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C. R. Acad. Sci. Paris, Ser. I 336 (2003) 135–140
COMPTES RENDUS MATHEMATIQUE

Partial Differential Equations

On the fundamental state for a Schrödinger operator with magnetic field in a domain with corners

Etat fondamental de l'opérateur de Schrödinger avec champ magnétique dans un domaine à coin

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Received and accepted 26 November 2002

Presented by Haïm Brézis

Abstract

We show that the Neumann realization for the Schrödinger operator with a constant magnetic field in a sector has at least one eigenvalue below the essential spectrum, when the angle is sufficiently small. We establish the complete asymptotics of the lowest eigenvalue as the angle tends to 0. This study is applied to the analysis of the bottom of the spectrum in the semi-classical case for domains with edges. ***To cite this article: V. Bonnaillie, C. R. Acad. Sci. Paris, Ser. I 336 (2003).***

Résumé

Nous démontrons que la réalisation de Neumann de l'opérateur de Schrödinger avec un champ magnétique constant sur un secteur $\Omega_\alpha \subset \mathbb{R}^2$ d'angle $\alpha \in]0, \pi]$ admet au moins une valeur propre, en dessous du spectre essentiel, quand l'angle est suffisamment petit. Nous établissons un développement limité de la plus petite valeur propre pour α proche de 0. Cette étude permet de donner des estimations du bas du spectre dans le cas semi-classique pour des domaines à coin. ***Pour citer cet article : V. Bonnaillie, C. R. Acad. Sci. Paris, Ser. I 336 (2003).***

Version française abrégée

Le résultat principal de cette Note est d'établir un développement limité du bas du spectre, noté $\mu(\alpha)$, de la réalisation de Neumann de l'opérateur de Schrödinger, noté P_{Ω_α}, avec un champ magnétique constant égal à 1, sur un secteur angulaire d'angle α pour α près de 0. L'expression de ce développement est donnée au Théorème 3.1. La construction de chaque terme du développement est explicite et permet aussi de donner des conditions suffisantes simples sur l'angle pour que le bas du spectre soit une valeur propre.

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1631-073X/03/$ – see front matter
doi:10.1016/S1631-073X(03)00008-6

Pour justifier cette construction, nous effectuons des changements de variables et de jauge pour transformer la forme sesquilinéaire associée à l'opérateur de Schrödinger sur un domaine dépendant de α, en une nouvelle forme dépendant de α mais sur un domaine fixe. Cette nouvelle forme, a_α, donnée en (1), utilise deux formes de poids par rapport à α différents : ℓ définie en (2) et la forme associée à la réalisation de Neumann de l'opérateur $-\partial_\eta^2$. Revenant à la forme a_α, une idée naturelle est de commencer par restreindre la forme aux fonctions indépendantes de η afin de supprimer la singularité $1/\alpha^2$. Ceci nous conduit à étudier une nouvelle forme ℓ^{moy}, déduite de la restriction de ℓ aux fonctions indépendantes de η. La définition de ℓ^{moy} est donnée en (4) et son étude est précisée au Lemme 2.1. L'état fondamental de l'opérateur L^{moy} associé donne un quasi-mode et permet de majorer le bas du spectre de l'opérateur de Schrödinger initial par $\alpha/\sqrt{3}$.

Nous établissons ensuite un développement limité complet. Pour cela, nous allons chercher une fonction u sous forme d'une série formelle $\sum_{k\geqslant 0}\alpha^{2k}u_k$ et une série $\mu(\alpha)\sim\alpha\sum_{k\geqslant 0}\alpha^{2k}m_k$ telles que : $P_{\Omega_\alpha}u\sim\mu(\alpha)u$. Pour résoudre cette égalité, nous regardons chaque coefficient de α^{2k} et utilisons les Lemmes 2.1 et 2.2 afin de déterminer chaque coefficient u_k et m_k par récurrence.

Cette méthode nous permet d'associer à la série formelle un quasi-mode et conduit donc une majoration du bas du spectre. Pour montrer que $\mu(\alpha)$ est effectivement le bon candidat pour être un développement limité, nous estimons l'écart entre les deux premières valeurs propres comme en (15). Il reste à appliquer le théorème spectral pour montrer que la majoration précédente nous donne en fait un développement limité de la plus petite valeur propre.

Ce résultat complète les travaux sur la détermination du bas du spectre dans le plan $\mathbb{R}^2$ ou le demi-plan $\mathbb{R}\times\mathbb{R}^+$ et permet d'étudier l'opérateur de Schrödinger avec un champ magnétique non constant et un paramètre semi-classique dans des domaines avec un nombre fini de coins en reprenant les techniques de Helffer et Morame [6]. En effet, les techniques utilisées par ces auteurs sont d'utiliser une partition de l'unité afin de se ramener aux modèles $\mathbb{R}^2$ et $\mathbb{R}\times\mathbb{R}^+$. Or, si nous souhaitons obtenir des informations sur le bas du spectre dans des domaines à coin, l'utilisation de ces mêmes techniques conduit naturellement à l'étude de l'opérateur de Schrödinger dans un secteur angulaire. L'étude menée précédemment permet donc d'obtenir l'estimation donnée au Théorème 4.1. Signalons enfin que cette étude constitue une étape dans l'analyse de l'apparition de la supraconductivité (cf. les travaux de Bernoff et Sternberg [2], Del Pino, Felmer et Sternberg [10]).

1. Introduction

Let $\Omega_\alpha\subset\mathbb{R}^2$ be an angular sector with angle $\alpha\in\,]0,\pi]$, ν the unit outer normal of the boundary $\Gamma=\partial\Omega_\alpha$ when it is well defined. Our goal is to determine the bottom of the spectrum of the Neumann realization of the operator P_{Ω_α} defined by: $P_{\Omega_\alpha}=-\nabla_{\mathcal{A}}^2$, with $\nabla_{\mathcal{A}}=\nabla-\mathrm{i}\mathcal{A}$ and $\mathcal{A}:=\frac{1}{2}\binom{y}{-x}$, on the domain:

$$\mathcal{D}^N(P_{\Omega_\alpha}):=\left\{u\in L^2(\Omega_\alpha)\mid \nabla_{\mathcal{A}}u\in L^2(\Omega_\alpha),\ \nabla_{\mathcal{A}}^2u\in L^2(\Omega_\alpha),\ \nu\cdot\nabla_{\mathcal{A}}u_{|\Gamma}=0\right\}.$$

We denote by a_{Ω_α} the sesquilinear form associated to the operator P_{Ω_α}, which is defined on $H^1_{\mathcal{A}}(\Omega_\alpha):=\{u\in L^2(\Omega_\alpha)\mid\nabla_{\mathcal{A}}u\in L^2(\Omega_\alpha)\}$ and by $\mu(\alpha)$ the bottom of the spectrum of P_{Ω_α}. The Persson's lemma [9] may be generalized for unbounded domains of $\mathbb{R}^2$ and Neumann realizations. Its application gives that the bottom of the essential spectrum of $-\nabla_{\mathcal{A}}^2$ is equal to $\Theta_0:=\mu(\pi)$, which corresponds to the bottom of the spectrum relative to the half plane. After a change of variables, a scaling and a gauge transformation, we get a new operator P_α which is now defined on a constant domain $\Omega:=\mathbb{R}^+\times\,]-\frac{1}{2},\frac{1}{2}[$ (instead of a constant operator $-\nabla_{\mathcal{A}}^2$ on an α-dependent domain Ω_α). This P_α is associated to the sequilinear form a_α, which is defined on:

$$\mathcal{V}^N:=\left\{u\in L^2(\Omega)\;\middle|\;\sqrt{t}(D_t-\eta)u\in L^2(\Omega),\ \frac{1}{\sqrt{t}}D_\eta u\in L^2(\Omega)\right\},$$

by:

$$a_\alpha(u,v) := \int\limits_\Omega \left(2t(D_t-\eta)u\overline{(D_t-\eta)v} + \frac{1}{2\alpha^2 t} D_\eta u \overline{D_\eta v}\right) \mathrm{d}t\,\mathrm{d}\eta, \quad \forall u,v \in \mathcal{V}^N. \tag{1}$$

Notice that the bottom of the spectrum of P_α, $\lambda(\alpha)$, satisfies the relation $\mu(\alpha) = \alpha\lambda(\alpha)$.

Remark 1. From the expression of the form a_α, we immediately see that $\alpha \mapsto \alpha\mu(\alpha)$ is increasing and $\alpha \mapsto \mu(\alpha)/\alpha$ is decreasing. It would be interesting to show that $\alpha \mapsto \mu(\alpha)$ is monotonous from $]0,\pi]$ onto $]0,\Theta_0]$.

2. Analysis of the two key-operators

In the expression of a_α in (1), two forms (and two associated operators) appear. The first one is ℓ (with associated L) which will be defined just below and the second is associated to the Neumann realization of $-\partial_\eta^2$ in $]-\frac{1}{2},+\frac{1}{2}[$. We define the sesquilinear form ℓ, on $\mathcal{V}_\ell^N := \{u \in L^2(\Omega) \mid \sqrt{t}(D_t-\eta)u \in L^2(\Omega)\}$, by:

$$\ell(u,v) = \int\limits_\Omega 2t\big((D_t-\eta)u\big)\overline{(D_t-\eta)v}\,\mathrm{d}t\,\mathrm{d}\eta, \quad \forall u,v \in \mathcal{V}_\ell^N. \tag{2}$$

Let $\mathcal{P}\otimes\mathcal{S}(\overline{\mathbb{R}^+})$ be the space of polynomial functions in η whose coefficients are in $\mathcal{S}(\overline{\mathbb{R}^+})$. We define the operator $L := 2(D_t-\eta)(t(D_t-\eta))$ on $\mathcal{P}\otimes\mathcal{S}(\overline{\mathbb{R}^+})$, and verify:

$$\forall u \in \mathcal{P}\otimes\mathcal{S}\big(\overline{\mathbb{R}^+}\big),\ \forall v \in \mathcal{V}_\ell^N, \quad \ell(u,v) = \langle Lu, v\rangle_{L^2(\Omega)}. \tag{3}$$

The form a_α contains a term in $1/\alpha^2$, so when trying to minimize the associated quadratic form, it is quite natural to begin with studying the restriction of the form to functions which are independent of η. We define the form ℓ^{moy} which appears naturally when we restrict ℓ to functions independent of η. Then the sesquilinear form ℓ^{moy} is defined on $\mathcal{V}_{\mathrm{moy}}^N := \{f \in L^2(\mathbb{R}^+) \mid \sqrt{t}f \in L^2(\mathbb{R}^+),\ \sqrt{t}D_t f \in L^2(\mathbb{R}^+)\}$, by:

$$\ell^{\mathrm{moy}}(u,v) = \int\limits_0^\infty 2\left(D_t u\,\overline{D_t v} + \frac{1}{12}u\bar{v}\right)t\,\mathrm{d}t, \quad \forall u,v \in \mathcal{V}_{\mathrm{moy}}^N. \tag{4}$$

The associated operator

$$L^{\mathrm{moy}} = 2\left(D_t t D_t + \frac{t}{12}\right)$$

is self-adjoint and its domain can be characterized as being $\mathcal{W}_2^1(R^+) := \{u \in H^1(\mathbb{R}^+) \mid tu \in H^2(\mathbb{R}^+)\}$. Its spectrum is discrete and the eigenvalues are simple and given by $(2n-1)/\sqrt{3}$, $n \geqslant 1$. Moreover each eigen subspace is included in $\mathcal{S}(\overline{\mathbb{R}^+})$. So, we deduce by standard Fredholm theory and regularity (cf. [3]), the following lemma:

Lemma 2.1. *Let* $(\lambda_1^{\mathrm{moy}}, u_1^{\mathrm{moy}})$ *the fundamental state of* L^{moy}. *For all* $f \in \mathcal{S}(\overline{\mathbb{R}^+})$ *orthogonal to* u_1^{moy}, *there exists an unique* $u \in \mathcal{S}(\overline{\mathbb{R}^+})$ *such that*:

$$\begin{cases} (L^{\mathrm{moy}} - \lambda_1^{\mathrm{moy}})u = f, \\ \int_0^\infty u(t)\,\overline{u_1^{\mathrm{moy}}(t)}\,\mathrm{d}t = 0. \end{cases} \tag{5}$$

Furthermore, if u *is given by* (5), *then for all functions* $v \in \mathcal{V}_{\mathrm{moy}}^N$:

$$\ell^{\mathrm{moy}}(u,v) - \lambda_1^{\mathrm{moy}}\langle u,v\rangle_{L^2(\Omega)} = \langle f,v\rangle_{L^2(\Omega)}. \tag{6}$$

An elementary study of the Neumann realization of $-\partial_\eta^2$ leads to the following lemma:

Lemma 2.2. *Let f be a polynomial in η with coefficients in $\mathcal{S}(\overline{\mathbb{R}^+})$ such that for all $t \in \overline{\mathbb{R}^+}$, $\int_{-1/2}^{1/2} f(t,\eta)\,\mathrm{d}\eta = 0$. Then there exists an unique $\tilde{u} \in \mathcal{P} \otimes \mathcal{S}(\overline{\mathbb{R}^+})$ such that*:

$$\begin{cases} -\partial_\eta^2 \tilde{u} = 2tf, \\ \partial_\eta \tilde{u}_{|\eta=-1/2,1/2} = 0, \\ \int_{-1/2}^{1/2} \tilde{u}(t,\eta)\,\mathrm{d}\eta = 0. \end{cases} \tag{7}$$

Then, for all $v \in \mathcal{V}^N$, we have:

$$\frac{1}{2}\int_0^\infty \int_{-1/2}^{1/2} \frac{1}{t}\partial_\eta u \partial_\eta \bar{v}\,\mathrm{d}\eta\,\mathrm{d}t = \int_0^\infty \int_{-1/2}^{1/2} f\bar{v}\,\mathrm{d}\eta\,\mathrm{d}t. \tag{8}$$

3. Asymptotics

Theorem 3.1. *There exists a real sequence $(m_j)_{j\in\mathbb{N}}$, with $m_0 = 1/\sqrt{3}$ such that, as α tends to 0*:

$$\mu(\alpha) \sim \alpha \sum_{j=0}^{\infty} m_j \alpha^{2j}. \tag{9}$$

Let us emphasize that the proof gives an explicit algorithm for determining the coefficients m_j as we shall see now. Let us give the main lines for the proof of this theorem.

3.1. Upper bound

We look for two sequences: $(u_k)_{k\in\mathbb{N}} \in (\mathcal{P}\otimes\mathcal{S}(\overline{\mathbb{R}^+}))^{\mathbb{N}}$ and $(m_k)_{k\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$ such that for all $n \in \mathbb{N}^*$, if we define $U^{(n)} = \sum_{k=0}^n \alpha^{2k} u_k$ and $\mu^{(n)}(\alpha) = \sum_{k=0}^n \alpha^{2k} m_k$, then, modulo $\mathcal{O}_n(\alpha^{2n+2})$, we have:

$$a_\alpha\big(U^{(n)}, v\big) \equiv \mu^{(n)}(\alpha)\big\langle U^{(n)}, v\big\rangle_{L^2(\Omega)}, \quad \forall v \in \mathcal{V}^N.$$

We expand the equation in powers of α and express that the coefficients of α^{2k} ($k \geqslant -1$) should cancel. The cancellation of the coefficient of $1/\alpha^2$ gives:

$$\int_\Omega \frac{1}{t}\partial_\eta u_0 \partial_\eta \bar{v}\,\mathrm{d}t\,\mathrm{d}\eta = 0, \quad \forall v \in \mathcal{V}^N. \tag{10}$$

The unique condition coming from this relation is that u_0 depends only on t.
The vanishing of the coefficient of α^{2k}, for $k \geqslant 0$, gives:

$$\ell(u_k, v) + \int_\Omega \frac{1}{2t}\partial_\eta u_{k+1}\partial_\eta \bar{v}\,\mathrm{d}t\,\mathrm{d}\eta = \int_\Omega \left(\sum_{j=0}^k m_j u_{k-j}\right)\bar{v}\,\mathrm{d}t\,\mathrm{d}\eta, \quad \forall v \in \mathcal{V}^N. \tag{11}$$

This determines u_k and m_k recursively. Let us give some details for the treatment of the first term:

- Restrict the relation (11) for $k = 0$ to functions only depending on t.
- Determine u_0 and m_0 by solving the spectral problem $L^{\mathrm{moy}} u_0 = m_0 u_0$ and choosing the fundamental state $(\lambda_1^{\mathrm{moy}}, u_1^{\mathrm{moy}})$.

- Return to the initial relation (11) with $k=0$ and obtain informations about u_1 thanks to Lemma 2.2.
- Determine u_1 and m_1 by looking at the relation (11) with $k=1$ and Lemma 2.1.

This method can be applied to the other coefficients and leads to the following proposition:

Proposition 3.2. *For all $n \geqslant 1$, we can find m_n and $u_n = u_n^0 + \tilde{u}_n$, with $u_n^0 \in \mathcal{S}(\overline{\mathbb{R}^+})$, satisfying the relations*:

$$m_n = \langle \tilde{u}_n, L u_1^{\mathrm{moy}} \rangle_{L^2(\Omega)}. \tag{12}$$

$$\begin{cases} -\partial_\eta^2 \tilde{u}_n = 2t\left(\displaystyle\sum_{k=0}^{n-1} m_{n-1-k} u_k - L u_{n-1}\right), \\ \partial_\eta \tilde{u}_{n|_{\eta=-1/2,1/2}} = 0, \qquad \displaystyle\int_{-1/2}^{1/2} \tilde{u}_n \,\mathrm{d}\eta = 0. \end{cases} \tag{13}$$

$$\begin{cases} \left(L^{\mathrm{moy}} - \lambda_1^{\mathrm{moy}}\right) u_n^0 = -\displaystyle\int_{-1/2}^{1/2} L\tilde{u}_n \,\mathrm{d}\eta + \sum_{k=1}^{n} m_k u_{n-k}^0, \\ \displaystyle\int_0^\infty u_n^0(t)\, \overline{u_1^{\mathrm{moy}}(t)} \,\mathrm{d}t = 0. \end{cases} \tag{14}$$

By construction, $U^{(n)}$ and $\mu^{(n)}(\alpha)$ are such that the Rayleigh quotient of $U^{(n)}$ is less than $\mu^{(n)}(\alpha) + C\alpha^{2n+2}$. A change of variables gives exactly the upper bound for Theorem 3.1. The spectral theorem shows that there is at least one eigenvalue with the asymptotic expansion (9).

3.2. Lower bound

One proves that (9) is actually an asymptotic expansion. The proof uses a priori estimates on the decay of the first eigenfunction [1], a lower bound of $\mu(\alpha)$ with an error less than α^2 and a lower bound of the splitting between the two smallest eigenvalues. This last lower bound is established by using new operators. Indeed, by studying the behavior of the first eigenvector of P_α, we show that the eigenvalues, $\lambda(\alpha)$ and $\lambda^R(\alpha)$, are exponentially closed, where $\lambda^R(\alpha)$ is the first eigenvalue for the operator P_α on the domain $\Omega_{0,R} :=]0,R[\times]-1/2,1/2[$ with a Dirichlet condition on the boundary $t=R$. After that, we compare the new operator P_α^R with the operator $P^{R,\rho}$ which is the self-adjoint extension coming from the quadratic form $q^{R,\rho}(u) = \int_{\Omega_{0,R}} (2t|(D_t - \eta)u|^2 + \rho|D_\eta u|^2)\,\mathrm{d}t\,\mathrm{d}\eta$, with $\rho = 1/(2\alpha^2 R)$. These comparisons and the behavior of the eigenfunctions give a lower bound of the gap between the two first eigenvalues of $-\nabla_{\mathcal{A}}^2$:

Proposition 3.3. *Let $\mu_1(\alpha)$ and $\mu_2(\alpha)$ the two smallest eigenvalues of P_{Ω_α}. Denoting by λ_1^{moy} and λ_2^{moy} the smallest eigenvalues for L^{moy}, then there exists α_0 such that for $\alpha \in]0,\alpha_0]$*:

$$\frac{\mu_2(\alpha) - \mu_1(\alpha)}{\alpha} \geqslant \left(\lambda_2^{\mathrm{moy}} - \lambda_1^{\mathrm{moy}}\right) - C\alpha^{1/2}. \tag{15}$$

This estimate shows that there is at most one eigenvalue with the asymptotic expansion (9) and achieve the proof of Theorem 3.1.

4. Estimate in the semi-classical case

As in [6], we can give a lower bound and a upper bound for the fundamental state of the Schrödinger operator with a nonconstant magnetic field and a non smooth domain.

We consider $\mathcal{A}$ a magnetic potential with a nonconstant and positive field B, and $\Omega \subset \mathbb{R}^2$ a bounded open locally C^∞ except at vertex $S_1, \ldots, S_N$ with corresponding angles $\alpha_1, \ldots, \alpha_N$. For each vertex S_k, the boundary is locally C^1-diffeomorphic to a sector Ω_{α_k}. We define $b = \inf_{x \in \overline{\Omega}} B(x)$ and $b' = \inf_{x \in \partial\Omega} B(x)$.

Theorem 4.1. *Under these assumptions, there exists $h_0 > 0$ and two positive constants C and C' such that, for $h \leqslant h_0$:*

$$-Ch^{5/4} \leqslant \mu(h, B, \Omega) - h \inf\Big(b, \Theta_0 b', \inf_{j=1,\ldots,N} \mu(\alpha_j) B(S_j)\Big) \leqslant C' h^{3/2}.$$

To prove this last theorem, we use the same technics as Helffer and Morame [6]. A partition of unity permits to treat independently the contribution of every vertex and the size of the partition's support is of order h^γ, with $\gamma = 3/8$ for the lower bound and $\gamma = 1/2$ for the upper bound. If the support is in Ω, we compare with the model $\mathbb{R}^2$, if the support meets the boundary, we make a change of variables in order to compare either with the model $\mathbb{R} \times \mathbb{R}^+$, either with the sector's model Ω_α.

5. Conclusion

The relation (9) gives an expansion at any order and goes far beyond the works of Brosens, Devreese, Fomin and Moshchalkov (cf. [5]) who mention only the first term $\alpha/\sqrt{3}$ and a paper of Schweigert and Peeters (cf. [11]) who propose on the basis of numerical computations a two-terms formula. As usual in the physical literature, the best one can hope through their technics is an upper bound of $\mu(\alpha)$ because they propose only quasi-modes. An open problem is to prove the monotony of $\mu(\alpha)$. Computations by physicists (cf. [11]) seem indeed to suggest that $\mu(\alpha)$ is increasing with α. Jadallah [7] and Pan [8] give contributions for the study of $\mu(\alpha)$ in the particular case $\alpha = \pi/2$ and we tend to a more systematic analysis for all angles [4].

Acknowledgements

I am deeply grateful to Professor Bernard Helffer for constant attention to this work and his help, comments and suggestions.

References

[1] S. Agmon, in: Lectures on Exponential Decay of Solutions of Second Order Elliptic Equations, in: Math. Notes, Princeton University Press, 1982.
[2] A. Bernoff, P. Sternberg, Onset of superconductivity in decreasing fields for general domains, J. Math. Phys. 39 (3) (1998) 1272–1284.
[3] P. Bolley, J. Camus, Sur une classe d'opérateurs elliptiques et dégénérés à une variable, J. Math. Pures Appl. 51 (1972) 429–463.
[4] V. Bonnaillie, On the fundamental state for a Schrödinger operator with magnetic field in domains with corners, Preprint, Mittag–Leffler Institute, January 2003.
[5] F. Brosens, J.T. Devreese, V.M. Fomin, V.V. Moshchalkov, Superconductivity in a wedge: analytical variational results, Solid State Comm. 111 (12) (1999) 565–569.
[6] B. Helffer, A. Morame, Magnetic bottles in connection with superconductivity, J. Funct. Anal. 185 (2001) 604–680.
[7] H.-T. Jadallah, The Onset of superconductivity in a domain with a corner, Ph.D. thesis, Indiana University, 2001.
[8] X.-B. Pan, Upper Critical Field for superconductors with edges and corners, Calc. Var. Partial Differential Equations 14 (2002) 447–482.
[9] A. Persson, Bounds for the discrete part of the spectrum of a semi-bounded Schrödinger operator, Math. Scand. 8 (1960) 143–153.
[10] M. del Pino, P.L. Felmer, P. Sternberg, Boundary concentration for eigenvalue problems related to the onset of superconductivity, Comm. Math. Phys. 210 (2000) 413–446.
[11] V.A. Schweigert, F.M. Peeters, Influence of the confinement geometry on surface superconductivity, Phys. Rev. B 60 (5) (1999) 3084–3087.