_n, & \text{if } p_1=\dots=p_{m-1}=\mathbb{1},\ p_m=\mathbb{2},\\ p_1\dots p_mp_{m+1}p_{m+2}\dots p_n, & \text{otherwise.}\end{cases}$$

We then define W as the group of permutations of P^* generated by A and $\bar A$.

By $G \wr A$ we mean the wreath product $\{f : P \to G\} \rtimes A$. We introduce the notation $g = \langle\langle g_1,\dots,g_7\rangle\rangle a$ for the element (f,a) of $G\wr A$ defined by $f(p) = g_p$ for all $p \in P$, and we omit the "a" from the notation if it is trivial. If G acts on P^*, then $G \wr A$ also acts on P^* via the formula

$$(p_1p_2\dots p_n)g = (p_1a)\big((p_2\dots p_n)g_{p_1}\big). \tag{1}$$

In this notation, we have $\bar a = \langle\langle \bar a, a, 1, 1, 1, 1, 1\rangle\rangle$.

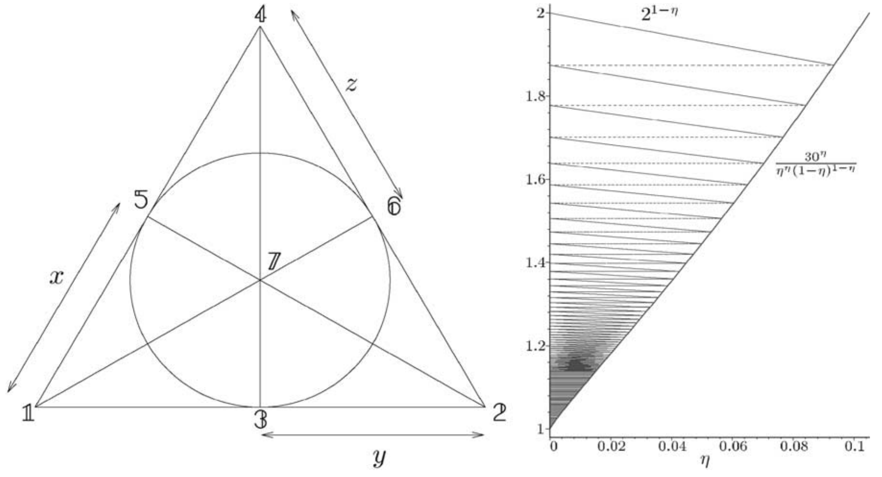

Fig. 1. The projective plane over $\mathbb{F}_2$, and the functions in the proof of Proposition 2.6.

Fig. 1. Le plan projectif sur $\mathbb{F}_2$, et les fonctions de la preuve de la Proposition 2.6.

Let G be a group generated by a finite set S. Its *growth rate* is, by definition, $\lambda(G,S) = \lim_{n\to\infty} \sqrt[n]{\#B_{G,S}(n)}$, where $B_{G,S}(n) = \{g \in G \mid g = s_1 \ldots s_n \text{ for some } s_i \in S\}$ is the ball of center 1 and radius n in G, with the word metric induced by S. (This limit exists because $\log \#B_{G,S}(n)$ is subadditive.)

The group G has *exponential growth* if $\lambda(G,S) > 1$ for one or, equivalently, for any generating set, and has *subexponential growth* otherwise. If $\lambda(G,S) = 1$ and $\#B_{G,S}(n)$ is not bounded by any polynomial function of n, then G has *intermediate growth*. It is non-trivial to construct groups of intermediate growth, and the first example was produced by Grigorchuk [3] in 1983.

Note that if $\lambda_{G,S} > 1$, then there exist other generating sets S' for G with $\lambda(G,S')$ arbitrarily large – for instance, $\lambda(G, B_{G,S}(k)) = \lambda(G,S)^k$. On the other hand, it is not obvious that $\lambda(G,S)$ can be made arbitrarily close to 1.

The group G has *uniformly exponential growth* if $\inf_{\{\text{finite } S\}} \lambda(G,S) > 1$.

Note that free groups, and more generally hyperbolic groups, have uniformly exponential growth as soon as they have exponential growth. Solvable groups [7], and linear groups [2] in characteristic 0, also have uniformly exponential growth as soon as they have exponential growth.

Mikhael Gromov asked in 1981 whether there exist groups of exponential, but non-uniformly exponential growth [4, Remarque 5.12]. This was answered positively by John Wilson [8] in 2002. The main result of this Note is the following:

Theorem 2.1. *The group W has exponential growth, but does not have uniformly exponential growth.*

The proof relies on the following propositions:

Proposition 2.2. *The group W satisfies the decomposition $W \cong W \wr A$; the isomorphism is given by* (1) *intertwining their actions on P^*.*

Proposition 2.3. *The group W contains a free monoid on two generators.*

Given a triple $\{a,b,c\}$ of involutions acting on P^*, we define a new triple $\{a',b',c'\}$ of involutions also acting on P^* by

$$a' = \langle\langle 1,1,1,a,1,1,1\rangle\rangle x, \qquad b' = \langle\langle b,1,1,1,1,1,1\rangle\rangle y, \qquad c' = \langle\langle 1,c,1,1,1,1,1\rangle\rangle z. \tag{2}$$

Proposition 2.4. *If G is a perfect group generated by three involutions a, b, c, then $\{a',b',c'\}$ generates $G \wr A$.*

Proposition 2.5. *The group W is generated by three involutions.*

Proposition 2.6. *Let G be generated by a triple of involutions $S=\{a,b,c\}$, and set $S'=\{a',b',c'\}$ and $H=\langle S'\rangle$. Then*

$$\lambda(H,S') \leqslant \inf_{\eta\in(0,1)} \max\left\{\lambda(G,S)^{1-\eta}, \frac{30^\eta}{\eta^\eta(1-\eta)^{1-\eta}}\right\}. \tag{3}$$

Fix the following sets of permutations of P^*: a generating set S_0 of W given by Proposition 2.5; and for all $n \geqslant 1$ a set S_n obtained from S_{n-1} by application of (2).

Proof of Theorem 2.1. W has exponential growth by Proposition 2.3.

Since S_0 contains three involutions, $\lambda(G,S_0) \leqslant 2$. By Propositions 2.6 and 2.4, the sets S_n also generate W for $n \geqslant 1$. Set $\Lambda_0 = 2$, and for all $n \geqslant 0$ define $\eta_n \in (0, \frac{1}{2})$ and $\Lambda_{n+1} > 1$ by

$$\Lambda_{n+1} = \Lambda_n^{1-\eta_n} = \frac{30^{\eta_n}}{\eta_n^{\eta_n}(1-\eta_n)^{1-\eta_n}}. \tag{4}$$

There is a unique solution η_n because the LHS of (4) is decreasing in η_n while the RHS is increasing. Clearly $0 < \eta_{n+1} < \eta_n$ and $1 < \Lambda_{n+1} < \Lambda_n$, so $\eta = \lim_{n\to\infty}\eta_n$ and $\Lambda = \lim_{n\to\infty}\Lambda_n$ exist. From (4) we have $\Lambda = \Lambda^{1-\eta}$, so either $\Lambda = 1$ or $\eta = 0$, which also implies $\Lambda = 1$. Since $\lambda(W,S_n) \leqslant \Lambda_n$ for all $n \in \mathbb{N}$, we have $\lim_{n\to\infty}\lambda(W,S_n) = 1$. □

3. A group of intermediate growth

Consider next the set $\widetilde{S} = \{\tilde{x},\tilde{y},\tilde{z}\}$ of permutations of P^* defined implicitly by $\tilde{x} = \langle\langle 1,1,1,\tilde{x},1,1,1\rangle\rangle x$, $\tilde{y} = \langle\langle \tilde{y},1,1,1,1,1,1\rangle\rangle y$, $\tilde{z} = \langle\langle 1,\tilde{z},1,1,1,1,1\rangle\rangle z$, and consider the group $V = \langle\widetilde{S}\rangle$.

Theorem 3.1. *The group V has intermediate growth.*

The group V is locally isomorphic to W, in that for any $R \in \mathbb{N}$, there exists $n \in \mathbb{N}$ such that $B_{V,\widetilde{S}}(R)$ and $B_{W,S_n}(R)$ are isomorphic graphs, seen as subsets of their respective groups' Cayley graphs, with $\tilde{x}$, $\tilde{y}$, $\tilde{z}$ identified with the elements $x^{(n)}$, $y^{(n)}$, $z^{(n)}$ of S_n respectively.

(Note that V is not perfect; indeed $V/V' \cong (\mathbb{Z}/2)^3$. Hence V does not decompose as a wreath product like W. Note also that $\tilde{x} = \lim x^{(n)}$ in the compact-open topology on $\operatorname{aut}(P^*)$; and similarly for $\tilde{y}$ and $\tilde{z}$.)

Proof. Apply Proposition 2.6 to $(V,\widetilde{S})$, and note $\widetilde{S}' = \widetilde{S}$; hence $\lambda(V,\widetilde{S}) = \lambda(V,\widetilde{S}')$, so $\lambda(V,\widetilde{S}) = 1$ by (3).

A group G acting on P^* is *contracting* if G identifies with a subgroup of $G \wr A$ via the decomposition (1), and there are constants $\rho < 1$ and M such that $B(G,R)$ identifies with a subset of $B(G,\rho R + M) \wr A$ for all $R \in \mathbb{N}$.

This property is independent of the chosen generating set, though the constants ρ, R do depend on it. For $(V,\widetilde{S})$ and for (W,S_n) for all $n \geqslant 0$ the property holds with $\rho = M = \frac{1}{2}$.

Pick now $R \in \mathbb{N}$. By contraction there exists $n \in \mathbb{N}$ such that the n-fold decomposition of $B(G,R)$ is a subset of $(\cdots(B(G,1) \wr A) \wr A \cdots \wr A)$. Since the generators $\tilde{S}$ and S_0 agree on a ball of radius 1, this implies that the generators $\widetilde{S}$ and S_n agree on a ball of radius R. □

4. Proof of the propositions

We will use repeatedly the following facts on A: it has order 168, and is simple, hence perfect. It is generated by $\{x,y,z\}$, and also by $\{xy,yz,zx\}$.

Proof of Proposition 2.2. Since A acts 2-transitively on P, there exists $u \in A$ that fixes $\mathbb{1}$ and moves $\mathbb{2}$ to another point, and $v \in A$ that fixes $\mathbb{2}$ and moves $\mathbb{1}$ to another point.

Then W contains $[\bar{a},\bar{b}^u] = \langle\langle[\bar{a},\bar{b}],1,\ldots,1\rangle\rangle$ for any $a,b \in A$, and since A is perfect, W contains $\langle\langle\bar{A},1,\ldots,1\rangle\rangle$. Similarly, W contains $[\bar{a},\bar{b}^v] = \langle\langle 1,[a,b],1,\ldots,1\rangle\rangle$ for any $a,b \in A$, so W contains $\langle\langle 1,A,1,\ldots,1\rangle\rangle$. Combining these, W contains $\langle\langle W,\ldots,W\rangle\rangle$ and A, so W contains $W \wr A$. The converse inclusion is obvious. □

Proof of Proposition 2.3. Pick $u \neq v \in A$ such that $\mathbb{1}u = \mathbb{1}v = \mathbb{2}$ and $\mathbb{2}u = \mathbb{2}v = \mathbb{1}$. Consider the elements $a = \bar{u}u, b = \bar{u}v, c = \bar{v}u, d = \bar{v}v$. They admit the decompositions

$$\begin{aligned} a &= \langle\langle \bar{u}, u, 1, \dots, 1\rangle\rangle(\mathbb{1}, \mathbb{2})\sigma, \qquad & b &= \langle\langle \bar{u}, u, 1, \dots, 1\rangle\rangle(\mathbb{1}, \mathbb{2})\tau, \\ c &= \langle\langle \bar{v}, v, 1, \dots, 1\rangle\rangle(\mathbb{1}, \mathbb{2})\sigma, \qquad & d &= \langle\langle \bar{v}, v, 1, \dots, 1\rangle\rangle(\mathbb{1}, \mathbb{2})\tau, \end{aligned} \tag{5}$$

for some permutations $\sigma \neq \tau$ of $P \setminus \{\mathbb{1}, \mathbb{2}\}$. We claim that $\{a, d\}$ generates a free monoid in W; actually, we will show slightly more: let $M = \{a, b, c, d\}^*$ be the submonoid of W generated by $\{a, b, c, d\}$. We claim that $M/(a = b, c = d)$ is free, freely generated by $\{a, d\}$.

Consider two words X, Y over a, b, c, d, that are inequivalent under $(a = b, c = d)$; we will prove by induction on $|X| + |Y|$ that they act differently on P^*. We may assume that X and Y are both non-empty, and that X starts by a or b, and Y starts by c or d. If $|X| = |Y| = 1$ then the decompositions (5) show that X and Y act differently on P^*. Otherwise, we have $|X| \equiv |Y| \bmod 2$, by considering their action on $\mathbb{1}$; and furthermore we may assume $|X| \equiv |Y| \equiv 0 \bmod 2$, if necessary by multiplying both X and Y by a on the right. Consider the decompositions $X = \langle\langle X_1, \dots, X_7\rangle\rangle\alpha,\ Y = \langle\langle Y_1, \dots, Y_7\rangle\rangle\beta$. Then $X_1, Y_1 \in M$, and X_1 starts by a or b, while Y_1 starts by c or d; so X_1 and Y_1 are inequivalent under $(a = b, c = d)$. We have $|X_1| = |X|/2$ and $|Y_1| = |Y|/2$, so by induction X_1 and Y_1 act differently on P^*, and hence so do X and Y. □

Proof of Proposition 2.4. Set $H = \langle a', b', c'\rangle$. Then H contains $(a'b'c'b')^3 = \langle\langle 1, 1, ac, ac, 1, 1, ca\rangle\rangle$, $(b'c'a'c')^3 = \langle\langle ba, 1, 1, 1, 1, ba, ab\rangle\rangle$ and $(c'a'b'a')^3 = \langle\langle 1, cb, 1, 1, cb, 1, bc\rangle\rangle$; so H contains $v = [(a'b'c'b')^3, (b'c'a'c')^3] = \langle\langle 1, \dots, 1, [ca, ab]\rangle\rangle \neq 1$. Now $G = \langle ca, ab, bc\rangle = \langle [ca, ab]^G, [ab, bc]^G, [bc, ca]^G\rangle$ because G is perfect, so conjugating v by all words in $(a'b'c'b')^3, (b'c'a'c')^3, (c'a'b'a')^3$ we see that H contains $\langle\langle G, \dots, G\rangle\rangle$.

Finally, H contains $x = \langle\langle 1, 1, 1, a, 1, 1, 1\rangle\rangle a'$, and similarly y and z, so $H = G \wr A$. □

Proof of Proposition 2.5. Define $a, b, c \in W$ by $a = \langle\langle 1, \bar{x}, 1, x, 1, 1, 1\rangle\rangle x,\ \ b = \langle\langle y, 1, 1, \bar{y}, 1, 1, 1\rangle\rangle y,\ \ c = \langle\langle \bar{z}, z, 1, 1, 1, 1, 1\rangle\rangle z$. Then $(ab)^4 = (\langle\langle 1, \bar{x}, 1, x\bar{y}, y, 1, 1\rangle\rangle xy)^4 = \langle\langle 1, \bar{x}, \bar{x}, (x\bar{y})^4, 1, \bar{x}, \bar{x}\rangle\rangle = \langle\langle 1, \bar{x}, \bar{x}, 1, 1, \bar{x}, \bar{x}\rangle\rangle$, and similarly $(bc)^4 = \langle\langle 1, 1, 1, \bar{y}, \bar{y}, \bar{y}, \bar{y}\rangle\rangle$ and $(ca)^4 = \langle\langle \bar{z}, 1, \bar{z}, 1, \bar{z}, 1, \bar{z}\rangle\rangle$.

Therefore the group $G = \langle a, b, c\rangle$ contains $u = [[(ab)^4, (bc)^4], (ca)^4] = \langle\langle 1, \dots, 1, [[\bar{x}, \bar{y}], \bar{z}]\rangle\rangle \neq 1$, so G contains all the conjugates of u by $(ab)^4, (bc)^4, (ca)^4$, and since A is simple, G contains $\langle\langle 1, \dots, 1, \bar{A}\rangle\rangle$. Since G acts transitively on P, it contains $\langle\langle \bar{A}, \dots, \bar{A}\rangle\rangle$.

Next, G contains $\langle\langle 1, \bar{x}, 1, 1, 1, 1, 1\rangle\rangle a = \langle\langle 1, 1, 1, x, 1, 1, 1\rangle\rangle x$, and similarly $\langle\langle y, 1, 1, 1, 1, 1, 1\rangle\rangle y$ and $\langle\langle 1, z, 1, 1, 1, 1, 1\rangle\rangle z$, so G contains $A \wr A$ by Proposition 2.4, and $G = W$. □

Proof of Proposition 2.6. Consider a word $w \in \{a', b', c'\}^*$ representing an element in H, and its decomposition $\langle\langle w_1, \dots, w_7\rangle\rangle\sigma$. Each of the w_i's is a word over $\{a, b, c\}$, and the total length of the w_i is at most the length of w, since by the definition (2) each S'-letter in w contributes a single S-letter to one of the w_i's.

A *reduced* word is a word with no two identical consecutive letters; we shall always assume that the words we consider are reduced. Therefore all aa-, bb- and cc-subwords of the w_i's are to be cancelled; and such subwords appear in a w_i whenever w has a subword belonging to

$$\Delta = \{a'b'a', b'c'b', c'a'c', a'c'b'a'c'a'b'c'a', b'a'c'b'a'b'c'a'b', c'b'a'c'b'c'a'b'c'\};$$

indeed Δ's elements are, up to the cyclic permutation $a' \mapsto b' \mapsto c' \mapsto a'$,

$$\begin{aligned} a'b'a' &= \langle\langle 1, 1, 1, aa, b, 1, 1\rangle\rangle xyx = \langle\langle 1, 1, 1, 1, b, 1, 1\rangle\rangle xyx, \\ a'c'b'a'c'a'b'c'a' &= \langle\langle a, cb, 1, aa, bc, a, c\rangle\rangle yzxzy = \langle\langle a, cb, 1, 1, bc, a, c\rangle\rangle yzxzy. \end{aligned}$$

Lemma 4.1. *For any $n \in \mathbb{N}$, there are at most 30 reduced words $w \in \{a', b', c'\}^*$ of length n that contain no subword belonging to Δ.*

Proof. If w contains $a'c'a'$, $b'a'b'$ or $c'b'c'$ as a subword, then this subword occurs either among the first five or the last five letters of w, and w is a subword of $(xyz)^\infty y(zyx)^\infty$, where x, y, z is a cyclic permutation of a', b', c'.

This gives 24 possibilities: 3 for the choice of the cyclic permutation and 8 for the position of the zyz subword in w.

If w does not contain any such subword, then w must be a subword of $(xyz)^\infty$ or $(zyx)^\infty$, and this gives 6 possibilities: 3 for the choice of the cyclic permutation and 2 for the choice of xyz or zyx. $\square$

Fix now for every $h \in H$ a word w_h of minimal length representing h; and for all $n \in \mathbb{N}$ let W_n denote the set of such words of length n. We wish to estimate $\#W_n$.

For any $\eta \in (0, 1)$, define the following sets:

$$W_n^{\geqslant\eta} = \{w \in W_n \mid w \text{ contains at least } \eta n \text{ subwords belonging to } \Delta\},$$
$$W_n^{\leqslant\eta} = \{w \in W_n \mid w \text{ contains at most } \eta n \text{ subwords belonging to } \Delta\}.$$

Any $w \in W_n^{\leqslant\eta}$ factors as a product of at most ηn non-empty pieces $u_1 \dots u_m$, where no u_i contains any subword belonging to Δ. We therefore have

$$\#W_n^{\leqslant\eta} \leqslant \sum_{m=1}^{\eta n} \binom{n-1}{m-1} 30^m \leqslant \eta n \binom{n}{\eta n} 30^{\eta n};$$

here m is the number of pieces u_i; $\binom{n-1}{m-1}$ is the number of possible lengths $|u_1|, \dots, |u_m|$ summing to n; and 30^m is (by Lemma 4.1) the number of possible $u_1, \dots, u_m$ with prescribed lengths.

Estimating the binomial coefficient $\binom{n}{\eta n} \approx (\eta^\eta(1-\eta)^{1-\eta})^{-n}$, we get $\lim_{n\to\infty} \sqrt[n]{\#W_n^{\leqslant\eta}} \leqslant \frac{30^\eta}{\eta^\eta(1-\eta)^{1-\eta}}$.

Consider next $w \in W_n^{\geqslant\eta}$, and decompose it as $w = \langle\langle w_1, \dots, w_7\rangle\rangle a$. The seven words $w_1, \dots, w_7$ have total length at most $(1-\eta)n$, after cancellation of the aa-, bb- and cc-subwords.

For any $\varepsilon > 0$, there is a constant K such that $\#B_{G,S}(n) \leqslant K(\lambda(G, S) + \varepsilon)^n$ for all $n \in \mathbb{N}$. Therefore

$$\#W_n^{\geqslant\eta} \leqslant \#A \sum_{\substack{n_1,\dots,n_7 \geqslant 0 \\ n_1+\cdots+n_7 \leqslant (1-\eta)n}} \prod_{p=1}^{7} K\big(\lambda(G, S) + \varepsilon\big)^{n_p} \leqslant \#A \binom{n+7}{7} K^7 \big(\lambda(G, S) + \varepsilon\big)^{(1-\eta)n};$$

the binomial term counts all possible lengths of the seven words w_i, and the other terms count the number of possible w_i's with prescribed lengths. It follows that $\lim_{n\to\infty} \sqrt[n]{\#W_n^{\geqslant\eta}} \leqslant (\lambda(G, S) + \varepsilon)^{1-\eta}$ for all $\varepsilon > 0$, and therefore $\lim_{n\to\infty} \sqrt[n]{\#W_n^{\geqslant\eta}} \leqslant \lambda(G, S)^{1-\eta}$.

Now $\#B_{H,S'}(n) \leqslant \sum_{j=0}^{n} (\#W_j^{\leqslant\eta} + \#W_j^{\geqslant\eta})$, and

$$\begin{aligned}\lambda(H, S') &= \lim_{n\to\infty} \sqrt[n]{\#B_{H,S'}(n)} \leqslant \lim_{n\to\infty} \sqrt[n]{\#W_n^{\leqslant\eta} + \#W_n^{\geqslant\eta}} \leqslant \lim_{n\to\infty} \max\left\{\sqrt[n]{\#W_n^{\leqslant\eta}}, \sqrt[n]{\#W_n^{\geqslant\eta}}\right\} \\ &\leqslant \max\left\{\lambda(G, S)^{1-\eta}, \frac{30^\eta}{\eta^\eta(1-\eta)^{1-\eta}}\right\}. \qquad \square\end{aligned}$$

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