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Probability Theory

Local self-similarity and the Hausdorff dimension

Auto-similarité locale et dimension de Hausdorff

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Abstract

Let X be a locally self-similar stochastic process of index $0 < H < 1$ whose sample paths are a.s. $C^{H-\varepsilon}$ for all $\varepsilon > 0$. Then the Hausdorff dimension of the graph of X is a.s. $2 - H$. ***To cite this article: A. Benassi et al., C. R. Acad. Sci. Paris, Ser. I 336 (2003).***

Résumé

Soit X un processus stochastique localement auto-similaire d'exposant $0 < H < 1$ dont les trajectoires sont p.s. $C^{H-\varepsilon}$ pour tout $\varepsilon > 0$. Alors la dimension de Hausdorff du graphe de X est p.s. $2 - H$. ***Pour citer cet article : A. Benassi et al., C. R. Acad. Sci. Paris, Ser. I 336 (2003).***

Version française abrégée

L'auto-similarité est souvent utilisée pour construire des objets dont la dimension de Hausdorff est non entière (e.g., [8]). Il est souvent délicat de comprendre si l'auto-similarité conduit nécessairement à des dimensions de Hausdorff non entières. Le but de cet article est de relier l'auto-similarité (locale) d'un processus stochastique et la dimension de Hausdorff du graphe de ses trajectoires. Soit $X(t),\ t \in [0,1]$, un processus stochastique centré. L'auto-similarité locale a été introduite dans [7,9]. Si un processus est localement auto-similaire d'exposant H au point t, alors il existe une variable aléatoire non nulle Y_t telle que :

$$\lim_{h\to 0} \frac{X(t+h)-X(t)}{|h|^H} \stackrel{\mathcal{L}}{=} Y_t,$$

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où $\stackrel{\mathcal{L}}{=}$ désigne la convergence en loi. Le premier résultat de cet article est le suivant : si X est localement auto-similaire d'exposant H en tout point, alors la dimension de Hausdorff $\dim_{\mathcal{H}} X$ du graphe de ses trajectoires est p.s. minorée par $2-H$. La question se pose alors de savoir si cette inégalité $\dim_{\mathcal{H}} X \geqslant 2-H$ peut être une égalité. La réponse ne peut être que négative : il suffit de penser aux processus α-stables, avec $\alpha<1$. Ces processus sont localement auto-similaires d'exposant $H=1/\alpha$ et la dimension de Hausdorff du graphe de leurs trajectoires ne peut être $2-H$ puisque $2-H<1$. Mais si l'on suppose en plus que les trajectoires de X sont $C^{H-\varepsilon}$ pour tout $\varepsilon>0$, alors on a l'égalité $\dim_{\mathcal{H}} X \stackrel{\text{p.s.}}{=} 2-H$.

Nous donnons ensuite une liste d'exemples de processus rentrant dans cette catégorie (i.e., localement auto-similaire et höldérien de même exposant H) : mouvements browniens fractionnaires, bruits blancs filtrés, processus de Lévy fractionnaires harmonisables, processus fractionnaires harmonisables stables. Nous montrons également que les mouvements browniens multifractionnaires, même s'ils ne rentrent pas stricto sensu dans cette catégorie, se traitent de la même façon.

1. Introduction

Self-similarity is often used to build objects with non-integer Hausdorff dimension (e.g., [8]). It is never straightforward to understand if self-similarity always leads to non-integer Hausdorff dimension. The aim of this paper is to link the (local) self-similarity and the Hausdorff dimension of the graph of a stochastic process. To clarify, let $X(t), t\in[0,1]$ be a centered stochastic process. The concept of local self-similarity has been introduced in [7,9]. If a stochastic process is self-similar at point t with index H then there exists a non-vanishing random variable Y_t such that:

$$\lim_{h\to 0}\frac{X(t+h)-X(t)}{|h|^H} \stackrel{\mathcal{D}}{=} Y_t,$$

where $\stackrel{\mathcal{D}}{=}$ stands for convergence in distribution. The first result of this paper is the following: if X is locally self-similar at every point t, then, a.s., $\dim_{\mathcal{H}} X \geqslant 2-H$, where $\dim_{\mathcal{H}} X$ stands for the Hausdorff dimension of the graph of X.

The following question then arises. Can the inequality $\dim_{\mathcal{H}} X \geqslant 2-H$ be an equality? The answer is negative and it is easy to construct counter-examples. There exists self-similar processes of index $H>1$ with stationary increments, for instance α-stable processes with $\alpha<1$. These processes are of course locally self-similar with index H, and the Hausdorff dimension of their graph cannot be $2-H$ since $2-H<1$. One needs an additional condition in order to obtain the equality $\dim_{\mathcal{H}} X = 2-H$. Numerous locally self-similar processes of index H at point t have sample paths that are, a.s., $(H-\varepsilon)$-Hölderian, for all $\varepsilon>0$. Standard results on Hausdorff dimension on graph of functions then imply the equality $\dim_{\mathcal{H}} X \stackrel{\text{a.s.}}{=} 2-H$.

The paper is organized as follows. Section 2 contains the statement of the results. Examples are provided in Section 3. Proofs are given in Section 4. The case of Multifractional Brownian Motions is given in Section 5.

2. Main result

Let $X(t)$ be a centered stochastic process from $[0,1]$ to $\mathbf{R}$. We assume that process X is locally self-similar with index H at every point $t\in[0,1]$. More precisely, we assume:

Assumption 1. There exists $0<H<1$, such that for all $t\in[0,1]$, there exists a random variable Y_t such that:

$$\lim_{h\to 0}\frac{X(t+h)-X(t)}{|h|^H} \stackrel{\mathcal{D}}{=} Y_t, \tag{1}$$

with $\mathbf{P}(Y_t = 0) = 0$. Moreover, one assumes the existence of a dominating function $\psi \in L^1([0,1] \times \mathbf{R})$ and the existence of $h_0 > 0$ such that:

$$\left|\mathbf{E}\left(\mathrm{e}^{\mathrm{i}\lambda(X(t+h)-X(t))/|h|^H}\right)\right| \leqslant \psi(t,\lambda), \quad t \in [0,1],\ \forall |h| \leqslant h_0\ s.t.\ t+h \in [0,1],\ \lambda \in \mathbf{R}.$$

As pointed out in [6, Proposition 3.3, p. 109], condition (1) implies that the Hölder exponent of process X at point t cannot be strictly greater than H. Nevertheless, Assumption 1 does not imply that the Hölder exponent of X is equal to H: α-stable processes are processes with stationary increments and are $1/\alpha$ self-similar; therefore, α-stable process satisfies Assumption 1. Nevertheless, except for Brownian motion, α-stable processes are discontinuous with probability 1.

We therefore assume.

Assumption 2. The sample paths of X are, a.s., $C^{H-\varepsilon}$, for all $\varepsilon > 0$.

We then state the main result of this paper.

Theorem 2.1. *Assuming Assumptions* 1 *and* 2*, then*:

$$\dim_{\mathcal{H}} X \overset{\text{a.s.}}{=} 2 - H.$$

3. Examples

3.1. Fractional Brownian Motions

Fractional Brownian Motion of index H is a self-similar process with stationary increments. Assumption 1 is fulfilled. Kolmogorov's theorem shows that Assumption 2 is fulfilled, so, as known for a long time, the Hausdorff dimension of the graph of a Fractional Brownian Motion is, a.s., $2 - H$.

3.2. Filtered White Noises

Filtered White Noises [5] are extensions of Fractional Brownian Motions, based on its harmonizable representation:

$$X(t) = \int_{\mathbf{R}} \frac{a(t,\lambda)(\mathrm{e}^{\mathrm{i}t\lambda} - 1)}{|\lambda|^{1/2+H}}\, \mathrm{d}W(\lambda),$$

where $\mathrm{d}W(\lambda)$ is the Brownian measure on L^2. Assume that $(t,\lambda) \to a(t,\lambda)$ is bounded on $[0,1] \times \mathbf{R}$ and that $\lim_{|\lambda| \to +\infty} a(t,\lambda)$ exists and is nonvanishing. Since X is a Gaussian process, one has only to evaluate $\mathbf{E}(X(t) - X(t'))^2$ in order to verify Assumption 1. For this purpose, one can use Lemma 1 of [5]. Then X satisfies Assumption 1 with index H. Kolmogorov's theorem shows that Assumption 2 is fulfilled with index H. The Hausdorff dimension of the graph of a Filtered White Noise is, a.s., $2 - H$.

3.3. Real Harmonizable Fractional Lévy Motions

Real Harmonizable Fractional Lévy Motions are obtained by a fractional harmonizable integration of a complex Lévy measure L which moments of order larger than two are finite [6]:

$$X(t) = \int_{\mathbf{R}} \frac{e^{it\lambda} - 1}{|\lambda|^{1/2+H}} \, dL(\lambda).$$

Fractional Brownian Motion is a particular case of Real Harmonizable Fractional Lévy Motion (when L is indeed the Brownian measure). Except in this case, Real Harmonizable Fractional Lévy Motions are non-Gaussian processes. According to Proposition 3.1 of [6], Assumption 1 is fulfilled with index H and Gaussian tangent variables Y_t, although X is itself not Gaussian. According to Proposition 3.3 of [6], Assumption 2 is fulfilled with index H. The Hausdorff dimension of the graph of a Real Harmonizable Fractional Lévy Motion is, a.s., $2 - H$.

3.4. Real Harmonizable Fractional Stable Motions

Real Harmonizable Fractional Stable Motions are obtained by a fractional harmonizable integration of a complex isotropic symmetric α-stable random measure M [10, Chapter 7.7]:

$$X(t) = \operatorname{Re} \int_{\mathbf{R}} \frac{e^{it\lambda} - 1}{|\lambda|^{1/\alpha+H}} \, dM(\lambda).$$

Real Harmonizable Fractional Stable Motions are self-similar with index H and have stationary increments: Assumption 1 is fulfilled with index H. One can check that their sample paths satisfy Assumption 2 with index H. The Hausdorff dimension of the graph of a Real Harmonizable Fractional Stable Motion is, a.s., $2 - H$.

4. Proofs

4.1. Upper bound

Assumption 2 and standard results on the Hölderian function (e.g., [8, Chapter 11.1]) imply that, a.s., $\dim_{\mathcal{H}} X \leqslant 2 - H$.

4.2. Lower bound

Following the Frostman criterion (e.g., [8, Chapter 4.3]), if one proves that the integral I_s

$$I_s = \int_{[0,1]\times[0,1]} \mathbf{E}\left(\left(X(t) - X(u)\right)^2 + |t-u|^2\right)^{-s/2} dt \, du, \tag{2}$$

is finite, then, a.s., $\dim_{\mathcal{H}} X \geqslant s$. Clearly, if the integral J_s

$$J_s = \int_{t\in[0,1],\ |t-u|\leqslant h_0} \mathbf{E}\left(\left(X(t) - X(u)\right)^2 + (t-u)^2\right)^{-s/2} dt \, du,$$

is finite, then, a.s., $\dim_{\mathcal{H}} X \geqslant s$.

Let $1 < s < 2$. To study the integral J_s, we need the following fundamental lemma, presented by Ayache et Roueff in [2,3] used in [1] and also obtained by the authors.

Lemma 4.1. *There exists a bounded function $f_s(\omega) \in L^1(\mathbf{R})$ such that*:

$$\left(x^2 + 1\right)^{-s/2} = \int_{\mathbf{R}} e^{i\omega x} f_s(\omega) \, d\omega.$$

It follows that:

$$\mathbf{E}\big(\big(X(t)-X(u)\big)^2+(t-u)^2\big)^{-s/2}=|t-u|^{-s}\int_{\mathbf{R}}\mathbf{E}\,\mathrm{e}^{\mathrm{i}(\omega(X(t)-X(u)))/|t-u|}f_s(\omega)\,\mathrm{d}\omega.$$

We use the change of variables $t=t,\ u-t=h$. Integral J_s can be rewritten as:

$$J_s=\int_{\Lambda}\mathbf{E}\big(\big(X(t+h)-X(t)\big)^2+|h|^2\big)^{-s/2}\,\mathrm{d}h\,\mathrm{d}t,$$

where $\Lambda=\{t\in[0,1],|h|\leqslant h_0 \text{ s.t. } t+h\in[0,1]\}$. One then deduces:

$$\mathbf{E}\big(\big(X(t+h)-X(t)\big)^2+|h|^2\big)^{-s/2}=|h|^{-s}\int_{\mathbf{R}}\mathbf{E}\,\mathrm{e}^{\mathrm{i}(\omega(X(t+h)-X(t)))/|h|}f_s(\omega)\,\mathrm{d}\omega.$$

The change of variable $\lambda=|h|^{H-1}\omega$ leads to:

$$\mathbf{E}\big(\big(X(t+h)-X(t)\big)^2+|h|^2\big)^{-s/2}=|h|^{1-H-s}\int_{\mathbf{R}}\mathbf{E}\,\mathrm{e}^{\mathrm{i}\lambda(X(t+h)-X(t))/|h|^H}f_s\big(h^{1-H}\lambda\big)\,\mathrm{d}\lambda.$$

By Assumption 1:

$$\mathbf{E}\big(\big(X(t+h)-X(t)\big)^2+|h|^2\big)^{-s/2}\leqslant\sup_{\omega\in\mathbf{R}}\big|f_s(\omega)\big||h|^{1-H-s}\int_{\mathbf{R}}\psi(t,\lambda)\,\mathrm{d}\lambda. \tag{3}$$

This leads to:

$$J_s\leqslant\sup_{\omega\in\mathbf{R}}\big|f_s(\omega)\big|\int_{\Lambda}\int_{\mathbf{R}}\psi(t,\lambda)|h|^{1-H-s}\,\mathrm{d}\lambda\,\mathrm{d}h\,\mathrm{d}t.$$

Therefore, J_s is finite a soon as $s<2-H$ and Theorem 2.1 is proved.

5. An extension: Multifractional Brownian Motions

Multifractional Brownian Motions are Gaussian processes which index H is no more constant [7,9]. Consider the harmonizable representation [4]:

$$X(t)=\int_{\mathbf{R}}\frac{a(t,\lambda)(\mathrm{e}^{\mathrm{i}t\lambda}-1)}{|\lambda|^{1/2+H(t)}}\,\mathrm{d}W(\lambda),$$

where $\mathrm{d}W(\lambda)$ is the Brownian measure on L^2. Assume that $(t,\lambda)\to a(t,\lambda)$ is bounded on $[0,1]\times\mathbf{R}$ and that $\lim_{|\lambda|\to+\infty}a(t,\lambda)$ exists and is nonvanishing. Assume moreover that function $t\to H(t)$ is C^1 and $0<H(t)<1$. Let $H_i=\inf_{t\in[0,1]}H(t)$.

Theorem 2.1 cannot be applied directly. Nevertheless, an inspection of the proof makes clear that inequality (3) is still available for process X. Since X is Gaussian, we only have to evaluate $\mathbf{E}(X(t)-X(t'))^2$ as we do for Filtered White Noises using Lemma 1 of [5] in order to have a function ψ. Consider now process X on $[a,b]$ where $0<a<b<1$. One then easily deduces that I_s, defined in the proof in (2), is finite as soon as $s<2-\sup_{t\in[a,b]}H(t)$. Let t_i be a point where $H(t_i)=H_i$. Since $\lim_{a\to t_i^-,b\to t_i^+}\sup_{t\in[a,b]}H(t)=H_i$, one deduces, a.s., $\dim_{\mathcal{H}}X\geqslant 2-H_i$.

Kolmogorov theorem proves that Assumption 2 is fulfilled with index H_i. The Hausdorff dimension of the graph of a Multifractional Brownian Motion is, a.s., $2 - H_i$. Note that, when $a \equiv 1$, this dimension has been computed in [9].

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