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# The incompressible limit of solutions of the two-dimensional compressible Euler system with degenerating initial data

# Limite incompressible de solutions du système d'Euler compressible correspondant à des données initiales dont la régularité dégénère

Alexandre Dutrifoy [a], Taoufik Hmidi [b]

[a] *Laboratoire Jacques-Louis Lions, Université Pierre et Marie Curie, boîte courrier 187, 75252 Paris cedex 05, France*
[b] *Centre de mathématiques, École polytechnique, 91128 Palaiseau cedex, France*

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**Abstract**

Using Strichartz estimates, it is possible to pass to the limit in the weakly compressible 2-D Euler system, when the Mach number $\varepsilon$ tends to zero, even if the initial data are not uniformly smooth. This leads to results of convergence to solutions of the incompressible Euler system whose regularity is critical, such as vortex patches or Yudovich solutions. ***To cite this article: A. Dutrifoy, T. Hmidi, C. R. Acad. Sci. Paris, Ser. I 336 (2003).***
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**Résumé**

En utilisant des inégalités de Strichartz, il est possible de passer à la limite dans le système d'Euler compressible 2-D, quand le nombre de Mach tend vers zéro, même si les données initiales ne sont pas uniformément régulières. Ceci mène à des résultats de convergence vers des solutions du système d'Euler incompressible dont la régularité est critique, comme des poches de tourbillon ou des solutions de Yudovich. ***Pour citer cet article : A. Dutrifoy, T. Hmidi, C. R. Acad. Sci. Paris, Ser. I 336 (2003).***
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## 1. Introduction

We consider a weakly compressible, isentropic fluid without viscosity, extended in the whole space $\mathbb{R}^2$. The Mach number, here a small positive parameter, is noted $\varepsilon$; eventually, we let $\varepsilon$ tend to 0. The state of the fluid is

*E-mail addresses:* dutrifoy@ann.jussieu.fr (A. Dutrifoy), hmidi@daphne.math.polytechnique.fr (T. Hmidi).

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described by the velocity field $v_\varepsilon(t,x) = \bar\gamma c_0 \tilde v_\varepsilon(\bar\gamma c_0 t, x)$ and the speed of sound $c_\varepsilon(t,x) = c_0 + \varepsilon\bar\gamma c_0 \tilde c_\varepsilon(\bar\gamma c_0 t, x)$, where $c_0$ and $\bar\gamma$ are positive constants. The scaled unknowns $\tilde v_\varepsilon$ and $\tilde c_\varepsilon$ then satisfy

$$\begin{cases} \partial_t \tilde v_\varepsilon + \tilde v_\varepsilon \cdot \nabla \tilde v_\varepsilon + \bar\gamma \tilde c_\varepsilon \nabla \tilde c_\varepsilon + \frac{1}{\varepsilon}\nabla \tilde c_\varepsilon = 0, \\ \partial_t \tilde c_\varepsilon + \tilde v_\varepsilon \cdot \nabla \tilde c_\varepsilon + \bar\gamma \tilde c_\varepsilon \operatorname{div} \tilde v_\varepsilon + \frac{1}{\varepsilon}\operatorname{div} \tilde v_\varepsilon = 0, \\ (\tilde v_\varepsilon, \tilde c_\varepsilon)|_{t=0} = (\tilde v_{0,\varepsilon}, \tilde c_{0,\varepsilon}). \end{cases} \tag{1}$$

We are interested in the convergence of solutions of (1) to solutions of the incompressible Euler system. This problem, of course, has already been studied in numerous articles; our reference list only contains a few of them [2,6,7].

The originality of this work is that we allow the initial data to be so ill-prepared that corresponding solutions can tend to a vortex patch or even a Yudovich solution. Precise statements require some preparation, but we can present right away a particular case of our results.

**Proposition 1.1.** *Suppose that* $(\tilde v_{0,\varepsilon}, \tilde c_{0,\varepsilon}) - (\tilde v_0, \tilde c_0) \to 0$ *in* $L^2(\mathbb{R}^2)$, *that* $\|\operatorname{rot} \tilde v_{0,\varepsilon}\|_{L^a\cap L^\infty} \leqslant C$, *where* $a < \infty$, *that* $\|\operatorname{rot} \tilde v_{0,\varepsilon}\|_{H^{s+1}} \leqslant C\exp((\ln \varepsilon^{-1})^\alpha)$ *and that* $\|(\tilde v_{0,\varepsilon}, \tilde c_{0,\varepsilon})\|_{H^{s+11/4}} \leqslant C\varepsilon^{-\beta}$, *for some* $s \in ]0,1[$, $\alpha < 1$ *and* $\beta < 1/8$.

*Then the times of existence for* (1) *tend to* $+\infty$; *the incompressible parts of the solutions tend to a Yudovich solution of the Euler system whose initial curl is* $\operatorname{rot}\tilde v_0$, *and their compressible and acoustic parts tend to zero in* $L^1_{\mathrm{loc}}(\mathbb{R}^+; \mathrm{Lip})$.

## 2. Reformulation of the problem

We consider $(\tilde v_\varepsilon^1, \tilde v_\varepsilon^2)$ in some space $\sigma' + L^2(\mathbb{R}^2)$, where $\sigma'(x^1,x^2) = (-x^2, x^1)\int_0^{|x|}\tau g(\tau)\,\mathrm{d}\tau/|x|^2$, with $g \in C_0^\infty(\mathbb{R}_0^+)$ – the existence of Yudovich solutions, indeed, is most naturally shown in those spaces [1]. If we set $\sigma = (\sigma'^1, \sigma'^2, 0)$, $U_\varepsilon = (\tilde v_\varepsilon^1, \tilde v_\varepsilon^2, \tilde c_\varepsilon) - \sigma$ and $U'_\varepsilon = (U_\varepsilon^1, U_\varepsilon^2)$, then (1) may be rewritten

$$\begin{cases} \partial_t U_\varepsilon + (U'_\varepsilon + \sigma')\cdot\nabla(U_\varepsilon + \sigma) + \left(\bar\gamma U_\varepsilon^3 + \frac{1}{\varepsilon}\right)B(D)^t U_\varepsilon = 0, \\ U_\varepsilon|_{t=0} = U_{0,\varepsilon}, \end{cases} \tag{2}$$

where

$$B(D) = \begin{pmatrix} 0 & 0 & \partial_1 \\ 0 & 0 & \partial_2 \\ \partial_1 & \partial_2 & 0 \end{pmatrix}.$$

Solving (1) in $\sigma + B^s_{2,q}$ is equivalent to solving (2) in $B^s_{2,q}$, for any $q \geqslant 1$ and $s \in \mathbb{R}$. (Since we always assume $s > 2$, doing this locally in time is not a problem, essentially because (2) is symmetric and all derivatives of $\sigma$ belong to $L^2$.)

## 3. Dispersive effects

Orthonormal eigenvectors of $B(\xi)$ are $V_0(\xi) = (-\xi^2, \xi^1, 0)/|\xi|$, corresponding to the eigenvalue 0, and $V_{\pm 1}(\xi) = (\pm\xi^1, \pm\xi^2, |\xi|)/(\sqrt{2}\,|\xi|)$, corresponding to the eigenvalues $\pm\mathrm{i}|\xi|$.

We shall use the projectors $P_0 = V_0(D)^t V_0(D)$ and $P_{\pm 1} = V_{\pm 1}(D)^t V_{\pm 1}(D)$. If $U$ is a vector field, $P_0 U$ is its incompressible part and $(P_1 + P_{-1})U$ is the sum of its compressible and acoustic parts.

Since

$$\begin{cases} \partial_t P_{\pm 1}U_\varepsilon \pm \frac{\mathrm{i}}{\varepsilon}|D| P_{\pm 1}U_\varepsilon = -P_{\pm 1} I_\varepsilon, \\ (P_{\pm 1}U_\varepsilon)|_{t=0} = P_{\pm 1}U_{0,\varepsilon}, \end{cases}$$

with

$$I_\varepsilon = (U'_\varepsilon + \sigma')\cdot\nabla(U_\varepsilon + \sigma) + \bar\gamma U_\varepsilon^3 B(D) U_\varepsilon,$$

we have for $P_{\pm 1}U_\varepsilon$ Strichartz estimates similar to those valid for solutions of the wave equation [5] – as, for example,

$$\|P_{\pm 1}U_\varepsilon\|_{L^4_t(C^{s+1})} \lesssim \varepsilon^{1/4}\left(\|P_{\pm 1}U_{0,\varepsilon}\|_{B^{s+7/4}_{2,q}} + \int_0^t \left(C_0 + C_0\left\|U_\varepsilon(t')\right\|_{B^{s+11/4}_{2,q}} + \left\|U_\varepsilon(t')\right\|_{\mathrm{Lip}}\left\|U_\varepsilon(t')\right\|_{B^{s+11/4}_{2,q}}\right)\mathrm{d}t'\right),$$

for $q \in [1,4]$ and $s \in ]0,1[$. The results presented just below are obtained thanks to the power of $\varepsilon$ in the right member of such inequalities. On the other hand, unfortunately, the presence of derivatives of $U_\varepsilon$ in $I_\varepsilon$ forces us to assume more smoothness on the data than one might wish to.

## 4. Results

The first theorem is a statement about the growth of the lifespans.

Although we are in dimension 2, the formalism of 3-D vortex patches [4], slightly adapted, is convenient. So we say that a system of continuous vector fields $W = \{w^\nu;\ \nu = 1,\dots,N\}$ is *admissible* if $[W]^{-1} \stackrel{\text{def}}{=} (\frac{1}{N}\sum_{\nu=1}^N |w^\nu|)^{-1}$ is bounded.

**Theorem 4.1.** *Let $W_0 = \{w_0^\nu;\ \nu = 1,\dots,N\}$ be an admissible system of $C^s$ vector fields, with $s \in ]0,1[$. Let $\Omega_{0,\varepsilon} = \partial_1\tilde v^2_{0,\varepsilon} - \partial_2\tilde v^1_{0,\varepsilon}$ denote the* curl *of $U'_{0,\varepsilon} + \sigma'$. Suppose that for some constant $C_0$ greater than*

$$\|\sigma'\cdot\nabla\sigma\|_{B^{s+11/4}_{2,q}} + \|\nabla\sigma\|_{B^{s+11/4}_{2,q}} + \|\sigma\|_{C^{s+11/4}} + 1,$$

*the initial data of* (2) *are bounded as follows*:

$$\|U_{0,\varepsilon}\|_{L^2} \leqslant C_0, \qquad \|\Omega_{0,\varepsilon}\|_{L^\infty} + \left\|[W_0]^{-1}\right\|_{L^\infty} + \sum_{\nu=1}^N \left\|w_0^\nu\right\|_{C^s} \leqslant C_0,$$

$$\sum_{\nu=1}^N \left\|\operatorname{div}\left(w_0^\nu\Omega_{0,\varepsilon}\right)\right\|_{C^{s-1}} \leqslant C_\varepsilon, \qquad \|U_{0,\varepsilon}\|_{B^{11/4}_{2,1}} \leqslant C_0\varepsilon^{-\alpha}, \qquad \|U_{0,\varepsilon}\|_{B^{s+11/4}_{2,q}} \leqslant C_0\varepsilon^{-\beta},$$

*with $q \in [1,4]$ and $0 \leqslant \alpha \leqslant \beta < 1/4$, and where $C_\varepsilon$ is an arbitrary, non-increasing function of $\varepsilon$.*

*Then, for all $\mu \in ]0, \frac{1}{4} - \alpha[$, the lifespan $T_\varepsilon$ of the solution $U_\varepsilon$ is bounded from below by*

$$T_\varepsilon^{(\mu)} = \min\left(C_0^{-8/3}\varepsilon^{4/3(\beta-1/4)}, \frac{1}{CC_0}\ln\frac{\ln(\varepsilon^{\alpha+\mu-1/4})}{\ln(\mathrm{e}+C_\varepsilon)} - 1\right).$$

*Moreover,*

$$\int_0^t \left\|U_\varepsilon(t')\right\|_{\mathrm{Lip}}\mathrm{d}t' \leqslant \mathrm{e}^{CC_0(t+1)}\ln(\mathrm{e}+C_\varepsilon) \tag{3}$$

*for all $t \in [0, T_\varepsilon^{(\mu)}]$, and*

$$\|P_{\pm 1}U_\varepsilon\|_{L^1_{T_\varepsilon^{(\mu)}}(\mathrm{Lip})} \leqslant \varepsilon^\mu. \tag{4}$$

This theorem is interesting if $C_\varepsilon$ does not grow too fast as $\varepsilon \to 0$, so that $T_\varepsilon^{(\mu)} \to \infty$. In particular, it applies neatly if the data are regularizations of a vortex patch, because then $C_\varepsilon \leqslant C_0'$ for some constant $C_0'$ independent of $\varepsilon$. (If the data are regularizations of a field without tangential regularity, as in the case of general Yudovich solutions, we can still gain control on $C_\varepsilon$ by reducing the speed of regularization.)

Our convergence result is described by a second theorem.

**Theorem 4.2.** *In addition to the hypotheses of Theorem* 4.1, *suppose that* $P_0 U_{0,\varepsilon} \to U_0$ *in* $L^2(\mathbb{R}^2)$, *that* $\|\Omega_{0,\varepsilon}\|_{L^a} \leqslant C_0$ *for some* $a < \infty$, *that*

$$\frac{\ln(\varepsilon^{-1})}{\ln(\mathrm{e} + C_\varepsilon)} \to +\infty \quad \text{as } \varepsilon \to 0,$$

*and that*

$$\|U_{0,\varepsilon}\|_{H^1} \leqslant C_0 \varepsilon^{-\gamma} \quad \text{and} \quad \|U_{0,\varepsilon}\|_{B^{7/4}_{2,1}} \leqslant C_0 \varepsilon^{-\delta}$$

*with* $\gamma \leqslant \delta \leqslant \alpha$ *and* $\gamma + \delta < 1/4$.

*Then* $\sigma' + (P_0 U_\varepsilon)'$ *converges in* $L^\infty_{\mathrm{loc}}(\mathbb{R}^+; \sigma' + L^2)$ *to the unique solution* $v \in C(\mathbb{R}^+; \sigma' + L^2)$ *of the incompressible Euler system*

$$\begin{cases} \partial_t v + v \cdot \nabla v = -\nabla p, \\ \operatorname{div} v = 0, \\ v|_{t=0} = \sigma' + U_0' \end{cases}$$

*such that* $p \in L^\infty_{\mathrm{loc}}(\mathbb{R}^+; L^2)$ *and* $\operatorname{rot} v \in L^\infty(\mathbb{R}^+; L^\infty \cap L^a)$.

## 5. Proofs

Detailed proofs for both theorems are avalaible in a preprint [3].

The proof of Theorem 4.1 is a mixing of two kinds of techniques. In order to estimate $\int_0^t \|P_0 U_\varepsilon(t')\|_{\mathrm{Lip}}\,\mathrm{d}t'$, we use the static, logarithmic estimate relating the Lipschitzian norm of a divergence free vector field to the striated regularity of its curl, and the dynamic estimates describing the evolution of this striated regularity [4]. Then we can dispose of all compressible and acoustic quantities thanks to dispersive estimates (see Section 3) in which we also introduce inequalities proved by classical energy methods.

The proof of Theorem 4.2 is basically an adaptation of the proof of Yudovich's theorem as presented in the book of Jean-Yves Chemin [1].

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