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Algebraic Geometry

Invariants of real rational symplectic 4-manifolds and lower bounds in real enumerative geometry

Invariants des variétés symplectiques rationnelles réelles de dimension quatre, et bornes inférieures en géométrie énumérative réelle

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Abstract

Following the approach of Gromov and Witten, we construct invariants under deformation of real rational symplectic 4-manifolds. These invariants provide lower bounds for the number of real rational J-holomorphic curves in a given homology class passing through a given real configuration of points. ***To cite this article: J.-Y. Welschinger, C. R. Acad. Sci. Paris, Ser. I 336 (2003).***

Résumé

Suivant l'approche de Gromov et Witten, nous construisons des invariants par déformation des variétés symplectiques réelles rationnelles de dimension quatre. Ces invariants fournissent des bornes inférieures pour le nombre de courbes J-holomorphes rationnelles réelles de classe d'homologie donnée passant par une configuration réelle de points donnée. ***Pour citer cet article : J.-Y. Welschinger, C. R. Acad. Sci. Paris, Ser. I 336 (2003).***

1. The invariant χ

Let (X, ω, c_X) be a *real symplectic* 4-*manifold*, that is a triple made of a 4-manifold X, a symplectic form ω on X and an involution c_X on X such that $c_X^*\omega = -\omega$, all of them being of class C^∞. The fixed point set of c_X is called the *real part of* X and is denoted by $\mathbb{R}X$. It is assumed here to be non-empty. Let $d \in H_2(X; \mathbb{Z})$ be a homology class satisfying $c_1(X)d > 0$, where $c_1(X)$ is the first Chern class of the symplectic 4-manifold (X, ω). From Corollary 1.5 of [5], we know that the existence of such a class forces the 4-manifold X to be rational or

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ruled, as soon as d is not the class of an exceptional divisor. Hence, from now on, we will assume (X,ω) to be rational. Let $x \subset X$ be a *real configuration* of points, that is a subset invariant under c_X, made of $c_1(X)d-1$ distincts points. Denote by r the number of such points which are real. Let $\mathcal{J}_\omega$ be the space of almost complex structures of X tamed by ω and which are of Hölder class $C^{l,\alpha}$ where $l \geqslant 2$ and $\alpha \in]0,1[$ are fixed. This space is a contractible Banach manifold of class $C^{l,\alpha}$. Denote by $\mathbb{R}\mathcal{J}_\omega \subset \mathcal{J}_\omega$ the subspace consisting of those $J \in \mathcal{J}_\omega$ for which c_X is J-antiholomorphic. It happens to be a contractible Banach submanifold of class $C^{l,\alpha}$ of $\mathcal{J}_\omega$. If $J \in \mathbb{R}\mathcal{J}_\omega$ is generic enough, then there are only finitely many J-holomorphic rational curves in X passing through x in the homology class d (see Theorem 3.1). These curves are all nodal and irreducible. The total number of their double points is $\delta = \frac{1}{2}(d^2 - c_1(X)d + 2)$. Let C be such a curve which is assumed to be real. The real double points of C are of two different natures. They are either the local intersection of two real branches, or the local intersection of two complex conjugated branches. In the first case they are called *non-isolated* and in the second case they are called *isolated*. We define the *mass* of the curve C to be the number of its real isolated double points, it is denoted by $m(C)$. For every integer m ranging from 0 to δ, denote by $n_d(m)$ the total number of real J-holomorphic rational curves of mass m in X passing through x and realizing the homology class d. Then define: $\chi_r^d(x,J) = \sum_{m=0}^{\delta}(-1)^m n_d(m)$.

The main result to be presented in this Note is:

Theorem 1.1. *Let (X,ω,c_X) be a real rational symplectic* 4*-manifold, and $d \in H_2(X;\mathbb{Z})$. Let $x \subset X$ be a real configuration of $c_1(X)d-1$ distincts points and r be the cardinality of $x \cap \mathbb{R}X$. Finally, let $J \in \mathbb{R}\mathcal{J}_\omega$ be an almost complex structure generic enough, so that the integer $\chi_r^d(x,J)$ is well defined. Then, this integer $\chi_r^d(x,J)$ neither depends on the choice of J nor on the choice of x* (*provided the cardinality of $x \cap \mathbb{R}X$ is r*).

For convenience, this integer will be denoted by χ_r^d, and when r does not have the same parity as $c_1(X)d-1$, we put χ_r^d to be 0. We then denote by $\chi^d(T)$ the polynomial $\sum_{r=0}^{c_1(X)d-1}\chi_r^d T^r \in \mathbb{Z}[T]$. It follows from Theorem 1.1 that the function $\chi : d \in H_2(X;\mathbb{Z}) \mapsto \chi^d(T) \in \mathbb{Z}[T]$ only depends on the real symplectic 4-manifold (X,ω,c_X) and is invariant under deformation of this real symplectic 4-manifold. As an application of this invariant, we obtain the following lower bounds:

Corollary 1.2. *Under the hypothesis of Theorem* 1.1*, the integer $|\chi_r^d|$ gives a lower bound for the total number of real rational J-holomorphic curves of X in the homology class d passing through x, independently of the choice of a generic $J \in \mathbb{R}\mathcal{J}_\omega$.*

Note that this number of real curves is always bounded from above by the total number N_d of rational J-holomorphic curves of X passing through x in the homology class d, which does not depend on the choice of J. This number N_d is a Gromov–Witten invariant of the symplectic 4-manifold (X,ω) and was computed by Kontsevich in [4]. One of the main problems of real enumerative geometry nowadays is, in this context, to know if there exists a generic real almost-complex structure $J \in \mathbb{R}\mathcal{J}_\omega$ such that all these rational J-holomorphic curves are real. The following corollary provides a criterium for the existence of such a structure.

Corollary 1.3. *Under the hypothesis of Theorem* 1.1*, assume that $\chi_r^d \geqslant 0$* (*resp. $\chi_r^d \leqslant 0$*)*. Assume that there exists a generic $J \in \mathbb{R}\mathcal{J}_\omega$ such that X has $\frac{1}{2}(N_d - |\chi_r^d|)$ real J-holomorphic curves of odd* (*resp. even*) *mass passing through x in the homology class d. Then, all of the rational J-holomorphic curves of X passing through x in the homology class d are real.*

Example 1. Let (X,ω,c_X) be the complex projective plane equipped with its standard symplectic form and real structure. We denote the homology classes of $\mathbb{C}P^2$ by integers. Then $\chi_2^1 = 1$, $\chi_5^2 = 1$ and $\chi_r^3 = r$ for an even r in between 0 and 8. The latter can be obtained computing the Euler caracteristic of the real part of the blown up projective plane at the nine base points of a pencil of cubics, as was noticed by V. Kharlamov (see [1], Proposition 4.7.3, or [7], Theorem 3.6).

Example 2. Let (X, ω, c_X) be the real maximal smooth cubic surface of $\mathbb{C}P^3$, and l be the homology class of a line. Then $\chi_0^l = 27$.

Further computations of this invariant χ seem to require some recursion formula analogous to the one obtained by Kontsevich in [4]. Also, is it possible to obtain similar invariants for any real symplectic 4-manifold using higher genus curves?

2. The invariant θ

Now, let $y = (y_1, \ldots, y_{c_1(X)d-2})$ be a real configuration of $c_1(X)d - 2$ distinct points of X, and s be the number of those which are real. We assume that $y_{c_1(X)d-2}$ is real, so that s does not vanish. If $J \in \mathbb{R}\mathcal{J}_\omega$ is generic enough, then there are only finitely many J-holomorphic rational curves in X in the homology class d passing through y and having a node at $y_{c_1(X)d-2}$. These curves are all nodal and irreducible. For every integer m ranging from 0 to δ, denote by $\hat{n}_d^+(m)$ (resp. $\hat{n}_d^-(m)$) the total number of these curves which are real, of mass m and with a non-isolated (resp. isolated) real double points at $y_{c_1(X)d-2}$. Define then: $\theta_s^d(y, J) = \sum_{m=0}^{\delta}(-1)^m(\hat{n}_d^+(m) - \hat{n}_d^-(m))$.

Theorem 2.1. *Let (X, ω, c_X) be a real rational symplectic 4-manifold, and $d \in H_2(X; \mathbb{Z})$. Let $y \subset X$ be a real configuration of $c_1(X)d - 2$ distincts points and $s \neq 0$ be the cardinality of $y \cap \mathbb{R}X$. Finally, let $J \in \mathbb{R}\mathcal{J}_\omega$ be an almost complex structure generic enough, so that the integer $\theta_s^d(y, J)$ is well defined. Then, this integer $\theta_s^d(y, J)$ neither depends on the choice of J nor on the choice of y (provided the cardinality of $y \cap \mathbb{R}X$ is s).*

Once more, for convenience, the integer $\theta_s^d(y, J)$ will be denoted by θ_s^d, and we put $\theta_s^d = 0$ when s does not have the same parity as $c_1(X)d$. This invariant makes it possible to give relations between the coefficients of the polynomial χ^d, namely:

Theorem 2.2. *Let (X, ω, c_X) be a real rational symplectic 4-manifold, $d \in H_2(X; \mathbb{Z})$ and r be an integer in between 0 and $c_1(X)d - 3$. Then $\chi_{r+2}^d = \chi_r^d + 2\theta_{r+1}^d$.*

3. Outline of the proof of Theorem 1.1

First, we construct the moduli space $\mathcal{M}_0^d(x)$ of genus 0 pseudo-holomorphic curves in the homology class d passing through x. This space is equipped with a $\mathbb{Z}/2\mathbb{Z}$-action induced by the real structure c_X. The fixed point set $\mathbb{R}\mathcal{M}_0^d(x)$ of this action is a Banach submanifold of $\mathcal{M}_0^d(x)$ consisting of the real curves, that is of the curves which are invariant under c_X. The index zero Fredholm projection $\pi : \mathcal{M}_0^d(x) \to \mathcal{J}_\omega$ is $\mathbb{Z}/2\mathbb{Z}$-equivariant and induces an index zero Fredholm projection $\pi_\mathbb{R} : \mathbb{R}\mathcal{M}_0^d(x) \to \mathbb{R}\mathcal{J}_\omega$. We first prove the following theorem.

Theorem 3.1. *The set of regular values of the projection $\pi : \mathcal{M}_0^d(x) \to \mathcal{J}_\omega$ intersects $\mathbb{R}\mathcal{J}_\omega$ in a dense set of the second category of $\mathbb{R}\mathcal{J}_\omega$.*

This theorem follows from the fact that a point of $\mathcal{M}_0^d(x)$ (resp. of $\mathbb{R}\mathcal{M}_0^d(x)$) is regular for π (resp. for $\pi_\mathbb{R}$) if and only if it corresponds to an immersed curve (see [2,3]) and from the following proposition.

Proposition 3.2. *The submanifold $\mathbb{R}\mathcal{J}_\omega$ of $\mathcal{J}_\omega$ is transversal to the restriction of π to $\mathcal{M}_0^d(x) \setminus \mathbb{R}\mathcal{M}_0^d(x)$.*

It is – essentially – a consequence of Theorem 3.1 that the integers $\chi_r^d(x, J)$ and $\theta_s^d(y, J)$ are well defined, as soon as $J \in \mathbb{R}\mathcal{J}_\omega$ is generic enough.

Then, let $J_0, J_1 \in \mathbb{R}\mathcal{J}_\omega$ be two regular values of the projection $\pi : \mathcal{M}_0^d(x) \to \mathcal{J}_\omega$ such that the integers $\chi_r^d(x, J_1)$ and $\chi_r^d(x, J_2)$ are well defined. Let $\gamma : [0, 1] \to \mathbb{R}\mathcal{J}_\omega$ be a path transversal to the projections $\pi_\mathbb{R} : \mathbb{R}\mathcal{M}_0^d(x) \to \mathbb{R}\mathcal{J}_\omega$ and $\pi : (\mathcal{M}_0^d(x) \setminus \mathbb{R}\mathcal{M}_0^d(x)) \to \mathcal{J}_\omega$ (see Proposition 3.2), joining J_0 to J_1. Thus, $\mathbb{R}\mathcal{M}_\gamma = \pi_\mathbb{R}^{-1}(\mathrm{Im}(\gamma))$ is a submanifold of dimension one of $\mathbb{R}\mathcal{M}_0^d(x)$, equipped with a projection $\pi_\gamma : \mathbb{R}\mathcal{M}_\gamma \to [0,1]$ induced by $\pi_{\mathbb{R}}$. Using genericity arguments, we prove that the path γ can be chosen in order that every element of $\mathbb{R}\mathcal{M}_\gamma$ is a nodal curve, except a finite number of them which may have a unique real ordinary cusp, a unique real triple point or a unique real tacnode. Moreover, this path can be chosen so that when a sequence of elements of $\mathbb{R}\mathcal{M}_\gamma$ converges in Gromov topology to a reducible curve of X, then this curve has only two irreducible components, both real, and only nodal points as singularities. Finally, this path can be chosen so that the critical points of the projection π_γ, which correspond to the cuspidal curves, are all non-degenerate. To obtain these genericity results, we make a strong use of the results and techniques developed in [3] and [6].

At this point, the integer $\chi_r^d(x, \gamma(t))$ is well defined for all but a finite number of values of t, and is obviously constant between these parameters. The only thing to prove is that it also does not change while crossing these values which correspond to a real triple point or tacnode, to a cuspidal curve, or to a reducible curve. In the case of a curve having a real triple point or a real tacnode, it is not hard to check. In the case of a reducible curve, it follows from the following proposition.

Proposition 3.3. *Let C_0 be a real reducible J_0-holomorphic curve of X passing through x and limit of a sequence of elements of $\mathbb{R}\mathcal{M}_\gamma$. Let $J_0 = \gamma(t_0)$ for $t_0 \in]0,1[$ and C_1, C_2 be the two irreducible components of C_0. Let R be the number of real intersection points between C_1 and C_2. Then there exist a neighborhood W of C_0 in the Gromov compactification $\overline{\mathbb{R}\mathcal{M}_0^d(x)}$ and $\eta > 0$ such that for every $t \in]t_0 - \eta, t_0 + \eta[\setminus \{t_0\}$, $\pi_\gamma^{-1}(t) \cap W$ consists exactly of R real $\gamma(t)$-holomorphic curves, each of them obtained topologically by smoothing a different real intersection point of $C_1 \cap C_2$.*

Finally, in the case of a cuspidal curve, it follows from the following proposition.

Proposition 3.4. *Let $C_0 \in \mathbb{R}\mathcal{M}_\gamma$ be a critical point of π_γ which is a local maximum* (*resp. minimum*). *Then there exist a neighborhood W of C_0 in $\mathbb{R}\mathcal{M}_\gamma$ and $\eta > 0$ such that for every $t \in]t_0 - \eta, t_0[$ (resp. for every $t \in]t_0, t_0 + \eta[$), $\pi_\gamma^{-1}(t) \cap W$ consists of two curves C_t^+ and C_t^- satisfying $m(C_t^+) = m(C_t^-) + 1$, and for every $t \in]t_0, t_0 + \eta[$ (resp. for every $t \in]t_0 - \eta, t_0[$), $\pi_\gamma^{-1}(t) \cap W = \emptyset$.*

In Proposition 3.4, the curve C_0 has a unique cuspidal point which is a real ordinary cusp. Thus both C_t^+ and C_t^- have a real node in a neighborhood of this cusp. To get Proposition 3.4, one has to prove that for one of these curves, this real node is non-isolated and for the other one, it is isolated. Note that in contrast to the previous cases, the coefficient $(-1)^m$ in the definition of χ_r^d plays here a crucial rôle to get the invariance.

The proofs of Theorems 2.1 and 2.2 are based on the same kind of arguments.

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