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Mathematical Problems in Mechanics

Liquid-like behavior of shape memory alloys

Comportement liquide d'alliages à mémoire de forme

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Abstract

In this Note, we prove that the identity matrix is an inner point of the quasiconvex hull K^{qc} of a compact set $K \subset \{X \in \mathbb{M}^{3,3}\colon \det X = 1\}$ whenever K^{qc} contains a three-well configuration. This is in particular the case for the cubic to tetragonal and the cubic to orthorhombic phase transformations, and answers a question discussed in S. Müller, Microstructures, phase transitions and geometry, in: A. Balog et al. (Eds.), Proceedings European Congress of Mathematics, Progr. Math., Birkhäuser, 1998. ***To cite this article: G. Dolzmann, B. Kirchheim, C. R. Acad. Sci. Paris, Ser. I 336 (2003).***

Résumé

Dans cette Note, nous démontrons que la matrice identité est un point intérieur de l'envelope quasiconvexe K^{qc} d'un ensmble compact $K \subset \{X \in \mathbb{M}^{3,3}\colon \det X = 1\}$ lorsque K^{qc} contient une configuration de type triple puits. Ceci est le cas en particulier pour les transformations de phase de cubique à tétragonal et de cubique à orthorhombique, et répond à une question discutée dans S. Müller, Microstructures, phase transitions and geometry, in: A. Balog et al. (Eds.), Proceedings European Congress of Mathematics, Progr. Math., Birkhäuser, 1998. ***Pour citer cet article : G. Dolzmann, B. Kirchheim, C. R. Acad. Sci. Paris, Ser. I 336 (2003).***

Version française abrégée

Lors d'expériences de traction, des alliages à mémoire de forme subissant une transformation de phase solide–solide présentent un réponse élastique molle pour des taux de déformation faibles. Dans le cadre de modèles pour de tels matériaux, basés sur des intégrales variationelles avec une densité d'énergie libre positive adéquate W, la gamme de comportements idéalement mous est caractérisée par l'envelope quasiconvexe K^{qc} de l'ensemble nul

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K de W (voir par exemple [1,5] pour les notations importantes). Notre principal résultat indique que la matrice identité est un point intérieur de l'envelope quasiconvexe pour la transition de phase de cubique vers tétragonal, moyennant la contrainte que les déformations préservent le volume, ce qui découle du fait que les phases sont liées par symétries. Ce résultat prédit un comportement liquide du matériau, c'est-à-dire que le matériau ne résiste pas aux déformations petites. Plus généralement, nous démontrons le théorème suivant :

Théorème 0.1. *Soit* $K \subset \{X \in \mathbb{M}^{3,3}\colon \det X = 1\}$ *un compact et supposons que* K^{qc} *contienne une configuration de type triple-puits* (1) *avec* $\alpha > 0$, $\alpha \neq 1$. *Alors* $\mathrm{Id} \in \operatorname{rel\,int} K^{\mathrm{qc}}$.

L'ingrédient principal pour démontrer ce résultat est une subtile séparation de la matrice $X \in \mathbb{M}^{2,2}$.

Proposition 0.2. *Soit* $X \in \mathbb{M}^{2,2}$ *tel que* $\|X - \mathrm{Id}\| < \sqrt{2}/3$. *Alors il existe quatre vecteurs* $\boldsymbol{u}_i, \boldsymbol{v}_i \in \mathbb{R}^2$, $i = 1, 2$, *tels que*

(i) $\det(X + t\boldsymbol{u}_i \otimes \boldsymbol{v}_i) = \det X$ *pour* $t \in \mathbb{R}$;
(ii) $X - \boldsymbol{u}_i \otimes \boldsymbol{v}_i$ *est conforme, c'est-à-dire que* $X - \boldsymbol{u}_i \otimes \boldsymbol{v}_i = \sqrt{\det X}\, R_i$ *avec* $R_i \in \mathrm{SO}(2)$;
(iii) $|\langle \boldsymbol{u}_1, \boldsymbol{u}_2\rangle| \leqslant |\boldsymbol{u}_1||\boldsymbol{u}_2|/2$, *c'est-à-dire que l'angle entre* $\boldsymbol{u}_1$ *et* $\boldsymbol{u}_2$ *est uniformément borné inférieurement.*

Afin de démontrer cette proposition, nous supposons que $K = K_{\mathrm{ct}}$ dans (1). Supposons que $\|X - \mathrm{Id}\| \leqslant \varepsilon$. Alors il existe une rotation $Q \in \mathrm{SO}(3)$ telle que

$$QX = \begin{pmatrix} \widehat{X} & \hat{\boldsymbol{u}} \\ \boldsymbol{0}^{\mathrm{T}} & 1/\Delta \end{pmatrix}, \quad \|Q - \mathrm{Id}\| = \mathcal{O}(\varepsilon), \quad |\Delta - 1| = \mathcal{O}(\varepsilon), \quad |\hat{\boldsymbol{u}}| = \mathcal{O}(\varepsilon), \quad \big\|\widehat{X} - \sqrt{\Delta}\,\widehat{\mathrm{Id}}\big\| = \mathcal{O}(\varepsilon).$$

Soit $\tilde{\boldsymbol{u}}_i = \pm \boldsymbol{u}_i/|\boldsymbol{u}_i|$, $\boldsymbol{u} = \gamma_1 \tilde{\boldsymbol{u}}_1 + \gamma_2 \tilde{\boldsymbol{u}}_2$ avec $\gamma_i \geqslant 0$, et $\boldsymbol{w}_i = (\gamma_1 + \gamma_2)\tilde{\boldsymbol{u}}_i$. Alors $\gamma_i, |\boldsymbol{w}_i| = \mathcal{O}(\varepsilon)$, et

$$\begin{pmatrix} \widehat{X} & \hat{\boldsymbol{u}} \\ \boldsymbol{0}^{\mathrm{T}} & 1/\Delta \end{pmatrix} = \frac{\gamma_1}{\gamma_1+\gamma_2}\begin{pmatrix} \widehat{X} & \boldsymbol{w}_1 \\ \boldsymbol{0}^{\mathrm{T}} & 1/\Delta \end{pmatrix} + \frac{\gamma_2}{\gamma_1+\gamma_2}\begin{pmatrix} \widehat{X} & \boldsymbol{w}_2 \\ \boldsymbol{0}^{\mathrm{T}} & 1/\Delta \end{pmatrix} = \frac{\gamma_1}{\gamma_1+\gamma_2}P_1 + \frac{\gamma_2}{\gamma_1+\gamma_2}P_2.$$

Puisque K^{qc} est fermé par lamination [5] et $\operatorname{rank}(P_1 - P_2) = 1$, il suffit de montrer que $P_i \in K^{\mathrm{qc}}$. Or

$$P_i = \frac{1}{2}\begin{pmatrix} \widehat{X} - \boldsymbol{u}_i \otimes \boldsymbol{v}_i & 2\boldsymbol{w}_i \\ \boldsymbol{0}^{\mathrm{T}} & 1/\Delta \end{pmatrix} + \frac{1}{2}\begin{pmatrix} \widehat{X} + \boldsymbol{u}_i \otimes \boldsymbol{v}_i & \boldsymbol{0} \\ \boldsymbol{0}^{\mathrm{T}} & 1/\Delta \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \sqrt{\Delta}R_i & 2\boldsymbol{w}_i \\ \boldsymbol{0}^{\mathrm{T}} & 1/\Delta \end{pmatrix} + \frac{1}{2}Z_i.$$

Soit Y_i défini par $2P_i = Y_i + Z_i$. Alors $\operatorname{rank}(Y_i - Z_i) = 1$ et il suffit de montrer que $Y_i, Z_i \in K^{\mathrm{qc}}$. Ceci peut être fait par des constructions explicites basées sur le résultat bidimensionnel [1,3]. On utilise le fait que chaque couple de puits dans (1) correspond essentiellement à une situation bidimensionelle, puisque les matrices U_i et U_j ont une valeur propre commune avec le même vecteur propre.

Proposition 0.3. *Supposons que* $K = \mathrm{SO}(2)\operatorname{diag}(\lambda, \mu) \cup \mathrm{SO}(2)\operatorname{diag}(\mu, \lambda)$ *avec* $0 < \mu < 1 < \lambda$. *Alors*

$$K^{\mathrm{qc}} = \big\{F \in \mathbb{M}^{2,2}\colon \det F = \mu\lambda,\ \big|F(\boldsymbol{e}_1 \pm \boldsymbol{e}_2)\big|^2 \leqslant \lambda^2 + \mu^2\big\},$$

où $\{\boldsymbol{e}_1, \boldsymbol{e}_2\}$ *est la base canonique dans* $\mathbb{R}^2$. *Ainsi* $\operatorname{diag}(s,t) \in K^{\mathrm{qc}}$ *si et seulement si* $\mu \leqslant |s|, |t| \leqslant \lambda$, $st = \lambda\mu$, *et la matrice* $\sqrt{\lambda\mu}\,\mathrm{Id}$ *est un point intérieur de* K^{qc}.

1. Introduction and statement of the results

Shape memory alloys undergoing solid to solid phase transformations show in stretching experiments a soft elastic response for small strains. The reason for these soft modes is the ability of the material to accommodate small deformations by a rearrangement of the different phases in different parts of the solid. In the framework of the models based on nonlinear elasticity and minimization of a free energy functional with a suitable nonnegative free energy density W, the range of ideally soft behavior is characterized by the quasiconvex hull K^{qc} of the zero set K of W (see, e.g., [1,5] for the relevant notations). The results in [2] characterize the phase transformations for which the identity matrix Id is contained in K^{qc} (self-accommodation). In this Note, we prove that the identity matrix is an inner point of the quasiconvex hull for the cubic to tetragonal phase transition for which the zero set of the energy density is given by

$$K_{\mathrm{ct}} = \mathrm{SO}(3)\,\mathrm{diag}\left(\alpha^2, \frac{1}{\alpha}, \frac{1}{\alpha}\right) \cup \mathrm{SO}(3)\,\mathrm{diag}\left(\frac{1}{\alpha}, \alpha^2, \frac{1}{\alpha}\right) \cup \mathrm{SO}(3)\,\mathrm{diag}\left(\frac{1}{\alpha}, \frac{1}{\alpha}, \alpha^2\right), \quad \alpha > 0,\ \alpha \neq 1, \tag{1}$$

relative to the constraint that the deformations are volume preserving which follows from the fact that the phases are symmetry related. More generally, this result applies to all phase transformations for which the quasiconvex hull of the set K contains a three-well configuration. For example, using the constructions in the proof of the main theorem, it is easy to see that this is the case for all self-accomodating transformations characterized in [2]. The precise description of K^{qc} remains a challenging open problem. A first step to get information about the quasiconvex hull is to characterize the polyconvex hull of K which provides an important outer bound. For the cubic to tetragonal phase transformation, a formula for K^{pc} was announced in [4].

The main theorem can be stated as follows.

Theorem 1.1. *Assume that $K \subset \{X\colon \det X = 1\}$ is compact and that K^{qc} contains a three-well configuration* (1) *with $\alpha > 0$, $\alpha \neq 1$. Then* $\mathrm{Id} \in \mathrm{rel\,int}\, K^{\mathrm{qc}}$.

In order to prove this result, it is sufficient to establish the following special case.

Theorem 1.2. *Assume that $\alpha > 0$, $\alpha \neq 1$ and that $K = K_{\mathrm{ct}}$ is given by* (1). *Suppose that $\varepsilon \in (0, 1/25)$ and that there exists a $\lambda \geqslant 1$ such that*

$$\left(138 + \frac{2}{3}\left(\lambda^2 - 1\right)\alpha^2\right)\varepsilon \leqslant (\alpha - 1)^2\left(\frac{\alpha + 1}{\alpha}\right)^2\left(1 - \frac{1}{\lambda^2}\right).$$

Then all $X \in \mathbb{M}^{3,3}$ with $\det(X) = 1$ and $\|X - \mathrm{Id}\| \leqslant \varepsilon$ are contained in K^{qc}. In particular, for $\alpha < 3/2$ we have

$$B\left(\mathrm{Id}, \frac{(\alpha - 1)^2}{62}\right) \cap \{X\colon \det X = 1\} \subset K^{\mathrm{qc}}.$$

2. Preliminaries

The proof of the theorem becomes more transparent if one identifies the vector $\boldsymbol{z} = (z_1, z_2)^{\mathrm{T}} \in \mathbb{R}^2$ with the complex number $z = z_1 + \mathrm{i}z_2$ and rewrites the 2×2 matrix X in terms of its conformal and anti-conformal part given by the complex numbers

$$X_{\mathrm{H}} = \frac{1}{2}\big(x_{11} + x_{22} + \mathrm{i}(x_{21} - x_{12})\big), \quad X_{\overline{\mathrm{H}}} = \frac{1}{2}\big(x_{11} - x_{22} + \mathrm{i}(x_{21} + x_{12})\big).$$

Then we can express the action of X as

$$X\boldsymbol{z} = X_{\mathrm{H}}z + X_{\overline{\mathrm{H}}}\bar{z} \quad \text{for all } \boldsymbol{z} \in \mathbb{R}^2 \tag{2}$$

and we notice that

$$\det X = |X_{\mathrm{H}}|^2 - |X_{\overline{\mathrm{H}}}|^2, \quad \|X\|^2 = 2\big(|X_{\mathrm{H}}|^2 + |X_{\overline{\mathrm{H}}}|^2\big), \quad \big(\mathrm{cof}\, X^{\mathrm{T}}\big)_{\mathrm{H}} = \overline{X_{\mathrm{H}}} \quad \text{and} \quad \big(\mathrm{cof}\, X^{\mathrm{T}}\big)_{\overline{\mathrm{H}}} = -X_{\overline{\mathrm{H}}}. \tag{3}$$

Proposition 2.1. *Suppose that $X \in \mathbb{M}^{2,2}$ satisfies $\|X - \mathrm{Id}\| < \sqrt{2}/3$. Then there exist vectors $\boldsymbol{u}_i, \boldsymbol{v}_i \in \mathbb{R}^2$, $i = 1, 2$, such that*

(i) $X_{\overline{\mathrm{H}}} = (\boldsymbol{u}_i \otimes \boldsymbol{v}_i)_{\overline{\mathrm{H}}}$;
(ii) $\det(X + t\boldsymbol{u}_i \otimes \boldsymbol{v}_i) = \det X$ *for* $t \in \mathbb{R}$;
(iii) $|\langle \boldsymbol{u}_1, \boldsymbol{u}_2\rangle| \leqslant |\boldsymbol{u}_1||\boldsymbol{u}_2|/2$.

Proof. Recall that $\det(X + t\boldsymbol{u} \otimes \boldsymbol{v}) = \det X + t \operatorname{cof} X : (\boldsymbol{u} \otimes \boldsymbol{v}) = \det X + t\langle \operatorname{cof} X^{\mathrm{T}}\boldsymbol{u}, \boldsymbol{v}\rangle$ and thus condition (ii) is satisfied if $\langle \operatorname{cof} X^{\mathrm{T}}\boldsymbol{u}, \boldsymbol{v}\rangle = 0$. Let J denote the counterclockwise rotation through $\pi/2$ which corresponds in view of our identifications with complex numbers to a multiplication by the imaginary unit i. We consider the following two possible choices for $\boldsymbol{v}$, namely

$$\boldsymbol{v} = \sigma J \operatorname{cof} X^{\mathrm{T}}\boldsymbol{u}, \quad \sigma \in \{-1, 1\}. \tag{4}$$

In complex notation this is by (2) and (3) equivalent to $v = \sigma\mathrm{i}(\overline{X_{\mathrm{H}}}u - X_{\overline{\mathrm{H}}}\bar{u})$ and since $(\boldsymbol{u} \otimes \boldsymbol{v})_{\overline{\mathrm{H}}} = \frac{1}{2}uv$, condition (i) can be rewritten as

$$2X_{\overline{\mathrm{H}}} = uv = \sigma\mathrm{i}\big(\overline{X_{\mathrm{H}}}u^2 - X_{\overline{\mathrm{H}}}|u|^2\big) \quad \text{or} \quad \frac{X_{\overline{\mathrm{H}}}}{\overline{X_{\mathrm{H}}}} = \frac{\sigma u^2}{\sigma|u|^2 - 2\mathrm{i}} = \frac{u^2}{|u|^2 - 2\sigma\mathrm{i}}.$$

We first solve for $|u|$ and get with $\xi = |X_{\overline{\mathrm{H}}}|/|\overline{X_{\mathrm{H}}}| = |X_{\overline{\mathrm{H}}}|/|X_{\mathrm{H}}|$ that $\xi = |u|^2/\sqrt{|u|^4 + 4}$. The solution for $|u|$ is unique, and does exist if $\xi < 1$, i.e., if $\det X > 0$. In order to solve for the polar angle, we note that $(u_{1,2}/|u|)^2$ satisfies

$$\left(\frac{u_{(\sigma+3)/2}}{|u|}\right)^2 = \frac{X_{\overline{\mathrm{H}}}}{\overline{X_{\mathrm{H}}}} \, \frac{|u|^2 - 2\mathrm{i}\sigma}{\sqrt{|u|^4 + 4}} \bigg/ \left|\frac{X_{\overline{\mathrm{H}}}}{\overline{X_{\mathrm{H}}}}\right| \quad \text{for } \sigma \in \{-1, 1\}.$$

This shows

$$\left|\left(\frac{u_1}{u_2}\right)^2 - (-1)\right| = \left|\frac{|u|^2 + 2\mathrm{i}}{|u|^2 - 2\mathrm{i}} + 1\right| = \frac{2|u|^2}{||u|^2 - 2\mathrm{i}|} = 2\xi.$$

Since our assumption ensures that $|(X - \mathrm{Id})_{\mathrm{H}}| = |X_{\mathrm{H}} - 1|$, $|(X - \mathrm{Id})_{\overline{\mathrm{H}}}| = |X_{\overline{\mathrm{H}}}| < 1/3$ we see that $2\xi < 1$. Therefore, the complex unit $(u_1/u_2)^2$ has argument strictly between $2\pi/3$ and $4\pi/3$. This proves the assertion (iii) that the unoriented angle between the vectors $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ is at least $\pi/3$. □

The following proposition is an immediate consequence of the characterization of the quasiconvex hull of two wells in [1,3].

Proposition 2.2. *Suppose that* $\lambda, \mu \in \mathbb{R}$ *with* $0 < \mu < 1 < \lambda$ *and that*

$$K = \mathrm{SO}(2)\operatorname{diag}(\lambda, \mu) \cup \mathrm{SO}(2)\operatorname{diag}(\mu, \lambda). \tag{5}$$

Then $K^{\mathrm{qc}} = \{F \in \mathbb{M}^{2,2}\colon \det F = \mu\lambda,\ |F(\boldsymbol{e}_1 \pm \boldsymbol{e}_2)|^2 \leqslant \lambda^2 + \mu^2\}$,*where* $\{\boldsymbol{e}_1, \boldsymbol{e}_2\}$ *denotes the standard basis in* $\mathbb{R}^2$.

This result allows us to quantify perturbations of matrices in the interior of the quasiconvex hull of K.

Corollary 2.3. *Let* $\alpha > 0$, $\varepsilon \in (0, 1/25)$ *and* $|\Delta - 1| \leqslant 4\varepsilon$.

(a) *If* $\widehat{Z} \in \mathbb{M}^{2,2}$ *with* $\det \widehat{Z} = \Delta$, $\|\widehat{Z} - \sqrt{\Delta}\,\widehat{\mathrm{Id}}\| \leqslant (1 + \sqrt{2})9\varepsilon$ *and if there exists a* $\lambda \geqslant 1$ *such that*

$$\left(138 + \frac{2}{3}(\lambda^2 - 1)\alpha^2\right)\varepsilon \leqslant (\alpha - 1)^2\left(\frac{\alpha + 1}{\alpha}\right)^2\left(1 - \frac{1}{\lambda^2}\right)$$

then

$$\widehat{Z} \in \left(\mathrm{SO}(2)\operatorname{diag}\left(\Delta\alpha, \frac{1}{\alpha}\right) \cup \mathrm{SO}(2)\operatorname{diag}\left(\frac{1}{\alpha}, \Delta\alpha\right)\right)^{\mathrm{qc}}.$$

(b) *If* $|w| \leqslant 31\,\varepsilon$ *and if there exists a* $\lambda \geqslant 1$ *such that*

$$\left(106+\frac{1}{2}(\lambda^2-1)\alpha^2\right)\varepsilon \leqslant (\alpha-1)^2\left(\frac{\alpha+1}{\alpha}\right)^2\left(1-\frac{1}{\lambda^2}\right)$$

then

$$\begin{pmatrix}\sqrt{\Delta} & w\\ 0 & 1/\Delta\end{pmatrix} \in \left(\mathrm{SO}(2)\,\mathrm{diag}\left(\frac{\alpha}{\sqrt{\Delta}}, \frac{1}{\alpha}\right) \cup \mathrm{SO}(2)\,\mathrm{diag}\left(\frac{1}{\alpha}, \frac{\alpha}{\sqrt{\Delta}}\right)\right)^{\mathrm{qc}}.$$

Proof. This follows from a careful analysis of the conditions in Proposition 2.2. □

Proposition 2.4. *Let* K *be given by* (1) *with* $\alpha > 1$. *Suppose that* $(\Delta - 1/\alpha^2)(\Delta-\alpha) \leqslant 0$. *Then* $\mathrm{diag}(\Delta\alpha, 1/\alpha, 1/\Delta) \in K^{\mathrm{qc}}$.

Proof. This follows from Proposition 2.2 by considering

$$K_0 = \mathrm{SO}(3)\,\mathrm{diag}\left(\alpha^2, \frac{1}{\alpha}, \frac{1}{\alpha}\right) \cup \mathrm{SO}(3)\,\mathrm{diag}\left(\frac{1}{\alpha}, \frac{1}{\alpha}, \alpha^2\right)$$

and by restricting all constructions to the x_1, x_3 plane.

3. Proof of Theorem 1.2

Suppose $X \in \mathbb{M}^{3,3}$ satisfies the assumptions in Theorem 1.2. We claim the existence of a rotation $Q \in \mathrm{SO}(3)$ such that

$$QX = \begin{pmatrix}\widehat{X} & \hat{\boldsymbol{u}}\\ \boldsymbol{0}^{\mathrm{T}} & 1/\Delta\end{pmatrix} \quad \text{with } \|Q-\mathrm{Id}\| \leqslant \frac{7\varepsilon}{3},\ |\Delta-1| \leqslant 4\varepsilon,\ |\hat{\boldsymbol{u}}| \leqslant \frac{10\varepsilon}{3} \text{ and } \left\|\widehat{X}-\sqrt{\Delta}\,\widehat{\mathrm{Id}}\right\| \leqslant 9\varepsilon.$$

For this we note the following two simple geometric observations

(i) If $\boldsymbol{v} \in \mathbb{S}^2$ then there is a $Q \in \mathrm{SO}(3)$ with $Q\boldsymbol{v} = \boldsymbol{e}_3$ and $\|Q-\mathrm{Id}\| = \sqrt{2}|\boldsymbol{v}-\boldsymbol{e}_3|$. Indeed, in a suitable coordinate system $\boldsymbol{v} \in \mathrm{span}\{\boldsymbol{e}_2, \boldsymbol{e}_3\}$, thus we can have $(Q-\mathrm{Id})(\boldsymbol{e}_1) = 0$ and $(Q-\mathrm{Id})$ acts on $\boldsymbol{e}_1^{\perp}$ just as a conformal linear map of operator norm $|\boldsymbol{v}-\boldsymbol{e}_3|$ and thus Euclidean norm $\sqrt{2}|\boldsymbol{v}-\boldsymbol{e}_3|$.
(ii) For $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^2 \setminus \{0\}$ we have $|(\boldsymbol{u}/|\boldsymbol{u}|) - (\boldsymbol{v}/|\boldsymbol{v}|)| \leqslant |\boldsymbol{u}-\boldsymbol{v}|/\min(|\boldsymbol{u}|, |\boldsymbol{v}|)$ since we can suppose $|\boldsymbol{u}| = 1 \leqslant |\boldsymbol{v}|$ and use that $\frac{\mathrm{d}}{\mathrm{d}\lambda}\|\lambda\frac{\boldsymbol{v}}{|\boldsymbol{v}|} - \boldsymbol{u}\|^2 = 2\lambda - 2\langle\frac{\boldsymbol{v}}{|\boldsymbol{v}|}, \boldsymbol{u}\rangle \geqslant 0$ if $\lambda \geqslant 1$.

Now, writing $\boldsymbol{x}_j = X(\cdot, j) \in \mathbb{R}^3$ for the columns of X we set $\boldsymbol{y} = \boldsymbol{x}_1 \times \boldsymbol{x}_2$ and estimate

$$\begin{aligned}|\boldsymbol{y}-\boldsymbol{e}_3| &\leqslant \left|(\boldsymbol{x}_1-\boldsymbol{e}_1)\times\boldsymbol{x}_2\right| + \left|\boldsymbol{e}_1\times(\boldsymbol{x}_2-\boldsymbol{e}_2)\right| \leqslant \max\bigl(1, |\boldsymbol{x}_2|\bigr)\bigl(|\boldsymbol{x}_1-\boldsymbol{e}_1| + |\boldsymbol{x}_2-\boldsymbol{e}_2|\bigr)\\ &\leqslant \sqrt{2}\frac{26}{25}\|X-\mathrm{Id}\| < \frac{1}{16}.\end{aligned}$$

Thus (ii) tells us that $\boldsymbol{v} = \boldsymbol{y}/|\boldsymbol{y}|$ satisfies $|\boldsymbol{v}-\boldsymbol{e}_3| \leqslant \frac{16}{15}\sqrt{2}\,\frac{26}{25}\|X-\mathrm{Id}\| \leqslant \frac{7\sqrt{2}}{6}\|X-\mathrm{Id}\|$ and (i) now allows us to choose $Q \in \mathrm{SO}(3)$ with $Q\boldsymbol{v} = \boldsymbol{e}_3$ and $\|Q-\mathrm{Id}\| \leqslant \frac{7}{3}\|X-\mathrm{Id}\|$. Of course, $\langle \boldsymbol{e}_3, QX\boldsymbol{e}_j\rangle = \langle Q^{\mathrm{T}}\boldsymbol{e}_3, X\boldsymbol{e}_j\rangle = \langle \boldsymbol{v}, X\boldsymbol{e}_j\rangle = 0$ for $j = 1, 2$ and hence QX has the required block structure. Moreover,

$$\|QX-\mathrm{Id}\| \leqslant \|QX-Q\| + \|Q-\mathrm{Id}\| \leqslant \|X-\mathrm{Id}\| + \|Q-\mathrm{Id}\| \leqslant \frac{10}{3}\|X-\mathrm{Id}\| \leqslant \frac{1}{7}.$$

Hence the first three estimates follow, and the last one stated in our claim is implied by

$$\left\|\widehat{X}-\sqrt{\Delta}\,\widehat{\mathrm{Id}}\right\| \leqslant \left(\frac{10}{3}+4\sqrt{2}\right)\|X-\mathrm{Id}\| \leqslant 9\|X-\mathrm{Id}\| < \frac{2}{\sqrt{3}}.$$

Therefore, Proposition 2.1 now ensures the existence of vectors $\boldsymbol{u}_i$ and $\boldsymbol{v}_i$ in $\mathbb{R}^2$ with

(i) $\widehat{X}_{\overline{\mathrm{H}}} = (\boldsymbol{u}_i \otimes \boldsymbol{v}_i)_{\overline{\mathrm{H}}}$;
(ii) $\det(\widehat{X} + t\boldsymbol{u}_i \otimes \boldsymbol{v}_i) = \Delta$ for $t \in \mathbb{R}$;
(iii) $|\langle \boldsymbol{u}_1, \boldsymbol{u}_2\rangle| \leqslant |\boldsymbol{u}_1||\boldsymbol{u}_2|/2$.

We now define $\widehat{Y}_i = \widehat{X} - \boldsymbol{u}_i \otimes \boldsymbol{v}_i$, $\widehat{Z}_i = \widehat{X} + \boldsymbol{u}_i \otimes \boldsymbol{v}_i$. Note that $\det(\boldsymbol{u}_i \otimes \boldsymbol{v}_i) = 0$ implies $\|\boldsymbol{u}_i \otimes \boldsymbol{v}_i\| = \sqrt{2}\|\widehat{X}_{\overline{\mathrm{H}}}\| = \sqrt{2}\|(\widehat{X} - \sqrt{\Delta}\,\widehat{\mathrm{Id}})_{\overline{\mathrm{H}}}\| \leqslant 9\sqrt{2}\varepsilon$ and so $\|\widehat{Z}_i - \sqrt{\Delta}\,\widehat{\mathrm{Id}}\| \leqslant (1+\sqrt{2})9\varepsilon$. Thus Corollary 2.3(a), used in the (x_1, x_2)-plane, now gives

$$Z_i = \begin{pmatrix} \widehat{Z}_i & \mathbf{0} \\ \mathbf{0}^{\mathrm{T}} & 1/\Delta \end{pmatrix} \in \left(\mathrm{SO}(3)\,\mathrm{diag}\left(\Delta\alpha, \frac{1}{\alpha}, \frac{1}{\Delta}\right) \cup \mathrm{SO}(3)\,\mathrm{diag}\left(\frac{1}{\alpha}, \Delta\alpha, \frac{1}{\Delta}\right)\right)^{\mathrm{qc}} \subset K^{\mathrm{qc}}$$

because our assumption also ensures that $(\Delta - \frac{1}{\Delta})^2 < (\alpha - \frac{1}{\alpha})^2$ and so Proposition 2.4 applies. By condition (iii) we find $\gamma_i \in [0, 2|\hat{\boldsymbol{u}}|/\sqrt{3}]$ with $\hat{\boldsymbol{u}} = \gamma_1\tilde{\boldsymbol{u}}_1 + \gamma_2\tilde{\boldsymbol{u}}_2$ and $\tilde{\boldsymbol{u}}_i = \pm\boldsymbol{u}_i/|\boldsymbol{u}_i|$. We set

$$Y_i = \begin{pmatrix} \widehat{Y}_i & 2(\gamma_1+\gamma_2)\tilde{\boldsymbol{u}}_i \\ \mathbf{0}^{\mathrm{T}} & 1/\Delta \end{pmatrix}, \qquad P_i = \frac{1}{2}(Y_i + Z_i) = \begin{pmatrix} \widehat{X}_i & (\gamma_1+\gamma_2)\tilde{\boldsymbol{u}}_i \\ \mathbf{0}^{\mathrm{T}} & 1/\Delta \end{pmatrix}.$$

Then $\mathrm{rank}(Y_i - Z_i), \mathrm{rank}(P_1 - P_2) \leqslant 1$ and hence $QX = (\gamma_1 P_1 + \gamma_2 P_2)/(\gamma_1+\gamma_2) \in (\{Y_1, Y_2\} \cup K)^{\mathrm{qc}}$. It remains to show that the matrices Y_i belong to K^{qc}. Then the same holds for QX and by invariance of K^{qc} under multiplication by $\mathrm{SO}(3)$ from the left also for X.

But $\widehat{Y}_i$ satisfies $\det\widehat{Y}_i = \Delta$ and $(\widehat{Y}_i)_{\overline{\mathrm{H}}} = 0$. Thus $\widehat{R}_i\widehat{Y}_i = \sqrt{\Delta}\,\widehat{\mathrm{Id}}$ for some $\widehat{R}_i \in \mathrm{SO}(2)$. Let $R_i = \mathrm{diag}(\widehat{R}_i, 1)$. Again by the invariance of K^{qc} under multiplication by $\mathrm{SO}(3)$ from the left it suffices to show that

$$R_i Y_i = \mathrm{diag}(\sqrt{\Delta}, \sqrt{\Delta}, 1/\Delta) + \widehat{R}_i\big(2(\gamma_1+\gamma_2)\tilde{\boldsymbol{u}}_i\big) \otimes \boldsymbol{e}_3 \in K^{\mathrm{qc}}.$$

We notice that $\widehat{R}_i(2(\gamma_1+\gamma_2)\tilde{\boldsymbol{u}}_i) = (s_1^i\boldsymbol{e}_1 + s_2^i\boldsymbol{e}_2)/2$ with $|s_1^i|, |s_2^i| \leqslant 8\frac{2}{\sqrt{3}}|\hat{\boldsymbol{u}}| \leqslant 31\,\varepsilon$. Therefore, Corollary 2.3(b) applied in the (x_1, x_3)-plane implies

$$W_1^i = \mathrm{diag}\left(\sqrt{\Delta}, \sqrt{\Delta}, \frac{1}{\Delta}\right) + s_1^i\boldsymbol{e}_1 \otimes \boldsymbol{e}_3 \in \left(\mathrm{SO}(3)\,\mathrm{diag}\left(\frac{\alpha}{\sqrt{\Delta}}, \sqrt{\Delta}, \frac{1}{\alpha}\right) \cup \mathrm{SO}(3)\,\mathrm{diag}\left(\frac{1}{\alpha}, \sqrt{\Delta}, \frac{\alpha}{\sqrt{\Delta}}\right)\right)^{\mathrm{qc}}.$$

Since we already observed that Δ and hence also $\Delta^{-1/2}$ is between α and $1/\alpha$, Proposition 2.4 implies $W_1^i \in K^{\mathrm{qc}}$. Because the same argument also gives that $W_2^i = \mathrm{diag}(\sqrt{\Delta}, \sqrt{\Delta}, 1/\Delta) + s_1^i\boldsymbol{e}_1 \otimes \boldsymbol{e}_3 \in K^{\mathrm{qc}}$, the center R_iY_i of these two rank-one connected matrices is in K^{qc} as well.

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