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Towards Tikhonov regularization of non-linear ill-posed problems: a dc programming approach

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Abstract The Tikhonov regularization method for non-linear ill-posed problems requires us to globally solve non-convex optimization problem which have been very little studied in the inverse problems community. In this paper we suggest a method which is applicable to the Tikhonov method for a wide class of non-linear ill-posed problems. This is a class of problems when the Tikhonov functional for them can be represented by the difference of two convex functionals. Our method for these problems is a combination of the recently developed algorithm DCA in dc programming with the branch-and-bound techniques. ***To cite this article: Le Thi Hoai An et al., C. R. Acad. Sci. Paris, Ser. I 335 (2002) 1073–1078.***
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Vers la régularisation de Tikhonov pour des problèmes non linéaires mal posés : une approche en optimisation dc

Résumé La méthode de régularisation de Tikhonov pour les problèmes non linéaires mal posés requiert une solution optimale globale des problèmes d'optimisation non convexe qui ont été très peu étudiés dans la communauté des problèmes inverses. Dans ce papier nous suggérons une méthode qui est applicable à une large classe des problèmes non linéaires mal posés. C'est une classe de problèmes dans lesquels la fonctionnelle de Tikhonov peut être représentée comme différences de fonctionnelles convexes (dc). Notre méthode pour ces problèmes est une combinaison de l'algorithme DCA, récemment développé en optimisation dc, et les techniques de séparation et évaluation. ***Pour citer cet article : Le Thi Hoai An et al., C. R. Acad. Sci. Paris, Ser. I 335 (2002) 1073–1078.***
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Beaucoup de problèmes en Science naturelle, ingénierie, médecine, etc., requièrent la résolution des problèmes non linéaires

$$F(x) = y, \tag{1}$$

où X et Y sont deux espaces de Banach et $F : D(F) \subset X \to Y$ est un opérateur non linéaire de X dans Y. Ces problèmes sont souvent mal posés, dans le sens que (i) problème (1) peut ne pas avoir de solution, (ii) ou s'il y a une solution, elle peut ne pas être unique, (iii) et s'il y a une solution, elle peut ne pas dépendre avec continuité des données (voir, par exemple, [2,14] et les références incluses). Ici comme notion de

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« solution » de (1), nous choisissons le concept d'une solution x_0, avec la norme de $x_0 - x^*$ minimale (x^*-MNS (x^*-minimum-norm solution)) in $\mathcal{C}$, c'est-à-dire,

$$F(x_0) = y_0, \quad x_0 \in \mathcal{C}, \tag{2}$$
$$\|x_0 - x^*\|_X = \min_{x \in D(F)} \{\|x - x^*\|_X : F(x) = y_0\}. \tag{3}$$

Nous supposons l'existence d'une x^*-MNS pour les données exactes y_0. Puisque F est non linéaire, cette solution peut ne pas être unique. L'élément x^* in (3) joue le rôle d'un critère de sélection ou d'une estimation pour elle. Si (1) est mal posé dans le sens d'absence de continuité de ses solutions par rapport aux données, les techniques de régularisation sont requises. Une des plus populaires techniques est la régularisation de Tikhonov qui approxime la solution de (1) par une solution du problème de minimisation [1,2,14]

$$\min_{\mathcal{C}} \{J_\alpha(x)\} \quad \text{avec } J_\alpha(x) := \frac{1}{2}\|F(x) - y_\varepsilon\|_Y^2 + \frac{\alpha}{2}\|L(x - x^*)\|_Z^2. \tag{4}$$

Ici L est un opérateur linéaire de X dans un espace de Hilbert Z, $\alpha > 0$ is un petit paramètre et $y_\varepsilon \in Y$ sont des données bruitées disponibles. Nous supposons avoir l'information que $\|y - y_\varepsilon\| \leqslant \varepsilon$.

Il y a diverses méthodes de choix du paramètre α pour garantir la convergence de la méthode quand le niveau du bruit ε tend vers zéro [2,14], cependant une des plus importantes questions, à savoir comment résoudre *globalement* le problème d'optimisation non convexe (4) (i.e. déterminer une solution globale de (4) ou un minimum global de $J_\alpha(x)$ sur $\mathcal{C}$), a été très peu étudiée dans la communauté des problèmes inverses. A notre connaissance seules les méthodes locales telles que Gauss–Newton, Levenberg–Marquardt, etc., ont été utilisées pour résoudre (4) qui ne peuvent pas garantir que leurs solutions numériques soient des bonnes approximations d'une solution optimale globale de (4). Le but de ce papier est de suggérer une méthode globale pour résoudre une large classe de problèmes non linéaires mal posés. C'est la classe de problèmes mal posés dont la fonctionnelle J_α est dc. Cela signifie que J_α peut être représentée par **d***ifférence de fonctions* **c***onvexes*. Pour cette classe nous allons utiliser les techniques de la programmation dc, à savoir un robuste algorithme, appelé DCA, introduit par Pham Dinh Tao [8,9] et développé par Le Thi Hoai An et Pham Dinh Tao [5,10–13] et références incluses) pour résoudre les problèmes d'optimisation dc. DCA est une méthode locale, et même si les solutions locales obtenues par DCA sont en pratique aussi globales, on ne peut garantir théoriquement la convergence vers une solution optimale globale de (4). Par conséquent nous devons combiner DCA avec les techniques de l'optimisation globale, en fait nous allons utiliser les techniques d'évalutaion et séparation.

1. Introduction

Many problems in natural science, engineering, medicine, etc., require us to solve non-linear equations of the form

$$F(x) = y, \tag{5}$$

where $F : D(F) \subset X \to Y$ is a non-linear operator between normed spaces X and Y. These problems are often ill-posed, i.e., (i) there may be no solution of (5); (ii) or if there is a solution, it may not be unique; (iii) and if there is a solution, it may not depend continuously on the data (see, e.g., [2,14] and the references therein). Here as the notion of "a solution" to (5), we choose the concept of an x^*-minimum-norm solution, (x^*-MNS), x_0 in $\mathcal{C}$, a closed bounded convex set in X (see [1,2,14]), i.e.,

$$F(x_0) = y_0, \quad x_0 \in \mathcal{C}, \tag{6}$$
$$\|x_0 - x^*\|_X = \min_{x \in D(F)} \{\|x - x^*\|_X : F(x) = y_0\}. \tag{7}$$

We assume the existence of an x^*-MNS for the exact data y_0. Since F is non-linear, this solution need not be unique. The element x^* in (7) plays the role of a selection criterion or of a guess for the actual solution x_0.

If (5) is ill-posed in the sense of lack of continuity of its solutions with respect to the data, regularization techniques are required. One of the most popular techniques is Tikhonov regularization which approximates the solution of (5) by a solution of the minimization problem [1,14]

$$\min_{\mathcal{C}}\{J_\alpha(x)\} \quad \text{with } J_\alpha(x) := \frac{1}{2}\|F(x) - y_\varepsilon\|_Y^2 + \frac{\alpha}{2}\|L(x - x^*)\|_Z^2. \tag{8}$$

Here L is a linear operator from X to a Hilbert space Z, $\alpha > 0$ is a small parameter and $y_\varepsilon \in Y$ are the available noisy data. We assume to have the information that $\|y - y_\varepsilon\|_Y \leqslant \varepsilon$.

There are various methods of choosing α to guarantee the convergence of the method when the noise level ε tends to zero (see, e.g., [2,14]). However, one of the most important questions, how to *globally* solve the nonlinear optimization problem (8), has been very little studied in the literature of the inverse problems community. To our knowledge only local methods such as Gauss–Newton, Levenberg–Marquardt type methods, to name a few, are used to solve (8). These methods do not guarantee that their numerical solutions of (8) are good approximations to a global solution of it. The aim of this paper is to suggest a method for solving *globally* (8) for a wide class of non-linear ill-posed problems. This is the class of the ill-posed problems where the functional (8) is dc. The last means that J_α can be represented by the **d***ifference* of **c***onvex* functions. For this class of problems we use a robust technique from dc programming, the DCA, which was suggested by Pham Dinh Tao [8,9] and developed by Le Thi Hoai An and Pham Dinh Tao [5,10–13]. As the DCA is also a local method, it cannot, in general, guarantee the convergence to a global solution of (8). We therefore combine it with techniques from global optimization; in our case, we shall use the branch-and-bound techniques.

2. A class of non-linear ill-posed problems

It is well known that if a functional J defined in a bounded convex set U of a Hilbert space H is twice Fréchet differentiable and its second Fréchet derivative is bounded, then it can be represented by the difference of two convex functionals. For example, if there is a positive constant ρ such that $|(J''(x)h,h)| \leqslant \rho/2\|h\|_{\mathrm{H}}^2$ for $x \in U,\ h \in \mathrm{H}$, then $J(x) + \rho\|x\|_{\mathrm{H}}^2$ is convex and so $J(x) = (J(x) + \rho\|x\|_{\mathrm{H}}^2) - \rho\|x\|_{\mathrm{H}}^2$ is the difference of two convex functionals. The last representation is of course not unique, there are infinitely many other representations and these strongly influence our algorithm's efficiency in Section 3 as well as make it abundant. In the application sometimes we use the representation $J(x) = \rho\|x\|_{\mathrm{H}}^2 - (\rho\|x\|_{\mathrm{H}}^2 - J(x))$ rather than the just mentioned one.

We give now two examples of ill-posed problems the output least squares functional J_0 of which are dc, and are so the Tikhonov functionals J_α (8). We choose these two examples in this short note since they represent two different classes of nonlinear ill-posed problems: the first one is an ill-posed problem with a constraint on total variation, the second one is typical in inverse coefficient problems. Other examples on various coefficient inverse problems for elliptic and parabolic equations where Tikhonov's regularization method leads to minimization of a dc functional can be found in [6]. We note further that for every inverse (and ill-posed) problem one should use the properties of its very structure to develop an efficient numerical method.

2.1. Autoconvolution equation

For solving the autoconvolution equation $\int_0^s x(s-t)x(t)\,\mathrm{d}t = y(s),\ 0 \leqslant s \leqslant 1$, with total variation constraints we use the non-linear discrete least squares problem $J_0(x) = \frac{1}{2}h\sum_{i=1}^n |f_i(x) - y_i|^2 \to \min$, subject to $x \in \mathcal{C}$, where $h = 1/n$, $f_i(x) := \sum_{j=1}^{\mathrm{i}} hx_{i-j+1}x_j$, $x = (x_1,\ldots,x_n)^{\mathrm{T}}$, and $\mathcal{C} := \{x \in \mathbb{R}^n : 0 < a \leqslant x_i \leqslant b,\ i = 1,2,\ldots,n,\ \sum_{i=1}^{n-1}|x_{i+1} - x_i| \leqslant c\}$. Here a, b, c are given positive numbers. It is proved that J_0 is twice differentiable and $\|\nabla^2 J_0(x)\| \leqslant (n+1)/n^2b^2 + 1/n\max_{i=1,\ldots,n}|y_i| + 4b^2$. Thus, J_0 is dc. Furthermore, a very simple formula for the first gradient has been found [6].

2.2. Identification problems in an elliptic equation

Let Ω be a bounded domain in $\mathbb{R}^n (n = 1, 2, 3)$. For $n = 1$ we set $\Omega = (0, 1)$. For $n = 2, 3$ we suppose that Ω is a sphere, a sphere shell, a parallelepiped, or a domain that can be transformed into one of these domains by a regular mapping $y = y(x) \in C^2(\overline{\Omega})$. Consider the problem of determining the coefficient $a(x)$ from the observation z of u over the whole domain Ω from the system

$$-\operatorname{div}(a \operatorname{grad} u) = f \quad \text{in } \Omega, \qquad u = g \quad \text{in } \partial\Omega, \tag{9}$$

where f and g are functions in $H^{-1}(\Omega)$ and $H^{1/2}(\Omega)$, respectively. We suppose that $a \in \mathcal{U} := \{a \in W^1_p(\Omega),\ 0 < a_1 \leqslant a(x) \leqslant a_2 < \infty \text{ a.e.}\}$, where $p > n$, a_1 and a_2 are given positive numbers. When $n = 1$, we take $p = 2$. The output least squares functional $J_0(a) = \frac{1}{2}\|u(a) - z\|^2_{L^2(\Omega)}$ is twice Fréchet differential and there is a positive constant ρ such that $|(J_0''(a)h, h)| \leqslant \rho\|h\|^2_{H^2(\Omega)}$. The first gradient $J_0'(a) = -\operatorname{grad} u(a) \operatorname{grad} \varphi$, where φ is the weak solution of the adjoint problem $-\operatorname{div}(a \operatorname{grad} \varphi) = u(a) - z$ in Ω, $\varphi = 0$ in $\partial\Omega$. Again we see that J_0 is dc.

3. DCA and branch-and-bound techniques

We see that (Section 2 and more detail in [6]) for many non-linear ill-posed problems Tikhonov's functional is dc in a bounded convex set. In particular, when a least squares method is applied to bilinear inverse problems, then the corresponding least squares functional is dc in a bounded convex set. Further, discretizing a bilinear inverse problem we get the system $Au + B(q)u = f$ and the observations $u_{\text{obs}} = Cu$. Here A, B and C are appropriate matrices, q is the parameter to find, and B linearly depends on q. The output least squares functional $J_0(q) = \|Cu - u_{\text{obs}}\|^2$ is infinitely differentiable, and thus if we restrict q to a bounded convex domain, then J_0 is dc on it. The last follows that corresponding Tikhonov's functional J_α is also dc. Thus, dc programming techniques can be applied to the above inverse problem.

3.1. Difference of convex functions algorithm (DCA).

Denote the set of all lower semicontinuous proper convex functions on $\mathbb{R}^n$ by $\Gamma_0(\mathbb{R}^n)$. For $g \in \Gamma_0(\mathbb{R}^n)$, the *conjugate function* g^* of g is a function belonging to $\Gamma_0(\mathbb{R}^n)$ defined by $g^*(y) = \sup\{\langle x, y\rangle - g(x) : x \in \mathbb{R}^n\}$.

Let g and h be in $\Gamma_0(\mathbb{R}^n)$. A general dc program is that of the form

$$(\mathrm{P_{dc}}) \qquad \alpha = \inf\big\{f(x) := g(x) - h(x) : x \in \mathbb{R}^n\big\}.$$

Its dual program is given by

$$(\mathrm{D_{dc}}) \qquad \alpha = \inf\big\{h^*(y) - g^*(y) : y \in \mathbb{R}^n\big\}.$$

For solving these two dc programs we shall use the DCA, a primal–dual subdifferential method. Based on the dc duality and the local optimality, the DCA consists in the construction of two sequences $\{x^k\}$ and $\{y^k\}$ such that x^{k+1} (resp. y^k) is a solution to the convex program (P_k) (resp. (D_k)) defined by

$$(\mathrm{P}_k) \qquad \min\big\{g(x) - \big[h\big(x^k\big) + \big\langle x - x^k, y^k\big\rangle\big] : x \in \mathbb{R}^n\big\},$$

$$(\mathrm{D}_k) \qquad \min\big\{h^*(y) - \big[g^*\big(y^{k-1}\big) + \big\langle x^k, y - y^{k-1}\big\rangle\big] : y \in \mathbb{R}^n\big\}.$$

In view of the relation: (P_k) (resp. (D_k)) is obtained from $(\mathrm{P_{dc}})$ (resp. $(\mathrm{D_{dc}})$) by replacing h (resp. g^*) with its affine minorization defined by $y^k \in \partial h(x^k)$ (resp. $x^k \in \partial g^*(y^{k-1})$), the DCA yields the next scheme:

$$y^k \in \partial h\big(x^k\big); \qquad x^{k+1} \in \partial g^*\big(y^k\big), \tag{10}$$

while for the *complete* DCA we impose the following natural choice

$$x^{k+1} \in \arg\min\big\{g(x) - h(x) : x \in \partial g\big(y^k\big)\big\} = \arg\min\big\{\big\langle x, y^k\big\rangle - h(x) : x \in \partial g\big(y^k\big)\big\}$$

and

$$y^k \in \arg\min\big\{h^*(y) - g^*(y) : y \in \partial h\big(x^k\big)\big\} = \arg\min\big\{\big\langle x^k, y\big\rangle - g^*(y) : y \in \partial h\big(x^k\big)\big\}.$$

It is proved in Pham Dinh Tao [9], Pham Dinh Tao and Le Thi Hoai An [10,11] that

(i) The sequences $\{g(x^k) - h(x^k)\}$ and $\{h^*(y^k) - g^*(y^k)\}$ are decreasing and
- $g(x^{k+1}) - h(x^{k+1}) = g(x^k) - h(x^k)$ if and only if $y^k \in \partial g(x^k) \cap \partial h(x^k)$, $y^k \in \partial g(x^{k+1}) \cap \partial h(x^{k+1})$ and $[\rho(g) + \rho(h)]\|x^{k+1} - x^k\| = 0$.
- $h^*(y^{k+1}) - g^*(y^{k+1}) = h^*(y^k) - g^*(y^k)$ if and only if $x^{k+1} \in \partial g^*(y^k) \cap \partial h^*(y^k)$, $x^{k+1} \in \partial g^*(y^{k+1}) \cap \partial h^*(y^{k+1})$ and $[\rho(g^*) + \rho(h^*)]\|y^{k+1} - y^k\| = 0$.

($\rho(g)$ denotes the modulus of strong convexity of g.) In such a case the DCA terminates at the k-th iteration.

(ii) Every limit point x^* (resp. y^*) of the sequence $\{x^k\}$ (resp. $\{y^k\}$) is a critical point of $g - h$ (resp. $h^* - g^*$). (A point x is called *a critical point* of $g - h$ if $\partial g(x) \cap \partial h(x) \neq \emptyset$.)

(iii) The complete DCA ensures in addition local optimality conditions: $\partial h(x^k) \subset \partial g(x^k)$ and $\partial g^*(y^k) \subset \partial h^*(y^k)$ in (i) and $\partial h(x^*) \subset \partial g(x^*)$ and $\partial g^*(y^*) \subset \partial h^*(y^*)$ in (ii).

(iv) If DCA converges to a point x^* that admits a neighbourhood in which the objective function f is finite and convex (i.e., the function f is locally convex at x^*) and if the second dc component h is differentiable at x^*, then x^* is a local minimizer for (P_{dc}). For the complete DCA the second condition is not needed. Property (iv) has of course its dual part.

(v) DCA converges to a global minimizer for (P_{dc}) if $f = g - h$ is actually convex (f then is called "false" dc function).

It is well known that Tikhonov's functional (8) is strongly convex in a neighbourhood of x_0 [3, p. 23], so if we start our DCA in an appropriate subset of this neighbourhood, we can get a global solution (see (iv)). However, it is hard to find this neighbourhood, and since DCA is a local method, in general, we should combine it with some other techniques to find global solutions of the problem. We shall use the branch-and-bound technique for this purpose in this note.

3.2. The basic branch-and-bound method

The branch-and-bound method is widely used in global optimization. To solve a concrete problem by a branch-and-bound method one has to specify three basic operations: the subdivision (branching), the estimation of lower bounds, and the computation of upper bounds (bounding). We present below the basic branch-and-bound scheme to solve the non-convex program

$$\alpha = \min\{f(x) : x \in \mathcal{C}\}, \tag{11}$$

where f is a real-valued continuous function and $\mathcal{C}$ is a compact set.

Initialization:

Let $S \supseteq \mathcal{C}$. Set $S \leftarrow \mathcal{R}$. Compute a lower bound $\beta(S)$ for α by the relaxation of either the objective function or the feasible set. Compute an upper bound γ of $\alpha : \gamma := f(\overline{x}), \overline{x} \in \mathcal{C}$ (if a feasible point in S has not yet been determined, then $\gamma := -\infty$).

Let stop $\leftarrow$ false; $k \leftarrow 1$, $\beta \leftarrow \beta(S)$

while stop $=$ false **do**

If $\gamma = \beta$ **then** stop $\leftarrow$ true (($\alpha, \overline{x}$) is an optimal solution of problem (11))

else Divide S into r subsets $S_1, \ldots, S_r$ satisfying $\bigcup_{i=1}^r S_i = S$, $\operatorname{int} S_i \cap \operatorname{int} S_j = \emptyset$ for $i \neq j$. For each $i = 1, \ldots, r$ compute a lower bound $\beta(S_i)$ for the optimal value of the problem $\min\{f(x) : x \in S_i\}$. Delete all $S_i \in \mathcal{R}$ such that $\beta(S_i) \geqslant \gamma$, and rename the remaining set $\mathcal{R}$, i.e., set $\mathcal{R} \leftarrow \{S_i \in \mathcal{R} : \beta(S_i) < \gamma\}$.

If a point $x' \in \mathcal{C}$ such that $f(x') < f(\overline{x})$ is detected, then update the upper bound by setting $\gamma \leftarrow f(x')$, $\overline{x} \leftarrow x'$. If $\mathcal{R} \neq \emptyset$, set $\beta \leftarrow \min\{\beta(S) : S \in \mathcal{R}\}$; Choose $S_i \in \mathcal{R}$ such that $\beta(S_i) = \beta$.

endif

$k \leftarrow k + 1$

endwhile

To find the lower and upper in branch-and-bound procedures again we can use the DCA. In many inverse problems the constraints are boxes, say, $\mathcal{C} := \prod_1^n [a_i, b_i],\ -\infty < a_i \leqslant b_i < \infty,\ i = 1, 2, \ldots, n$. It is natural to use the rectangular bisection in the partitions of $\mathcal{C}$ (see, e.g., [5]). To find lower and upper bounds for the dc problem $\alpha := \min\{f(x) : x \in \mathcal{C}\}$ we can proceed as follows. Take a $\rho > 0$ such that $\rho/2\|x\|^2 - f(x)$ is convex and decompose the function f in the form $f(x) = \rho/2\|x\|^2 - (\rho/2\|x\|^2 - f(x))$. It is easily seen that DCA applied to this dc program yields the sequence $x^{k+1} = P_{\mathcal{C}}(x^k - (1/\rho)\nabla f(x^k))$, where $P_{\mathcal{C}}$ is the orthogonal projection into the box $\mathcal{C}$. Clearly, $\lim f(x^k)$ is an upper bound of α. To find a lower bound of α we use the decomposition $f(x) = (\rho/2\|x\|^2 + f(x)) - \rho/2\|x\|^2 := g(x) - h(x)$. We approximate the function h by a linear functional. Namely, by $\sum_{i=1}^n l_i(x_i) = \sum_{i=1}^n [(a_i + b_i)x_i - a_i b_i] \geqslant h(x)$. (We note that over a bounded set such an approximation can always be done.) Then the function $\tilde{f}(x) = g(x) - \sum_{i=1}^n l_i(x_i)$ is convex and $\tilde{f}(x) \leqslant f(x)$ in $\mathcal{C}$. Thus, if we can find a bound for its global minimum on $\mathcal{C}$, then we have a lower bound of α. Since $\tilde{f}(x)$ is convex in $\mathcal{C}$ we can use various techniques to find its minimum, DCA is a candidate among them (see (v)).

DCA is a first order descent method without line-search and to check whether a critical point is the local minimizer we need not to use the second derivative like the other local minimization methods (see (iv)). Further, it is very well suited for branch-and-bound procedures.

In the concrete ill-posed problems as in Section 2 and [6], since the gradient J_0' can be found and $J_\alpha(x) = (J_0(x) + \alpha/2\|L(x - x^*)\|_Z^2 + \rho/2\|x\|^2) - \rho/2\|x\|^2$, the DCA applied to them has the form of (P_k), (D_k) which are explicit and very simple. The numerical results for ill-posed problems in Section 2 show that our method is efficient and robust [6,7].

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