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C. R. Acad. Sci. Paris, Ser. I 336 (2003) 219–224
COMPTES RENDUS MATHEMATIQUE

Partial Differential Equations

On the summability of the formal solutions for some PDEs with irregular singularity

Sur la sommabilité des solutions formelles de certaines EDP à singularité irrégulière

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Received and accepted 8 January 2003

Presented by Bernard Malgrange

Abstract

In this Note, we consider some classes of nonlinear partial differential equations with regular singularity with respect to $t = 0$ and irregular one with respect to $x = 0$. Our purpose is to establish a result which is similar to the k-summability property, known in the case of singular ordinary differential equations. We can prove that, except at most a countable set, the formal solution is Borel summable or k-summable with respect to x in all other directions. ***To cite this article: Z. Luo et al., C. R. Acad. Sci. Paris, Ser. I 336 (2003).***

Résumé

Dans la présente Note, nous considérons des classes d'équations aux dérivées partielles, non linéaires et qui sont toutes singulières régulières en $t = 0$ et irrégulières en $x = 0$. Notre but est d'établir un résultat similaire à la k-sommabilité connue pour des équations différentielles méromorphes à points singuliers. Nous montrons que, sous certaines conditions de généricité, toutes les solutions formelles sont Borel sommables ou k-sommables dans toutes les directions du plan des x sauf éventuellement un nombre dénombrable. ***Pour citer cet article : Z. Luo et al., C. R. Acad. Sci. Paris, Ser. I 336 (2003).***

Version française abrégée

Soit m, n des entiers > 0 ; on pose $\mathcal{F} = \{(i, j) \in \mathbf{N}^2\colon\ in + jm \leqslant mn, i < m\}$; soit N le cardinal de $\mathcal{F}$. Soit aussi k un entier > 0 et $F(t, x, \{v_{i,j}\}_{(i,j)\in\mathcal{F}})$ un germe de fonction analytique au voisinage de l'origine de $\mathbf{C}^{2+N}$.

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doi:10.1016/S1631-073X(03)00023-2

On considère l'équation aux dérivées partielles (E) suivante :

$$(t\partial_t)^m u = F\big(t, x, \{(t\partial_t)^i (x^{k+1}\partial_x)^j u\}_{(i,j)\in\mathcal{F}}\big), \qquad u(0,x) \equiv 0. \tag{E}$$

Théorème. *Supposons que $\partial_{v_{0,n}} F(0,0,0) \neq 0$ et que $\hat{u}(t,x) \in \mathbf{C}[[t,x]]$ est une solution formelle de (E) avec $\hat{u}(t,0) \equiv \hat{u}(0,x) \equiv 0$. Alors si l'on désigne par $\tilde{u}(t,\xi)$ la transformée de Borel formelle d'ordre $1/k$ en x, on a $\tilde{u}(t,\xi) \in \xi^{1-k}\mathbf{C}\{t,\xi\}$ et il existe un ensemble dénombrable $DS \subset [0,2\pi[$ tel que, pour toute direction d'argument $d \in [0,2\pi[\,\backslash DS$, on peut trouver un disque ouvert $D_R = \{t \in \mathbf{C}\colon |t| < R\}$, un secteur ouvert $S(d,\theta) = \{\xi \in \mathbf{C}^*\colon |\arg\xi - d| < \theta/2\}$ et des constantes $C_0 > 0$, $K > 0$ avec ceci : la fonction somme de $\tilde{u}(t,\xi)$ peut être prolongée en une fonction analytique dans l'ouvert $D_R \times S(d,\theta)$ et vérifiant*

$$\sup_{t\in D_R} \big|\xi^{k-1}\tilde{u}(t,\xi)\big| \leqslant C_0\, \mathrm{e}^{K|\xi|^k}, \quad \forall \xi \in S(d,\theta).$$

On notera que toute direction singulière $d \in DS$ contient nécessairement l'une des racines des polynômes suivants en ξ :

$$P_0(\ell,\xi) = \ell^m - \sum_{\substack{(i,j)\in\mathcal{F}\\ im+jn=nm}} a_{i,j}\ell^i\xi^{kj}, \qquad P(\ell,\xi) = \ell^m - \sum_{(i,j)\in\mathcal{F}} a_{i,j}\ell^i\xi^{kj},$$

où ℓ décrit l'ensemble des entiers > 0 et où $a_{i,j} = k^j \partial_{v_{i,j}} F(0,0,0)$. En outre, les valeurs d'accumulation de DS se situent parmi les arguments des racines de $P_0(1,\xi)$, ces derniers étant en nombre kn au plus.

L'hypothèse « $\partial_{v_{0,n}} F(0,0,0) \neq 0$ » exprime le fait que le polygone de Newton relatif à la variable x de Éq. (E) admet, en un certain sens, une pente unique qui vaut k. Pour que (E) possède une série entière pour solution formelle, les conditions suivantes sont suffisantes : (1) $F(0,x,0) \equiv 0$; (2) pour tout entier $\ell > 0$, $\ell^m - \sum_{0\leqslant i\leqslant m-1} a_{i,0}\ell^i \neq 0$. La première est d'ailleurs nécessaire.

Faisant suite au travail [4], la présente Note commencera par l'étude d'une classe d'équations aux dérivées partielles extraite de ce dernier travail. Ce sont, en effet, des versions généralisées des équations du type Briot–Bouquet ; nous nous référons à [6], Chapitres 5 et 6, pour des détails.

Soulignons aussi que des idées de [1], p. 217, et de [5] nous ont été utiles pour la démonstration du théorème énoncé plus haut. Des généralisations et raffinements de ce théorème seront présentés dans l'article [3].

1. Introduction

Since the last decade, it is known that all power series solutions of meromorphic differential equation are multisummable (cf. [2]). But only a few results are known about the summability of divergent power series which satisfies partial differential equation; see [5,7] for example. In [4], the Gevrey index is given for the formal solutions of nonlinear PDEs in two complex variables while one variable is with regular singularity and the other with irregular singularity at the origin. The present work can be considered as a first step in the study of the summability of this kind of nonlinear singular PDEs.

As in [4], we consider the following equation

$$t\partial_t u = F(t, x, u, \partial_x u), \qquad u(0,x) \equiv 0, \tag{1}$$

where $(t,x) \in \mathbf{C}^2$, $u = u(t,x)$ is an unknown function and $F(t,x,u,v)$ is a function given in an open polydisc Δ centered at the origin of $\mathbf{C}^4$. We suppose that

(F1) $F(t,x,u,v)$ is a holomorphic function on Δ;
(F2) $F(0,x,0,0) \equiv 0$ on Δ_0, where Δ_0 denotes the projection of Δ on the x-plane.

In other words, the function $F(t,x,u,v)$ is supposed to be written in the following form:

$$F(t,x,u,v) = a(x)t + b(x)u + \gamma(x)v + \sum_{i+j+\alpha \geqslant 2} a_{i,j,\alpha}(x) t^i u^j v^\alpha, \tag{2}$$

where $a, b, \gamma, a_{i,j,\alpha}$ are holomorphic on Δ_0. The existence and uniqueness of a holomorphic solution of (1) depend mainly on the valuation $\mathrm{val}(\gamma)$ of the function γ at $x=0$. For more details, see [6], Chapters 5, 6, and [4] and the references therein.

Let $p = \mathrm{val}(\gamma)$, the valuation of $\gamma(x)$ at $x=0$, and $k = p-1$; we suppose that $2 \leqslant p < \infty$. According to Theorem 1.2 in [4], if $b(0) \notin \mathbf{N}^* = \{1,2,3,\ldots\}$ and $a_{i,j,\alpha}(0) = 0$ for all $\alpha > 0$, then there is a unique power series solution and it is convergent in t and Gevrey of order $1 + 1/k$ (or $1/k$ for some authors) in x. In this Note, we shall establish the k-summability property of this formal solution. Thus we need the following condition:

(F3) Consider (2) and set $p = \mathrm{val}(\gamma)$, $\nu = \min\{\mathrm{val}(a); \mathrm{val}(a_{i,0,0})\colon\ i \geqslant 2\}$. We assume that the inequality $\mathrm{val}(a_{i,j,\alpha}) + \nu j \geqslant p$ holds for each (i,j,α) such that $\alpha > 0$.

The rest of this Note contains three sections, in which we shall study the case $p = 2$ and the case of $p > 2$ respectively.

Theorem 1.1. *Consider Eq.* (1) *and set* $\gamma(x) = x^2 c(x)$, $b = b(0)$, $c = c(0) (\neq 0)$. *If* $b \notin \mathbf{N}^*$ *and conditions* (F1)–(F3) *are satisfied, then the unique formal solution* $\hat{u}(t,x)$ *of* (1) *is Borel summable in all directions of the* x*-plane except at most a countable set.*

Precisely, if

$$\hat{u}(t,x) = u_{-1}(t) + \sum_{\ell=0}^{\infty} u_\ell(t) x^{\ell+1}, \qquad \tilde{u}(t,\xi) = \sum_{\ell=0}^{\infty} \frac{u_\ell(t)}{\ell!} \xi^\ell,$$

then $u_{-1} \in \mathbf{C}\{t\}$, $\tilde{u} \in \mathbf{C}\{t,\xi\}$; *moreover, for each direction of argument* d ($d \in [0, 2\pi)$) *that starts from* $\xi = 0$ *and does not pass through any point of the set* $\{\frac{1}{c}, \frac{1-b}{c}, \frac{2-b}{c}, \frac{3-b}{c}, \ldots\}$, *then there exist a positive constant* θ *and an open disk* $D_R = \{t \in \mathbf{C}\colon |t| < R\}$ *such that in* $D_R \times S(d,\theta)$, $S(d,\theta) = \{\xi \in \mathbf{C}\colon |\arg\xi - d| < \theta/2\}$, $\tilde{u}(t,\xi)$ *can be analytically continued to a function with a growth of exponential type, i.e., there exist* $C_0 > 0$, $K > 0$ *such that*

$$\sup_{t \in D_R} \left|\tilde{u}(t,\xi)\right| \leqslant C_0\, \mathrm{e}^{K|\xi|}, \quad \forall \xi \in S(d,\theta) = \left\{\xi \in \mathbf{C},\ |\arg\xi - d| \leqslant \frac{\theta}{2}\right\}. \tag{3}$$

Some extensions of Theorem 1.1 will be given in Section 3. In fact, if $p \geqslant 2$, the k-summability in the sense of Ramis [8] can be used instead of the Borel summability. Moreover, if $(t\partial_t)^m u - c(x)(x^{k+1}\partial_x)^n u$ is the "dominant" linear part, the equation admits a unique positive slope in the variable x and remains Fuchsian type in t.

In the forthcoming paper [3], by means of a fine Nagumo norm estimates, it will be shown that the result of Theorem 1.1 remains valid if condition (F3) is replaced by the following one:

(F3B) The inequality $\mathrm{val}(a_{i,j,\alpha}) \geqslant 1$ holds for each (i,j,α) such that $\alpha > 0$.

2. The sketch of the proof of Theorem 1.1

Since $b(0) \notin \mathbf{N}^*$, there exists a unique formal power series solution $\hat{u}$ to Eq. (1). By a direct computation, one can show that if $\mathrm{val}(a_{i,0,\alpha}) \geqslant 1$ for all $\alpha > 0$, then $\mathrm{val}_{x=0}\hat{u} \geqslant \nu$, where ν is defined in (F3). Replacing u by $x^\nu u$ in Eq. (1) which gives an equation with $\nu = 0$; hence we can suppose $\nu = 0$ in Eq. (1). It is obvious that $u_{-1} \in \mathbf{C}\{t\}$,

$\tilde{u} \in \mathbf{C}\{t, \xi\}$ (cf. [4], Theorem 1.2). We replace $u(t,x)$ by $u_{-1}(t) + u_0(t)x + u_1(t)x^2 + x^2u(t,x)$ in Eq. (1), which yields

$$t\partial_t u = xa(x)t + b(x)u + c(x)x^2\partial_x u + \sum_{i+j+\alpha\geqslant 2} a_{i,j,\alpha}(x)t^i u^j (x^2\partial_x u)^\alpha, \tag{4}$$

where $a(x)$, $a_{i,j,\alpha}(x)$ are holomorphic at $x=0$ and where $b(x)$, $c(x)$ are the same functions as appeared in (2).

Let us consider the summability of the formal solution $\hat{u}(t,x)$ of (4), with $\hat{u}(0,x) \equiv \hat{u}(t,0) \equiv 0$. Let $\tilde{u}(t,\xi)$ be the formal Borel transform of $\hat{u}(t,x)$, which is given in Theorem 1.1; Eq. (4) is then transformed into the following convolution equation:

$$\begin{aligned}\big(t\partial_t - (b + c\xi)\big)\tilde{u} \;&=\; A(\xi)t + B(\xi) * \tilde{u} + C(\xi) * (\xi\tilde{u}) \\ &\quad + \sum_{i+j+\alpha\geqslant 2} t^i\big[A_{i,j,\alpha}(\xi) * \tilde{u}^{*j} * (\xi\tilde{u})^{*\alpha} + B_{i,j,\alpha}\tilde{u}^{*j} * (\xi\tilde{u})^{*\alpha}\big].\end{aligned} \tag{5}$$

Here, $*$ denotes the convolution with respect to ξ, $b = b(0)$, $c = c(0)$, $B_{i,j,\alpha} = a_{i,j,\alpha}(0)$, $B_{i,0,0} = 0$ and the functions A, B, C, $A_{i,j,\alpha}$ are the Borel transform of $xa(x)$, $b(x) - b$, $c(x) - c$ and $a_{i,j,\alpha}(x) - a_{i,j,\alpha}(0)$ respectively.

If we set $\tilde{u}(t,\xi) = \sum_{\ell\geqslant 1} \tilde{u}_\ell(\xi)t^\ell$, then $\tilde{u}_\ell(\xi)$ satisfies the following equation

$$(\ell - b - c\xi)\tilde{u}_\ell(\xi) = B(\xi) * \tilde{u}_\ell(\xi) + C(\xi) * \big(\xi\tilde{u}_\ell(\xi)\big) + F_\ell(\xi), \tag{6}$$

where $F_1(\xi) = A(\xi)$, $F_\ell(\xi)$ only depends on $\tilde{u}_m$, $m \leqslant \ell - 1$. It is easy to check that, for any direction considered in Theorem 1.1, there exist positive constants θ and σ such that for any $\ell \in \mathbf{N}^*$ and $\xi \in S(d,\theta)$, the following estimate holds:

$$|\ell - b - c\xi| \geqslant \sigma\big(\ell + |\xi|\big). \tag{7}$$

Definition 2.1 (cf. [5]). Let $S = S(d,\theta)$ and $\mu > 0$. We denote by $\mathcal{E}_\mu$ the space of holomorphic functions f in S such that there exists $C > 0$, satisfying

$$\big|f(\xi)\big(1 + |\xi|^2\big)\,\mathrm{e}^{-\mu|\xi|}\big| \leqslant C, \quad \forall\xi \in S.$$

We define the norm $\|\cdot\|_\mu$ in $\mathcal{E}_\mu$ by the formula

$$\|f\|_\mu = M_0 \sup_{\xi\in S}\big|f(\xi)\big(1 + |\xi|^2\big)\,\mathrm{e}^{-\mu|\xi|}\big|, \quad M_0 = \sup_{s>0}\left\{\frac{4(1+s^2)[\ln(1+s^2) + s\arctan s]}{s(s^2+4)}\right\}.$$

It is clear that $(\mathcal{E}_\mu, \|\cdot\|_\mu)$ is a Banach space and, in fact, a Banach algebra with respect to the convolution product. If $\mu_2 > \mu_1$, $\mathcal{E}_{\mu_1}$ can be considered as a sub-space of $\mathcal{E}_{\mu_2}$ and, for any $f \in \mathcal{E}_{\mu_1}$, $\|f\|_{\mu_2} \leqslant \|f\|_{\mu_1}$.

Lemma 2.2.

(a) (*cf.* [5]) *If* $f, g \in \mathcal{E}_\mu$, *then* $f * g \in \mathcal{E}_\mu$ *and* $\|f * g\|_\mu \leqslant \|f\|_\mu\|g\|_\mu$.
(b) *If* $\mu_2 > \mu_1$ *and* $f \in \mathcal{E}_{\mu_1}$, $g \in \mathcal{E}_{\mu_2}$, *then* $\|f * g\|_{\mu_2} \leqslant 4[M_0(\mu_2 - \mu_1)]^{-1}\|f\|_{\mu_1}\|g\|_{\mu_2}$.

By Lemma 2.2(b) and the Banach fixed point theorem, one can prove the following result.

Lemma 2.3. *Consider Eq.* (6) *and let* $\sigma > 0$ *as in* (7). *Let* μ_0 *be a sufficiently large number such that* $B, C \in \mathcal{E}_{\mu_0}$, *and let* $\mu = \mu_0 + 8(\sigma M_0)^{-1}(\|B\|_{\mu_0} + \|C\|_{\mu_0})$. *If* $F_\ell \in \mathcal{E}_\mu$, *then Eq.* (6) *has a unique solution* $\tilde{u}_\ell \in \mathcal{E}_\mu$ *and*

$$\big\|(\ell - b - c\xi)\tilde{u}_\ell\big\|_\mu \leqslant 2\|F_\ell\|_\mu. \tag{8}$$

Condition (F1) implies that there exists $\mu_0 > 0$ such that, in (5), all coefficient functions $A, B, C, A_{i,j,\alpha}$ are in $\mathcal{E}_{\mu_0}$ and that $\sum_{i+j+\alpha\geqslant 2} \|A_{i,j,\alpha}\|_{\mu_0} t^i u^j v^\alpha \in \mathbf{C}\{t,u,v\}$. So, by using Lemma 2.3 and induction on $\ell \in \mathbf{N}$, we get $(\ell - b - c\xi)\tilde{u}_\ell \in \mathcal{E}_\mu$.

Set $U(t) = \sum_{\ell\geqslant 1} \|\tilde{u}_\ell\|_\mu t^\ell$ and $W(t) = \sum_{\ell\geqslant 1} \|(\ell - b - c\xi)\tilde{u}_\ell\|_\mu t^\ell$. Let "$\ll$" be the "majorant series relationship" between a pair of power series. By Lemma 2.2(a) and estimate (8), we have:

$$U(t) \ll \sigma^{-1} W(t) \ll 2\sigma^{-1}\left[\|A\|_{\mu_0} t + \sum_{i+j+\alpha\geqslant 2} \bigl(|B_{i,j,\alpha}| + \|A_{i,j,\alpha}\|_{\mu_0}\bigr) t^i U^j(t) \bigl(\sigma^{-1} W(t)\bigr)^\alpha\right].$$

Then, we have $U(t) \ll Y(t)$ if we denote by $Y(t)$ the power series solution of the following analytic functional equation

$$Y(t) = 2\sigma^{-1}\left[\|A\|_{\mu_0} t + \sum_{i+j+\alpha\geqslant 2} \bigl(|B_{i,j,\alpha}| + \|A_{i,j,\alpha}\|_{\mu_0}\bigr) t^i Y^{j+\alpha}\right], \qquad Y(0) = 0.$$

By using the implicit function theorem, we know there exist a positive constant C and an open disk $D_R = \{t \in \mathbf{C}\colon |t| < R\}$ such that $\sup_{t\in D_R} |Y(t)| \leqslant C$; hence the conclusion (3) holds and we get Theorem 1.1.

3. Some extensions

By a similar manner, we can prove the following result:

Let $k = p - 1 \geqslant 1$. *If Eq.* (1) *satisfies* (F1)–(F3) *and* $b(0) \notin \mathbf{N}^*$, *then it has a unique formal solution* $\hat{u}(t,x)$ *which is* k*-summable in all directions of the* x*-plane except at most a countable set.*

More generally, let $k, m, n \in \mathbf{N}^*$ and consider the following equation:

$$(t\partial_t)^m u = F\bigl(t, x, \bigl\{(t\partial_t)^i \bigl(x^{k+1}\partial_x\bigr)^j u\bigr\}_{(i,j)\in\mathcal{F}}\bigr), \qquad u(0,x) \equiv 0, \tag{9}$$

where $(t,x) \in \mathbf{C}^2$, $\mathcal{F} = \{(i,j) \in \mathbf{N}^2\colon\ in + jm \leqslant mn,\ i < m\}$ and $\mathbf{N} = \{0, 1, 2, \ldots\}$. We suppose that

- (H1) $F(t, x, v_{i,j})$ is holomorphic near the origin of $\mathbf{C}^{2+N}$, where $N = \#\mathcal{F}$;
- (H2) $F(0, x, 0) \equiv 0$ near $x = 0$;
- (H3) $\partial_{v_{0,n}} F(0,0,0) \neq 0$;
- (H4) For any $\ell \in \mathbf{N}^*$, $\ell^m - \sum_{0\leqslant i\leqslant m-1} \partial_{v_{i,0}} F(0,0,0)\ell^i \neq 0$.

Remark that the condition (H3) guarantees the Newton polygon with respect to the variable x to have a unique slope which equals k. Similar to the hypothesis that $b \notin \mathbf{N}^*$ in Theorem 1.1, (H4) implies the existence of a power series solution of (9).

Theorem 3.1. *If Eq.* (9) *satisfies* (H1)–(H4), *then it has a unique formal solution* $\hat{u}(t,x)$ *which is* k*-summable in all directions except at most a countable set.*

Let us explain how to locate the singular directions in the case $k = 1$. Set $\mathcal{F}_0 = \{(i,j) \in \mathcal{F}\colon\ in + jm = mn\}$ and consider the polynomials

$$P_0(\ell,\xi) = \ell^m - \sum_{(i,j)\in\mathcal{F}_0} \partial_{v_{i,j}} F(0,0,0)\ell^i \xi^j, \qquad P(\ell,\xi) = \ell^m - \sum_{(i,j)\in\mathcal{F}} \partial_{v_{i,j}} F(0,0,0)\ell^i \xi^j,$$

and their zero point sets

$$\mathcal{Z}_0 = \bigcup_{\ell\in\mathbf{N}^*} \bigl\{\xi \in \mathbf{C}\colon\ P_0(\ell,\xi) = 0\bigr\}, \qquad \mathcal{Z} = \bigcup_{\ell\in\mathbf{N}^*} \bigl\{\xi \in \mathbf{C}\colon\ P(\ell,\xi) = 0\bigr\}.$$

Noticing that for any $\ell > 0$, $P_0(\ell, \xi) = \ell^m P_0(1, \ell^{-m/n}\xi)$, it follows that there exists a set Ω consisting of at most n distinct directions, such that $\mathcal{Z}_0 \subset \Omega$.

Lemma 3.2.

(a) *For any closed angular sector V with $V \cap \Omega = \emptyset$, $V \cap \mathcal{Z}$ is finite. In other words, $\mathcal{Z}$ accumulates to Ω.*
(b) *For any direction d that does not pass through any point of $\mathcal{Z} \cup \mathcal{Z}_0$, there exist $\theta > 0$ and $\sigma > 0$ such that*

$$\left|P(\ell, \xi)\right| \geqslant \sigma\left(\ell^m + |\xi|^n\right), \quad \forall \ell \in \mathbf{N}^*, \ \forall \xi \in S(d, \theta). \tag{10}$$

Observe here, the condition (7) is replaced by (10), and the singular directions, indicated by $\mathcal{Z} \cup \mathcal{Z}_0$, converge to the direction set Ω.

Acknowledgements

The research was supported by the NSFC and 973 project of the MOSTC.

A part of this work was started while C. Zhang was a visiting Professor at the Institute of Mathematics of Wuhan University in July, 2001; he would like to thank the Institute for the invitation and its hospitality. The work was finished while Z. Luo worked as a post-doc at University of Lille 1 in 2002; he would like to thank the university.

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