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A linearized Kuramoto–Sivashinsky PDE via an imaginary-Brownian-time-Brownian-angle process

Une version linéaire de L'EDP de Kuramoto–Sivashinsky via un processus de temps brownien imaginaire et d'angle brownien

Hassan Allouba

Department of Mathematical Sciences, Kent State University, Kent, OH 44242, USA

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Abstract

We introduce a new imaginary-Brownian-time-Brownian-angle process, which we also call the linear-Kuramoto–Sivashinsky process (LKSP). Building on our techniques in two recent articles involving the connection of Brownian-time processes to fourth order PDEs, we give an explicit solution to a linearized Kuramoto–Sivashinsky PDE in d-dimensional space: $u_t = -\frac{1}{8}\Delta^2 u - \frac{1}{2}\Delta u - \frac{1}{2}u$. The solution is given in terms of a functional of our LKSP. ***To cite this article: H. Allouba, C. R. Acad. Sci. Paris, Ser. I 336 (2003).***

Résumé

Nous introduisons un nouveau processus de temps brownien imaginaire et d'angle brownien, qu'on appelle aussi le processus de Kuramoto–Sivashinsky linéaire (PKSL). En étendant nos techniques développées dans deux articles récents sur la relation entre des processus de temps brownien et des EDPs du quatrième ordre, nous donnons une solution explicite à une version linéaire d'EDP d-dimensionnel de Kuramoto–Sivashinsky : $u_t = -\frac{1}{8}\Delta^2 u - \frac{1}{2}\Delta u - \frac{1}{2}u$. La solution est donnée par une fonctionnelle associée à notre PKSL. ***Pour citer cet article : H. Allouba, C. R. Acad. Sci. Paris, Ser. I 336 (2003).***

Version française abrégée

L'EDP de Kuramoto–Sivashinsky est une des équations proéminentes dans les mathématiques appliquées modernes. Cette équation a engendré beaucoup d'intérêt dans la littérature d'EDP non linéaire (e.g., [7–10,13,18] et beaucoup d'autres articles). Dans le champ des processus stochastiques, beaucoup d'intérêt est dirigé vers

E-mail address: allouba@math.kent.edu (H. Allouba).

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l'étude de processus pour lesquels le temps est remplacé d'une façon ou d'une autre par un mouvement brownien. Cet intérêt a pris une importance considérable (e.g., [3,1,4–6,16,17,14,15,11,12]) après le travail fondamental de Burdzy sur le mouvement brownien itéré [5,6]. Dans [3,1], nous avons fourni une structure unifiée pour un tel processus itéré de Burdzy aussi bien que pour d'autres nouveaux processus reliés, via une grande classe des processus à temps brownien. Et nous avons lié cette classe à quelques EDPs du quatrième ordre. Dans cet article, et comme annoncé dans [1], nous modifions notre processus figurant dans le Théorème 1.2 [1] et nous étendons nos méthodes de l'article [1] pour donner une solution explicite à une version linéaire de l'EDP de KS. Une modification nécessaire est l'introduction de $\mathrm{i} = \sqrt{-1}$ dans le temps brownien aussi bien que l'exponentielle du mouvement brownien. Cela mène à un nouveau processus intéressant qu'on appelle le processus de temps brownien imaginaire et d'angle brownien, avec une fonction initiale $f : \mathbb{R}^d \to \mathbb{R}$:

$$\mathbb{A}_B^{f,X}(t,x) \triangleq \begin{cases} f\big(X^x(\mathrm{i}B(t))\big)\exp\big(\mathrm{i}B(t)\big), & B(t) \geqslant 0; \\ f\big(\mathrm{i}X^{-\mathrm{i}x}(-\mathrm{i}B(t))\big)\exp\big(\mathrm{i}B(t)\big), & B(t) < 0. \end{cases} \tag{0.1}$$

Les processus de temps imaginaire $\{X^x(\mathrm{i}s),\ s \geqslant 0\}$ et $\{\mathrm{i}X^{-\mathrm{i}x}(-\mathrm{i}s),\ s \leqslant 0\}$ sont deux mouvements browniens indépendants X^x et $X^{-\mathrm{i}x}$ à valeurs dans $\mathbb{R}^d$ et $\mathrm{i}\mathbb{R}^d$ ($\mathrm{i}X^{-\mathrm{i}x}$ est à valeurs dans $\mathbb{R}^d$), avec $X^x(0) = x \in \mathbb{R}^d$ et $X^{-\mathrm{i}x}(0) = -\mathrm{i}x$ (i.e., $\mathrm{i}X^{-\mathrm{i}x}(0) = x$), où l'axe du temps traditionnel ($\mathbb{R}_+$) est remplacé par un axe de temps imaginaire positif ($\mathrm{i}\mathbb{R}_+$). Alors, on considère $\{X^x(\mathrm{i}s),\ s \geqslant 0\}$ et $\{\mathrm{i}X^{-\mathrm{i}x}(-\mathrm{i}s),\ s \leqslant 0\}$ comme deux processus qui possèdent la même distribution gaussienne complexe sur $\mathbb{R}^d$, avec la densité complexe correspondante

$$p_{\mathrm{i}s}^{(d)}(x,y) = \frac{1}{(2\pi\mathrm{i}s)^{d/2}}\,\mathrm{e}^{-|x-y|^2/2\mathrm{i}s}.$$

En outre, B est un mouvement brownien standard à valeurs dans $\mathbb{R}$ avec $B(0) = 0$, et indépendant des X^x et $X^{-\mathrm{i}x}$. Noter que le mouvement brownien B est aussi l'angle du processus $\mathbb{A}_B^{f,X}(t,x)$ et que $\mathbb{A}_B^{f,X}(0,x) = f(x)$. Motivé par les définitions de v_ε et u_ε dans la preuve du Théorème 1.2 de [1], nous définissons

$$\begin{aligned} v(s,x) &\triangleq \exp(\mathrm{i}s)\int_{\mathbb{R}^d} f(y)\frac{1}{(2\pi\mathrm{i}s)^{d/2}}\,\mathrm{e}^{-|x-y|^2/2\mathrm{i}s}\,\mathrm{d}y, \\ u(t,x) &\triangleq \int_{-\infty}^{0} v(s,x)p_t(0,s)\,\mathrm{d}s + \int_0^\infty v(s,x)p_t(0,s)\,\mathrm{d}s, \end{aligned} \tag{0.2}$$

où $p_t(0,s)$ est la densité du mouvement brownien interne et uni-dimensionnel B :

$$p_t(0,s) = \frac{1}{\sqrt{2\pi t}}\,\mathrm{e}^{-s^2/2t}.$$

Nous pouvons définir les fonctions v et u via une espérance complexe en définissant

$$v(s,x) \triangleq \mathbb{E}^{\mathbb{C}}\big[f\big(X^x(\mathrm{i}s)\big)\exp(\mathrm{i}s)\big] \quad \text{et} \quad u(t,x) \triangleq \mathbb{E}^{\mathbb{C}}\big[\mathbb{A}_B^{f,X}(t,x)\big].$$

Une étude plus détaillée des nombreuses relations entre notre processus à distribution complexe et l'EDP de KS et ses implications est le sujet d'un article prochain [2]. Nous pouvons maintenant présenter notre résultat principal.

Théorème 0.1. *Soit $f \in C_c^2(\mathbb{R}^d;\mathbb{R})$ et soit $D_{ij}f$ des fonctions höldériennes, $1 \leqslant i, j \leqslant d$. Alors, $u(t,x)$ donnée par* (0.2) *est une solution de l'EDP de Kuramoto–Sivashinsky linéarisée* :

$$\begin{cases} \dfrac{\partial}{\partial t}u(t,x) = -\dfrac{1}{8}\Delta^2 u(t,x) - \dfrac{1}{2}\Delta u(t,x) - \dfrac{1}{2}u(t,x), & t > 0,\ x \in \mathbb{R}^d; \\ u(0,x) = f(x), & x \in \mathbb{R}^d. \end{cases} \tag{0.3}$$

1. Statements and discussions of results

One of the prominent equations in modern applied mathematics is the celebrated Kuramoto–Sivashinsky (KS) PDE. This nonlinear equation has generated a lot of interest in the PDE literature (see, e.g., [7–10,13,18] and many other papers). In the field of stochastic processes, a great deal of interest is directed at the study of processes in which time is replaced in one way or another by a Brownian motion, and this interest has picked up considerably (see, e.g., [3,1,4–6,16,17,14,15,11,12]) after the fundamental work of Burdzy on iterated Brownian motion [5,6]. In [3,1], we provided a unified framework for such iterated processes (including the IBM of Burdzy) and introduced several interesting new ones, through a large class of processes that we called Brownian-time processes. We then related them to different fourth order PDEs. In this article, and as announced in [1], we modify our process in Theorem 1.2 [1] and build on our methods in [1] to give an explicit solution to a linear version of the KS PDE. One modification needed is the introduction of $\mathrm{i} = \sqrt{-1}$ in both the Brownian-time and the Brownian-exponential, and that leads to a new process we call imaginary-Brownian-time-Brownian-angle process IBTBAP, starting at $f:\mathbb{R}^d \to \mathbb{R}$:

$$\mathbb{A}_B^{f,X}(t,x) \triangleq \begin{cases} f\big(X^x\big(\mathrm{i}B(t)\big)\big)\exp\big(\mathrm{i}B(t)\big), & B(t) \geqslant 0; \\ f\big(\mathrm{i}X^{-\mathrm{i}x}\big(-\mathrm{i}B(t)\big)\big)\exp\big(\mathrm{i}B(t)\big), & B(t) < 0; \end{cases} \tag{1}$$

where X^x is an $\mathbb{R}^d$-valued Brownian motion starting from $x \in \mathbb{R}^d$, $X^{-\mathrm{i}x}$ is an independent $\mathrm{i}\mathbb{R}^d$-valued BM starting at $-\mathrm{i}x$ (so that $\mathrm{i}X^{-\mathrm{i}x}$ starts at x), and both are independent of the inner standard $\mathbb{R}$-valued Brownian motion B starting from 0. The time of the outer Brownian motions X^x and $X^{-\mathrm{i}x}$ is replaced by an imaginary positive Brownian time; and, when f is real-valued as we will assume here, the angle of $\mathbb{A}_B^{f,X}(t,x)$ is the Brownian motion B. We think of the imaginary-time processes $\{X^x(\mathrm{i}s),\ s \geqslant 0\}$ and $\{\mathrm{i}X^{-\mathrm{i}x}(-\mathrm{i}s),\ s \leqslant 0\}$ as having the same complex Gaussian distribution on $\mathbb{R}^d$ with the corresponding complex distributional density

$$p_{\mathrm{i}s}^{(d)}(x,y) = \frac{1}{(2\pi\mathrm{i}s)^{d/2}}\,\mathrm{e}^{-|x-y|^2/2\mathrm{i}s}.$$

We will also call the process given by (1) the d-dimensional Linear-Kuramoto–Sivashinsky process (LKSP) starting at f (clearly $\mathbb{A}_B^{f,X}(0,x) = f(x)$). The dimension in d-dimensional IBTBAP (or d-dimensional LKSP) refers to the dimension of the BMs X^x and $X^{-\mathrm{i}x}$, which is also the dimension of the spatial variable in the associated linearized KS PDE as we will see shortly.

Now, motivated by the definitions of v_ε and u_ε in the proof of Theorem 1.2 in [1], we let

$$\begin{aligned} v(s,x) &\triangleq \exp(\mathrm{i}s)\int_{\mathbb{R}^d} f(y)\frac{1}{(2\pi\mathrm{i}s)^{d/2}}\,\mathrm{e}^{-|x-y|^2/2\mathrm{i}s}\,\mathrm{d}y, \\ u(t,x) &\triangleq \int_{-\infty}^{0} v(s,x)p_t(0,s)\,\mathrm{d}s + \int_0^{\infty} v(s,x)p_t(0,s)\,\mathrm{d}s, \end{aligned} \tag{2}$$

where $p_t(0,s)$ is the transition density of the inner (one-dimensional) Brownian motion B:

$$p_t(0,s) = \frac{1}{\sqrt{2\pi t}}\,\mathrm{e}^{-s^2/2t}.$$

We may think of v and u in terms of complex expectation by defining $v(s,x) \triangleq \mathbb{E}^{\mathbb{C}}[f(X^x(\mathrm{i}s))\exp(\mathrm{i}s)]$ and $u(t,x) \triangleq \mathbb{E}^{\mathbb{C}}[\mathbb{A}_B^{f,X}(t,x)]$. A more detailed study of the rich connection between our process and its complex distribution to the KS PDE and its implications is the subject of an upcoming article [2]. We are now ready to state our main result.

Theorem 1.1. *Let* $f \in C_c^2(\mathbb{R}^d;\mathbb{R})$ *with* $D_{ij}f$ *Hölder continuous with exponent* $0<\alpha\leqslant 1$*, for all* $1\leqslant i,j\leqslant d$*. If* $u(t,x)$ *is given by* (2) *then* $u(t,x)$ *solves the linearized Kuramoto–Sivashinsky PDE*

$$\begin{cases} \dfrac{\partial}{\partial t}u(t,x) = -\dfrac{1}{8}\Delta^2 u(t,x) - \dfrac{1}{2}\Delta u(t,x) - \dfrac{1}{2}u(t,x), & t>0,\ x\in\mathbb{R}^d; \\ u(0,x)=f(x), & x\in\mathbb{R}^d. \end{cases} \tag{3}$$

2. Proof of the main result

Proof of Theorem 1.1. Let u and v be as given in (2). Differentiating $u(t,x)$ with respect to t and putting the derivative under the integral, which is easily justified by the dominated convergence theorem, then using the fact that $p_t(0,s)$ satisfies the heat equation

$$\frac{\partial}{\partial t}p_t(0,s) = \frac{1}{2}\frac{\partial^2}{\partial s^2}p_t(0,s)$$

and integrating by parts twice using the fact that the boundary terms vanish at $\pm\infty$ and that $(\partial/\partial s)p_t(0,s)=0$ at $s=0$, we obtain

$$\begin{aligned}
\frac{\partial}{\partial t}u(t,x) &= \int_{-\infty}^{0} v(s,x)\frac{\partial}{\partial t}p_t(0,s)\,\mathrm{d}s + \int_{0}^{\infty} v(s,x)\frac{\partial}{\partial t}p_t(0,s)\,\mathrm{d}s \\
&= \frac{1}{2}\left[\int_{-\infty}^{0} v(s,x)\frac{\partial^2}{\partial s^2}p_t(0,s)\,\mathrm{d}s + \int_{0}^{\infty} v(s,x)\frac{\partial^2}{\partial s^2}p_t(0,s)\,\mathrm{d}s\right] \\
&= \frac{1}{2}p_t(0,0)\left[\left(\frac{\partial}{\partial s}v(s,x)\right)\bigg|_{s=0^-} + \left(\frac{\partial}{\partial s}v(s,x)\right)\bigg|_{s=0^+}\right] \\
&\quad + \frac{1}{2}\int_{-\infty}^{0} p_t(0,s)\frac{\partial^2}{\partial s^2}v(s,x)\,\mathrm{d}s + \frac{1}{2}\int_{0}^{\infty} p_t(0,s)\frac{\partial^2}{\partial s^2}v(s,x)\,\mathrm{d}s \\
&= \frac{1}{2}\int_{-\infty}^{0} p_t(0,s)\left[-\frac{1}{4}\Delta^2 v(s,x) - \Delta v(s,x) - v(s,x)\right]\mathrm{d}s \\
&\quad + \frac{1}{2}\int_{0}^{\infty} p_t(0,s)\left[-\frac{1}{4}\Delta^2 v(s,x) - \Delta v(s,x) - v(s,x)\right]\mathrm{d}s \\
&= -\frac{1}{8}\Delta^2 u(t,x) - \frac{1}{2}\Delta u(t,x) - \frac{1}{2}u(t,x)
\end{aligned} \tag{4}$$

where for the last two equalities in (4) we have used the fact that

$$\begin{aligned}
\frac{\partial v}{\partial s} &= \frac{\mathrm{i}}{2}\Delta v(s,x) + \mathrm{i}v(s,x), \\
\frac{\partial^2 v}{\partial s^2} &= -\frac{1}{4}\Delta^2 v(s,x) - \Delta v(s,x) - v(s,x),
\end{aligned} \tag{5}$$

and the conditions on f to take the applications of the derivatives outside the integrals in (4) and (5) (the steps of Lemma 2.1 in [1] easily translates to our setting here, see the discussion below). Clearly $u(0,x) = f(x)$, and the proof is complete.

As we indicated above, only minor changes to Lemma 2.1 in [1] are needed to justify pulling the derivatives outside the integrals in (4) under the conditions on f of Theorem 1.1. We now adapt Lemma 2.1 [1] to our setting here, and we point out the necessary changes in its proof:

Lemma 2.1. *Let $v(s,x)$ be given by* (2) *and let f be as in Theorem* 1.1. *Let*

$$u_1(t,x) \triangleq \int_{-\infty}^{0} v(s,x)\, p_t(0,s)\,\mathrm{d}s \quad \textit{and} \quad u_2(t,x) \triangleq \int_{0}^{\infty} v(s,x)\, p_t(0,s)\,\mathrm{d}s, \tag{6}$$

then $\Delta^2 u_1(t,x)$ and $\Delta^2 u_2(t,x)$ are finite and

$$\Delta^2 u_1(t,x) = \int_{-\infty}^{0} \Delta^2 v(s,x)\, p_t(0,s)\,\mathrm{d}s \quad \textit{and} \quad \Delta^2 u_2(t,x) = \int_{0}^{\infty} \Delta^2 v(s,x)\, p_t(0,s)\,\mathrm{d}s. \tag{7}$$

Proof. As in the proof of Lemma 2.1 [1], letting $\mathring{\mathbb{R}}_+ = (0,\infty)$ and $\mathring{\mathbb{R}}_- = (-\infty,0)$, it suffices to show

$$\frac{\partial^4}{\partial x_j^4} \int_{\mathring{\mathbb{R}}_\pm} v(s,x)\, p_t(0,s)\,\mathrm{d}s = \int_{\mathring{\mathbb{R}}_\pm} \frac{\partial^4}{\partial x_j^4} v(s,x)\, p_t(0,s)\,\mathrm{d}s, \qquad j = 1,\ldots,d. \tag{8}$$

Letting $p_{\mathrm{i}s}^{(d)}(x,y) = (2\pi \mathrm{i}s)^{-d/2}\,\mathrm{e}^{-|x-y|^2/2\mathrm{i}s}$ and using the conditions on f, we easily get

$$\begin{aligned} \frac{\partial^4}{\partial x_j^4} v(s,x)\, p_t(0,s) &= \exp(\mathrm{i}s)\left(\int_{\mathbb{R}^d} f(y) \frac{\partial^4}{\partial y_j^4} p_{\mathrm{i}s}^{(d)}(x,y)\,\mathrm{d}y\right) p_t(0,s) \\ &= \exp(\mathrm{i}s)\left(\int_{\mathbb{R}^d} \frac{\partial^2}{\partial y_j^2} f(y) \frac{\partial^2}{\partial y_j^2} p_{\mathrm{i}s}^{(d)}(x,y)\,\mathrm{d}y\right) p_t(0,s). \end{aligned} \tag{9}$$

Rewriting the last term in (9), and letting $h_j(y) \triangleq \partial^2 f(y)/\partial y_j^2$, we have

$$\begin{aligned} &\left|\exp(\mathrm{i}s)\left(\int_{\mathbb{R}^d} (2\pi \mathrm{i}s)^{-d/2}\left(\frac{-(x_j-y_j)^2+\mathrm{i}s}{s^2}\right)\mathrm{e}^{-|x-y|^2/2\mathrm{i}s} h_j(y)\,\mathrm{d}y\right)\frac{\mathrm{e}^{-s^2/2t}}{\sqrt{2\pi t}}\right| \\ &\quad= \frac{\mathrm{e}^{-s^2/2t}}{\sqrt{2\pi t}}\left|\left(\int_{\mathbb{R}^d} (2\pi \mathrm{i}s)^{-d/2}\left(\frac{-(x_j-y_j)^2+\mathrm{i}s}{s^2}\right)\mathrm{e}^{-|x-y|^2/2\mathrm{i}s}\big(h_j(y)-h_j(x)\big)\,\mathrm{d}y\right)\right| \\ &\quad\leqslant \frac{\mathrm{e}^{-s^2/2t}}{\sqrt{2\pi t}} \int_{\mathbb{R}^d} \big(2\pi|s|\big)^{-d/2}\left|\frac{-(\tilde{x}_j-y_j)^2+|s|}{s^2}\right|\mathrm{e}^{-|\tilde{x}-y|^2/2|s|}\big|h_j(y)-h_j(\tilde{x})\big|\,\mathrm{d}y \\ &\quad= \frac{\mathrm{e}^{-s^2/2t}}{\sqrt{2\pi t}} \mathbb{E}_{\mathbb{P}}\left|\left(\frac{(\tilde{x}_j - W_j^{\tilde{x}}(|s|))^2-|s|}{s^2}\right)\Big(h_j\big(W^{\tilde{x}}\big(|s|\big)\big)-h_j(\tilde{x})\Big)\right|, \end{aligned} \tag{10}$$

for some $\tilde{x} \in \mathbb{R}^d$ where $\tilde{x}_j = \pm x_j$ for $j = 1, \ldots, d$; and where $W^{\tilde{x}} : \Omega \times \mathbb{R}_+ \to \mathbb{R}^d$ is a standard Brownian motion starting at $\tilde{x} \in \mathbb{R}^d$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and $W_j^{\tilde{x}}$ is its j-th component. The inequality in (10) follows easily if h_j is a polynomial, and standard approximation yields the inequality for $h_j \in C_c(\mathbb{R}^d; \mathbb{R})$. Now, exactly as in [1] (9) and (10); we use the Brownian motion scaling for $W^{\tilde{x}}$, the Cauchy–Shwarz inequality on the last term in (10), and the Hölder condition on h_j to deduce that the last term in (10) is bounded above by $K \exp(-s^2/2t)/(\sqrt{2\pi t}|s|^{1-\alpha/2}) \in L^1((-\infty, 0), \mathrm{d}s) \cap L^1((0, \infty), \mathrm{d}s)$; hence $|\partial^4/\partial x_j^4 v(s, x) p_t(0, s)| \in L^1((-\infty, 0), \mathrm{d}s) \cap L^1((0, \infty), \mathrm{d}s)$, which completes the proof by standard analysis. □

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