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# Geodesics and the Einstein-nonlinear wave system

# Courbes géodesiques et l'équation d'Einstein

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**Abstract**

Results concerning the problem of motion of test particles in the context of solitary wave solutions of the Einstein-nonlinear wave system are announced. ***To cite this article: D.M.A. Stuart, C. R. Acad. Sci. Paris, Ser. I 336 (2003).***


**Résumé**

On étude le problème du mouvement des ondes solitaires dans le système qui comprend l'équation d'Einstein et l'équation des ondes non linéaires. ***Pour citer cet article : D.M.A. Stuart, C. R. Acad. Sci. Paris, Ser. I 336 (2003).***


## 1. Introduction

The Einstein nonlinear wave system is the following system of evolution equations for a pseudo-Riemannian metric $\mathbf{g} = g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu$ on a space–time $\mathbb{M}$ and a function $\phi : \mathbb{M} \to \mathbb{C}$:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi \mathbf{G} T_{\mu\nu}, \qquad \Box_{\mathbf{g}}\phi + \mathcal{V}'(\phi) = 0, \tag{1}$$

where $R_{\mu\nu} = R_{\mu\nu}(\mathbf{g}),\ R = R(\mathbf{g})$ are, respectively, the Ricci and scalar curvatures of $\mathbf{g}$ and $\Box_{\mathbf{g}}$ is the covariant wave operator, $\Box_{\mathbf{g}}\phi = -|\det \mathbf{g}|^{-1/2}\partial_\mu(|\det \mathbf{g}|^{1/2} g^{\mu\nu}\partial_\nu\phi)$. The energy momentum tensor $T_{\mu\nu}$ is given by

$$T_{\mu\nu}(\mathbf{g}, \phi; \mathcal{V}) = \langle \partial_\mu\phi, \partial_\nu\phi\rangle - \frac{1}{2} g_{\mu\nu}\big(\langle \partial_\mu\phi, \partial_\nu\phi\rangle g^{\mu\nu} + 2\mathcal{V}(\phi)\big) \tag{2}$$

with $\langle a, b\rangle = (\bar{a}b + a\bar{b})/2$ for $a, b \in \mathbb{C}$, and $g^{\mu\nu}$ the inverse matrix of $g_{\mu\nu}$ (induced inner product on the cotangent space). In the coordinate system $\{x^\mu\}$ greek indices run through $0, 1, 2, 3$, latin indices through $1, 2, 3$, and $x^0 = t$. The metric is taken to have signature $(- + ++)$ so that the system is essentially hyperbolic (modulo diffeomorphism invariance).

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In general relativity the metric $\mathbf{g}$, obtained as a solution of the Einstein equation, represents the influence of the *matter field* $\phi$ on the space time, i.e., the gravitational field produced by $\phi$. In turn the metric influences the matter field via the appearance of $\mathbf{g}$ in the equation for $\phi$. Recall (e.g., [10, Chapter 3]) that it is a consequence of the principle of equivalence that a *test particle*, i.e., a particle of small size and mass, should move along a geodesic with respect to $\mathbf{g}$. More precisely if, at each time $t$, $\phi$ is zero, or close to zero, outside of a small region centered at $\xi(t)$, and if the energy is sufficiently small, then the curve $t \mapsto \xi(t)$ must be a geodesic to highest order; of course a proper understanding requires a precise formulation of the meaning of "small" and "to highest order". On the other hand for a specific matter field such as $\phi$, in which the time evolution is governed by a well-posed Cauchy problem for (1), this geodesic motion should be an *analytical* consequence of the equations (for appropriate initial data). A proof of this is here announced in the context of (1) under conditions which give a precise formulation of the notions mentioned above. (The full proof will appear in [7]; the simpler case of a given metric, corresponding to motion in an external gravitational field, is treated in [8], see the announcement [9].) To obtain a precise mathematical problem it is necessary to specify the type of initial data and solution which corresponds to a particle as well as the limiting process corresponding to the test particle limit. This is the purpose of the next section.

## 2. Solitons and the test-particle limit

In the case $\mathbf{g} = \eta = -\mathrm{d}t^2 + \sum_{i=1}^{3}(\mathrm{d}x^i)^2$, the Minkowski metric, and $\mathcal{V}(\phi) = \frac{m^2}{2}|\phi|^2 - G(|\phi|)$ the second equation of (1) becomes

$$\partial_t^2\phi - \Delta\phi + m^2\phi = \beta(|\phi|)\phi, \tag{3}$$

where $G'(s) = \beta(s)s$. For a large class of $G$ this equation admits a class of solitary wave solutions called *non-topological solitons* of the form: $\phi(t,x) = \mathrm{e}^{\mathrm{i}\omega t} f_\omega(x)$, with $f_\omega$ a positive, exponentially decreasing radial function and $0 \leqslant \omega^2 < m^2$. Existence, uniqueness and stability are well understood (see [1,5,3,6] and references therein). Application of the Poincaré group gives an 8-dimensional family of solutions parametrised (at each time) by frequency $\omega$, phase $\theta$, centre $\xi$ and momentum $p$ (or velocity). The stability of these solutions is determined entirely by $\omega$: the subset $\mathbf{I} = \{\omega\colon \frac{\partial}{\partial\omega}(-\omega\|f_\omega\|_{L^2}^2) > 0\} \subset (-m, m)$ is non-empty for some but not all $G$, and $\omega \in \mathbf{I}$ ensures modulational stability in the sense given in [6]. This characterisation first appeared in work of Strauss and Shatah (referenced in [3]).

Since the energy is exponentially localised in $x$ about $x = \xi$ the non-topological solitons can be regarded as particle-like and hence to provide an appropriate context in which to study the test-particle limit. As discussed, above this limit involves two small parameters: the particle size and the energy (or amplitude). Correspondingly it is necessary to introduce a two parameter family of potentials

$$\mathcal{V}_{\varepsilon,\delta}(\phi) = \delta^2\varepsilon^{-2}\mathcal{V}\big(\delta^{-1}\phi\big). \tag{4}$$

Under this scaling (1) becomes

$$\Box_{\mathbf{g}}\phi = \delta\varepsilon^{-2}\mathcal{V}'\big(\delta^{-1}\phi\big), \qquad R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi\mathbf{G}T_{\mu\nu}(\mathbf{g},\phi;\mathcal{V}_{\varepsilon,\delta}) \tag{5}$$

to be studied in the limit of small $\varepsilon,\ \delta$ (see below). Observe that, in the flat case $\mathbf{g} = \eta$, a function $\phi(t,x)$ solves (3) if and only if $\delta\phi(t/\varepsilon, x/\varepsilon)$ solves the first equation in (5), so that $\varepsilon$ determines the size of the soliton and $\delta$ its amplitude. (Also note that the scaling of the potential in (4) is equivalent to rescaling the metric $\mathbf{g} \mapsto \varepsilon^{-2}\mathbf{g}$ and Newton's constant $\mathbf{G} \mapsto \delta^2\mathbf{G}$.)

The aim will be to show that given a vacuum solution to (5) (i.e., a solution with $\phi \equiv 0$) there is, for $\varepsilon, \delta$ small, a nearby solution in which $\delta^{-1}\phi$ is close to a non-topological soliton concentrated along a curve which is close to a time-like geodesic. It is to be expected that restrictions on the manner in which $\varepsilon, \delta$ tend to zero will arise: in physical terms for geodesic motion it is necessary not only that the energy and size of the test particle approach zero but also that the energy density approach zero at a rate depending upon $\varepsilon$. In this regard the theorem will be proved under the assumption that $\varepsilon \to 0$ with

$$\delta \leqslant C\varepsilon^q \tag{6}$$

for certain $q$ to be specified. Analytically this type of restriction arises as follows:

(i) the geodesic equation itself involves first derivatives of the metric coefficients, so that it is to be expected that at least $C^1$ control of the deformation of the metric is required, which in turn follows from $H^s$ control, $s > 2.5$ which can be deduced from an estimate for $T_{\mu\nu}$ in $H^{s-1}$;
(ii) the standard local existence theory for the Einstein equation is for $\mathbf{g}$ in $H^s$, $s > 2.5$, so that uniform estimates of $T_{\mu\nu} \in H^{s-1}$ will be required. It thus turns out that any improvements in local existence theory (i.e., reducing the value of $s$ for which well-posedness holds) would probably not impact the present problem on account of the restriction entailed by (i).

In fact the present derivation of *uniform in* $\varepsilon, \delta$ estimates for $\phi \in H^s$ requires $s \geqslant 3 > 2.5$ which leads to the hypothesis $q \geqslant 7/4$ in the main theorem below, in contrast to the presumably optimal condition $q > 3/2$.

## 3. Results

Assume given a space time $\mathbb{M} = [0, t_0] \times \mathbb{R}^3$, foliated by space-like hypersurfaces $\Sigma_t \equiv \{x^0 = t = \mathit{constant}\} \approx \mathbb{R}^3$, with metric of the form $\boldsymbol{\gamma} = \gamma_{00}\,\mathrm{d}t^2 + \gamma_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j$ with $\gamma_{ij}$ the induced Riemannian metric on the space-like hypersurfaces given by the level sets of $t$. Assume further that $\boldsymbol{\gamma}$ is a solution of the vacuum Einstein equation, (i.e., it is Ricci flat $R_{\mu\nu}(\boldsymbol{\gamma}) = 0$), with regularity $\partial\boldsymbol{\gamma}(t) \in H^3(\Sigma_t)$ and there exists a constant $K_0$ such that

$$K_0^{-2} \leqslant -\gamma_{00} \leqslant K_0^2, \qquad K_0^{-2}\delta_{ij} \leqslant \gamma_{ij} \leqslant K_0^2\delta_{ij}.$$

Here and below $\partial$ stands for arbitrary space–time partial derivative.

The aim is to construct perturbatively a solution of (5) with $\delta^{-1}\phi$ close to a rescaled non-topological soliton. To this end consider a smooth potential function $\mathcal{V}(\phi)$ as above with the property that the corresponding wave equation (3) admits soliton solutions $f_\omega(x)\,\mathrm{e}^{i\omega t}$ which are stable in the sense of [6] for $\omega = \omega_0$, i.e., $\omega_0 \in \mathbf{I}$. The following definition gives a starting point for the perturbative construction, obtained by "freezing the coefficients". This means to construct a soliton centered at $\mathbf{x}(t)$ we evaluate the metric coefficients there: $q(t) = |\gamma_{00}(t, \mathbf{x}(t))|^{1/2}$, $h_{ij}(t) = \gamma_{ij}(t, \mathbf{x}(t))$. So $h_{ij}(t)$ is the inner product on $T_{\mathbf{x}(t)}\Sigma_t$, with corresponding norm $|\cdot|_h$; also write $h^{ij}(t)$ for the induced inner product on the cotangent space $T^*_{\mathbf{x}(t)}\Sigma_t$ (represented by the inverse matrix).

**Definition 3.1.** Given the metric $\boldsymbol{\gamma}$ and a curve $t \mapsto (\mathbf{x}(t), \mathbf{p}(t)) \in T^*\Sigma$ and a function $t \mapsto (\omega(t), \theta(t)) \in \mathbb{R}^2$ a soliton centred at $\mathbf{x}(t)$ with momentum $\mathbf{p}(t)$, frequency $\omega(t)$ and phase $\theta(t)$ is defined by the formula

$$\phi_S^\varepsilon(t, x) \equiv f_\omega\bigl(\varepsilon^{-1}\bigl|\overline{\gamma} P_t(x - \mathbf{x}) + Q_t(x - \mathbf{x})\bigr|_h\bigr) \exp\bigl[\mathrm{i}\varepsilon^{-1}\bigl(\theta - \omega\mathbf{p}_j(x - \mathbf{x})^j\bigr)\bigr], \tag{7}$$

where $P_t$ is the projection operator along the direction defined by the velocity $u^j(t) = q(t)h^{jk}(t)\mathbf{p}_k(t)/\overline{\gamma}(t)$ and $\overline{\gamma}(t)$ is the Lorentz contraction factor $\overline{\gamma}(t) = \sqrt{1 + \mathbf{p}_j(t)\mathbf{p}_k(t)h^{jk}(t)}$. Finally $Q_t$ is the orthogonal complement of $P_t$ with respect to the inner product defined by the metric at the point $\mathbf{x}(t)$.

**Remark.** This is a definition which is motivated by the fact that if $\boldsymbol{\gamma}$ is a flat constant metric and $\mathbf{x} = ut$ there is an exact solution of this form given by application of a Lorentz transformation to the basic solution $\phi(t, x) = \mathrm{e}^{\mathrm{i}\omega t} f_\omega(x)$.

The next theorem asserts the existence of solutions close to $\phi_S^\varepsilon$ in (5) with $\mathbf{g}$ close to $\boldsymbol{\gamma}$ as long as $\mathbf{x}(t)$ is close to a time-like geodesic. Thus let the curve $t \mapsto (\boldsymbol{\xi}(t), \boldsymbol{\pi}(t)) \in T^*\Sigma_t$ be a time-like geodesic with respect to $\boldsymbol{\gamma}$ (lifted to the contangent bundle). Given a (stable) frequency $\omega_0 \in \mathbf{I}$ and any $\theta_0$ define

$$\phi_0^\varepsilon(x) \equiv f_{\omega_0}\big(\varepsilon^{-1}\big|\overline{\gamma}P_0\big(x-\boldsymbol{\xi}(0)\big)+Q_0\big(x-\boldsymbol{\xi}(0)\big)\big|_{h_0}\big)\exp\big[\mathrm{i}\varepsilon^{-1}\big(\theta_0-\boldsymbol{\pi}(0)_j\big(x-\boldsymbol{\xi}(0)\big)^j\big)\big] \tag{8}$$

with $u_0,\ P_0,\ Q_0,\ \overline{\gamma}_0,\ q_0,\ h_0$ defined in the same way as $u(t),\ P_t,\ Q_t,\ \overline{\gamma}(t),\ q(t),\ h(t)$ above with $\boldsymbol{\xi}(0),\ \boldsymbol{\pi}(0)$ used instead of $\mathbf{x}(t), \mathbf{p}(t)$. Consider the initial value problem for (5) with the following initial data:

- $\delta^{-1}(\phi(0), \partial_t\phi(0))$ bounded in $H_\varepsilon^3 \times H_\varepsilon^2$ and with
$$\big\|\big(\delta^{-1}\phi(0)-\phi_0^\varepsilon, \varepsilon\delta^{-1}\partial_t\phi+\varepsilon u_0\cdot\partial_x\phi_0^\varepsilon-\mathrm{i}\omega_0 q_0\phi_0^\varepsilon/\overline{\gamma}_0\big)\big\|_{H_\varepsilon^1\times L_\varepsilon^2} \leqslant c_1\varepsilon.$$
Here $H_\varepsilon^s$ is the scaled Sobolev norm $\|f\|^2_{H_\varepsilon^s} = \sum_{|\alpha|=0}^s \varepsilon^{2|\alpha|-3}\|\partial^\alpha f\|^2_{L^2}$.
- $(\mathbf{g}(0), \partial_t\mathbf{g}(0))$ such that $\|\partial(\mathbf{g}-\boldsymbol{\gamma})\|_{H^2} \leqslant c_2\varepsilon$ at $t=0$ and the constraint equations are satisfied.

**Theorem 3.2.** *Assume* (6) *holds for $q \geqslant 7/4$ and $t \mapsto (\boldsymbol{\xi}(t), \boldsymbol{\pi}(t)) \in T^*\Sigma_t$ is a time-like geodesic. Then there exist positive numbers $c_*,\ t_*,\ \varepsilon_*$ such that for $\varepsilon < \varepsilon_*$ the solution to* (5) *with the initial data just described exists for $0 \leqslant t \leqslant t_*$, where $t_*$ is independent of $\varepsilon$, and there exists $t \mapsto (\omega, \theta, \mathbf{x}, \mathbf{p})(t) \in C^1([0, t_*]; \mathbb{R}^2 \times T^*\Sigma)$ such that, with $\phi_S^\varepsilon$ as above,*

$$\max_{0\leqslant t\leqslant t_*}\big[\big\|\partial(\mathbf{g}-\boldsymbol{\gamma})\big\|_{H^2(\Sigma_t)}+\big\|\delta^{-1}\phi-\phi_S^\varepsilon\big\|_{H_\varepsilon^1(\Sigma_t)}+\varepsilon\big\|\partial_t\big(\delta^{-1}\phi-\phi_S^\varepsilon\big)\big\|_{L_\varepsilon^2(\Sigma_t)}$$
$$+\big|\mathbf{x}(t)-\boldsymbol{\xi}(t)\big|+\big|\mathbf{p}(t)-\boldsymbol{\pi}(t)\big|+\big|\omega(t)-\omega_0\big|\big] \leqslant c_*\varepsilon.$$

**Remarks.** (i) The nonlinear wave equation for $\phi$ is treated using a generalisation (to allow perturbation of the metric) of the symplectic modulation approach [6,8] combined with new estimates for $(\phi, \partial_t\phi)$ in higher Sobolev norms uniform in $\varepsilon$ as $\varepsilon \to 0$. It is here that the restriction $q \geqslant 7/4$ comes in, as opposed to the range $q > 3/2$ which is presumed to be optimal in view of the remarks at the end of Section 2.

(ii) The assumption that the initial data satisfy the constraint equations is easily satisfied since in this situation the constraint equations can be solved using the implicit function theorem [2].

(iii) The Einstein equation is treated using the gauge condition $g^{\alpha\beta}(\Gamma^\mu_{\alpha\beta}(\mathbf{g})-\Gamma^\mu_{\alpha\beta}(\boldsymbol{\gamma}))=0$ to obtain a hyperbolic equation for $\mathbf{g}-\boldsymbol{\gamma}$ (essentially as in the original existence theorem of Choquet-Bruhat) with the sharpening of the estimates observed by [4].

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