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Complex Analysis

Subextension of plurisubharmonic functions with bounded Monge–Ampère mass

Sous-extension des fonctions plurisousharmoniques de masse de Monge–Ampère bornée

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Abstract

Let $\Omega \Subset \mathbb{C}^n$ be a hyperconvex domain. Denote by $\mathcal{E}_0(\Omega)$ the class of negative plurisubharmonic functions φ on Ω with boundary values 0 and finite Monge–Ampère mass on Ω. Then denote by $\mathcal{F}(\Omega)$ the class of negative plurisubharmonic functions φ on Ω for which there exists a decreasing sequence $(\varphi)_j$ of plurisubharmonic functions in $\mathcal{E}_0(\Omega)$ converging to φ such that $\sup_j \int_\Omega (dd^c\varphi_j)^n + \infty$.

It is known that the complex Monge–Ampère operator is well defined on the class $\mathcal{F}(\Omega)$ and that for a function $\varphi \in \mathcal{F}(\Omega)$ the associated positive Borel measure is of bounded mass on Ω. A function from the class $\mathcal{F}(\Omega)$ is called a plurisubharmonic function with bounded Monge–Ampère mass on Ω.

We prove that if Ω and $\widetilde{\Omega}$ are hyperconvex domains with $\Omega \Subset \widetilde{\Omega} \Subset \mathbb{C}^n$ and $\varphi \in \mathcal{F}(\Omega)$, there exists a plurisubharmonic function $\tilde{\varphi} \in \mathcal{F}(\widetilde{\Omega})$ such that $\tilde{\varphi} \leqslant \varphi$ on Ω and $\int_{\widetilde{\Omega}} (dd^c\tilde{\varphi})^n \leqslant \int_\Omega (dd^c\varphi)^n$. Such a function is called a subextension of φ to $\widetilde{\Omega}$.

From this result we deduce a global uniform integrability theorem for the classes of plurisubharmonic functions with uniformly bounded Monge–Ampère masses on Ω. ***To cite this article: U. Cegrell, A. Zeriahi, C. R. Acad. Sci. Paris, Ser. I 336 (2003).***

Résumé

Soit $\Omega \Subset \mathbb{C}^n$ un domaine hyperconvexe. On désigne par $\mathcal{E}_0(\Omega)$ la classe des fonctions plurisousharmoniques sur Ω avec valeurs au bord nulle et de masse de Monge–Ampère finie sur Ω. On désigne par $\mathcal{F}(\Omega)$ la classe des fonctions φ plurisousharmoniques négatives sur Ω, limite d'une suite décroissante (φ_j) de fonctions de $\mathcal{E}_0(\Omega)$ telle que $\sup_j \int_\Omega (dd^c\varphi_j)^n < +\infty$. On sait que l'opérateur de Monge–Ampère est bien défini sur $\mathcal{F}(\Omega)$ et que pour une fonction $\varphi \in \mathcal{F}(\Omega)$, la mesure de Monge–Ampère associée est une mesure de Borel sur Ω de masse totale bornée. Une telle fonction sera dite de masse de Monge–Ampère bornée sur Ω.

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On démontre alors que pour tout domaine hyperconvexe $\widetilde{\Omega}$, $\Omega \Subset \widetilde{\Omega} \Subset \mathbb{C}^n$ et tout $\varphi \in \mathcal{F}(\Omega)$ il existe une fonction $\tilde{\varphi} \in \mathcal{F}(\widetilde{\Omega})$ telle que $\tilde{\varphi} \leqslant \varphi$ sur Ω et $\int_{\widetilde{\Omega}} (dd^c\tilde{\varphi})^n \leqslant \int_{\Omega} (dd^c\varphi)^n$. Une telle fonction $\tilde{\varphi}$ est dite sous-extension de φ au domaine $\widetilde{\Omega}$. A partir de ce résultat, nous déduisons un théorème d'intégrabilté uniforme global pour les classes de fonction plurisousharmoniques sur $\Omega \Subset \mathbb{C}^n$ ayant des masses de Monge–Ampère uniformément bornées sur Ω. ***Pour citer cet article : U. Cegrell, A. Zeriahi, C. R. Acad. Sci. Paris, Ser. I 336 (2003).***

Version française abrégée

Soit $\Omega \Subset \mathbb{C}^n$ un domaine hyperconvexe. D'après [6], la mesure de Monge–Ampère $(dd^c\varphi)^n$ est bien définie pour $\varphi \in \mathcal{F}(\Omega)$ comme limite faible de la suite des mesures $((dd^c\varphi_j)^n)$, où $(\varphi)_j$ est une suite décroissante telle qu'elle est décrite dans la définition ; la limite étant indépendante de la suite choisie.

Voici le résultat principal de cette Note.

Théorème 1. *Soit $\Omega \Subset \widetilde{\Omega} \Subset \mathbb{C}^n$ deux domaines hyperconvexes et $\varphi \in \mathcal{F}(\Omega)$. Alors il existe une fonction $\tilde{\varphi} \in \mathcal{F}(\widetilde{\Omega})$ telle que $\tilde{\varphi} \leqslant \varphi$ sur Ω et $\int_{\widetilde{\Omega}} (dd^c\tilde{\varphi})^n \leqslant \int_{\Omega} (dd^c\varphi)^n$.*

Ce théorème de sous-extension a une application intéressante utilisant les résultats de [12,13].

Théorème 2. *Soit $\Omega \Subset \mathbb{C}^n$ un domaine hyperconvexe et $0 < \varepsilon \leqslant 2$ un nombre réel. Alors pour tout nombre réel α avec $0 < \alpha < \varepsilon$, il existe une constante $A = A(\alpha, \varepsilon, \Omega) > 0$ telle que pour tout $\varphi \in \mathcal{F}(\Omega)$ avec $\int_{\Omega} (dd^c\varphi)^n \leqslant 1$ on ait*

$$\int_{\Omega} \mathrm{e}^{-\varphi}\, \mathrm{d}\mu \leqslant A,$$

pour toute mesure de Borel μ sur Ω vérifiant la condition de régularité suivante :

$$\mu\big(\mathbb{B}(z,r)\big) \leqslant r^{2n-2+\varepsilon}, \quad \forall r \in [0,1],\ \forall z \in \Omega.$$

La preuve du Théorème 1 utilise des résultats récents sur le problème de Dirichlet pour l'opérateur de Monge–Ampère complexe (cf. [9,5]).

1. Introduction

Bedford and Burns and later Cegrell proved around 1978 that any smooth bounded domain satisfying certain boundary conditions is a domain of existence of a plurisubharmonic function (cf. [1,4]).

However, since plurisubharmonic functions occur in complex analysis through inequalities, it is more natural to ask for the subextension problem.

El Mir gave in 1980 an example of a plurisubharmonic function on the unit bidisc in $\mathbb{C}^2$ for which the restriction to any smaller bidisc admits no subextension to a larger domain (cf. [7]).

On the other hand, Fornaess and Sibony pointed out in 1987 that for a ring domain in $\mathbb{C}^2$, there exists a plurisubharmonic function which admits no subextension inside the hole (cf. [8]).

Finally Bedford and Taylor proved in 1988 that any smoothly bounded domain in $\mathbb{C}^n$ is a domain of existence of a smooth plurisubharmonic function (cf. [3]).

Here we prove that plurisubharmonic functions with bounded Monge–Ampère mass on a bounded hyperconvex domain admit a plurisubharmonic subextension to any larger bounded hyperconvex domain with a control of the Monge–Ampère mass.

2. The subextension theorem

As we pointed out in the introduction, on any smoothly bounded domain in $\mathbb{C}^2$ there is a smooth plurisubharmonic function which admits no subextention to any larger domain (cf. [3]). In contrast with this negative result, we prove here that for any hyperconvex domain $\Omega \Subset \mathbb{C}^n$, plurisubharmonic functions with bounded Monge–Ampère mass on Ω always admit a subextension to any larger bounded hyperconvex domain.

Let us first recall some useful facts (cf. [5,6]). Let $\Omega \Subset \mathbb{C}^n$ be a hyperconvex domain. By Cegrell [6] the Monge–Ampère measure $(dd^c\varphi)^n$ is well well defined for a function $\varphi \in \mathcal{F}(\Omega)$ as the weak*-limit of the sequence of measures $(dd^c\varphi_j)^n$, where (φ_j) is any decreasing sequence of plurisubharmonic functions from the class $\mathcal{E}_0(\Omega)$ converging to φ with uniformly bounded Monge–Ampère masses.

We will need the following result which follows from Corollary 3.4 and Proposition 5.1 in [6].

Lemma 2.1. *Let $\varphi \in \mathcal{F}(\Omega)$. Then*

$$e_0(\varphi) := \int_\Omega \left(dd^c\varphi\right)^n \tag{1}$$

is finite and for every sequence $(\varphi_j)_j$ of bounded plurisubharmonic functions which tends to 0 at the boundary such that $\varphi_j \searrow \varphi$ on Ω, the sequence of Monge–Ampère masses $\int_\Omega (dd^c\varphi_j)^n$ increases to $\int_\Omega (dd^c\varphi)^n$.

Now we can state our main result.

Theorem 2.2. *Let be $\Omega \Subset \widetilde{\Omega} \Subset \mathbb{C}^n$ two hyperconvex domains and $\varphi \in \mathcal{F}(\Omega)$. Then there exists a plurisubharmonic function $\tilde{\varphi} \in \mathcal{F}(\widetilde{\Omega})$ such that $\tilde{\varphi} \leqslant \varphi$ on Ω and $\int_{\widetilde{\Omega}} (dd^c\tilde{\varphi})^n \leqslant \int_\Omega (dd^c\varphi)^n$.*

Proof. By the definition of $\mathcal{F}(\Omega)$, there is a sequence (φ_j) of bounded plurisubharmonic functions which tends to 0 on the boundary such that $\varphi_j \searrow \varphi$ on Ω. Then $\int_\Omega (dd^c\varphi_j)^n$ increases to $\int_\Omega (dd^c\varphi)^n$ by the lemma. Fix an integer j and observe that the measure $\mu_j = \mathbf{1}_\Omega \cdot (dd^c\varphi_j)^n$ is a Borel measure with compact support in $\widetilde{\Omega}$ which puts no mass on pluripolar sets (cf. [2]). Then by Lemma 5.14 in [6], there is a unique function g_j in $\mathcal{F}(\widetilde{\Omega})$ such that $(dd^c g_j)^n = \mathbf{1}_\Omega \cdot (dd^c\varphi_j)^n$ as measures on $\widetilde{\Omega}$. We claim that $g_j \leqslant \varphi_j$ on Ω.

It follows from [5] that there exist $\psi_j \in \mathcal{E}_0(\widetilde{\Omega})$ and $0 \leqslant f_j \in L^1(\widetilde{\Omega}, (dd^c\psi_j)^n)$ such that $\mu_j = f_j(dd^c\psi_j)^n$ as measures on $\widetilde{\Omega}$.

Let us consider for each $k \in \mathbb{N}$ the Borel measure $\mu_{j,k} := \inf\{k, f_j\}(dd^c\psi_j)^n$ on $\widetilde{\Omega}$. Then by [9] or [5] there exists a function $\varphi_{j,k} \in \mathcal{E}_0(\Omega)$ such that $(dd^c\varphi_{j,k})^n = \inf\{k, f_j\}(dd^c\psi_j)^n$ on Ω and $g_{j,k} \in \mathcal{E}_0(\widetilde{\Omega})$ such that $(dd^c g_{j,k})^n = \inf\{k, f_j\}\mathbf{1}_\Omega \cdot (dd^c\varphi_j)^n$ on $\widetilde{\Omega}$. From the comparison principle, it follows that $(\varphi_{j,k})_k$ and $(g_{j,k})_k$ are decreasing sequences of plurisubharmonic functions in the class $\mathcal{E}_0(\Omega)$ and $\mathcal{E}_0(\widetilde{\Omega})$ respectively and that $g_{j,k} \leqslant \varphi_{j,k}$ on Ω. It now follows from Lemma 5.14 in [6] that $(g_{j,k})_k$ decreases to g_j on $\widetilde{\Omega}$ and $(\varphi_{j,k})_k$ decreases to φ_j on Ω as $k \to +\infty$. Thus $g_j \leqslant \varphi_j$ on Ω and the claim is proved.

Define now for each $j \in \mathbb{N}$, $\tilde{g}_j := [\sup_{j \leqslant k}(g_j)]^*$. Then $\tilde{g}_j \in \mathcal{E}_0(\widetilde{\Omega})$ and again by the comparison principle, it follows that $\int_{\widetilde{\Omega}} (dd^c\tilde{g}_j)^n \leqslant \int_\Omega (dd^c\varphi_j)^n$ and $\tilde{g}_j \leqslant \varphi_j$ on Ω.

Then the sequence $(\tilde{g}_j)_j$ decreases on $\widetilde{\Omega}$ to a function $\tilde{\varphi} \in \mathcal{F}(\widetilde{\Omega})$ which is smaller or equal to φ on Ω and satisfies $\int_{\widetilde{\Omega}} (dd^c\tilde{\varphi})^n \leqslant \int_\Omega (dd^c\varphi)^n$: the theorem is proved. □

It is interesting to observe that using our subextension theorem and results from [13], we can prove the following global uniform integrability theorem.

Theorem 2.3. *Let be $\Omega \Subset \mathbb{C}^n$ a hyperconvex domain and $0 < \varepsilon \leqslant 2$ a positive real number. Then for any real number α with $0 < \alpha < \varepsilon$, there exists a constant $A = A(\alpha, \varepsilon, \Omega) > 0$ such that for any function $\varphi \in \mathcal{F}(\Omega)$ with $\int_\Omega (dd^c\varphi)^n \leqslant 1$,*

$$\int_\Omega \mathrm{e}^{-\alpha\varphi}\,\mathrm{d}\mu \leqslant A, \tag{2}$$

where μ is any Borel measure on Ω satisfying the following regularity condition

$$\mu\big(\mathbb{B}(z; r)\big) \leqslant r^{2n-2+\varepsilon}, \quad \forall z \in \Omega,\ \forall r \in [0, 1],$$

$\mathbb{B}(z; r)$ beying the Euclidean ball of center z and radius $r > 0$.

In particular, under the same conditions, we have

$$\mu\big(\big\{z \in \Omega; \varphi(z) < -s\big\}\big) \leqslant A\,\mathrm{e}^{-\alpha s}, \quad \forall s > 0. \tag{3}$$

Proof. Indeed, take a hyperconvex domain $\widetilde{\Omega}$ such that $\Omega \Subset \widetilde{\Omega} \Subset \mathbb{C}^n$. Then by the subextension theorem, there is a function $\tilde{\varphi} \in \mathcal{F}(\widetilde{\Omega})$ such that $\tilde{\varphi} \leqslant \varphi$ on Ω and $\int_{\widetilde{\Omega}} (dd^c\tilde{\varphi})^n \leqslant \int_\Omega (dd^c\varphi)^n \leqslant 1$. Therefore it is enough to prove a uniform estimate of the integral $\int_\Omega \mathrm{e}^{-\alpha\tilde{\varphi}}\,\mathrm{d}V$ for the class $\widetilde{\mathcal{F}}_0$ of functions $\tilde{\varphi} \in \mathcal{F}(\widetilde{\Omega})$ such that $\int_{\widetilde{\Omega}} (dd^c\tilde{\varphi})^n \leqslant 1$ with $\Omega \Subset \widetilde{\Omega}$.

Now observe that $\widetilde{\mathcal{F}}_0$ is compact in $PSH(\widetilde{\Omega})$, $\Omega \Subset \widetilde{\Omega}$ (cf. [12]) and the Lelong numbers of such class satisfy the inequality $\nu_u(a) \leqslant (\int_\Omega (dd^c u)^n)^{1/n} \leqslant 1$ for any $a \in \overline{\Omega}$ and any $u \in \widetilde{\mathcal{F}}_0$ (cf. [11,6]). Therefore the stimates of the theorem follows from the results of [13]. □

In particular this theorem gives uniform estimates on volumes of sublevel sets of plurisubharmonic functions in the class $\widetilde{\mathcal{F}}_0$ which generalises results from [10] and [12,13].

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