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Équations aux dérivées partielles/*Partial Differential Equations*

Injectivity of the spherical means operator

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Abstract Let S be a surface in $\mathbb{R}^n$ which divides the space into two connected components D_1 and D_2. Let $f \in \mathrm{C}_0(\mathbb{R}^n)$ be some real-valued compactly supported function with supp $f \subset D_1$. Consider

$$Mf := m(y,r) := \int_{\mathbb{R}^n} f(z)\delta(|y-z|-r)\,\mathrm{d}z,$$

where δ is the delta-function, $y \in S$ and $r > 0$ are arbitrary. A general, local at infinity, condition on S is given, under which M is injective, that is, $Mf = 0$ implies $f = 0$. The injectivity result is extended to the case when the Fourier transform of f is quasianalytic, so that compactness of support of f is not assumed. A sufficient condition on S is given, under which M^{-1} can be analytically constructed. Two examples of inversion formulas are given: when S is a plane, and when S is a sphere. These formulas can be used in applications. ***To cite this article: A.G. Ramm, C. R. Acad. Sci. Paris, Ser. I 335 (2002) 1033–1038.***

Injectivité de l’operateur de moyen spherique

Résumé Soit S une surface de $\mathbb{R}^n$ qui divise l’espace en deux composantes connectées D_1 and D_2. Soit $f \in \mathrm{C}_0^\infty(\mathbb{R}^n)$ une fonction à valeurs réeles, supp $f \subset D_1$. Considérons

$$Mf := m(y,r) := \int_{\mathbb{R}^n} f(z)\delta(|y-z|-r)\,\mathrm{d}z,$$

où δ est la delta-fonction, $y \in S$ et $r > 0$ sont quelconques. Une condition générale, locale à l’infini, est donnée sur S, sous laquelle M est injective, c.a.d., $Mf = 0 \Rightarrow f = 0$. Le résultat d’injectivité est généralisé dans le cas où la transformée de Fourier de f est quasi-analytique, de façon à ne pas supposer que f est à support compact. Une condition suffisante sur S est donnée sous laquelle M^{-1} peut être construit analytiquement. Deux exemples de formules d’inversion sont donnés : dans le cas où S est plan, et dans le cas où S est une sphère. Ces formules peuvent etre utilisés dans les applications. ***Pour citer cet article : A.G. Ramm, C. R. Acad. Sci. Paris, Ser. I 335 (2002) 1033–1038.***

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Soit S une surface de $\mathbb{R}^n$ qui divise l'espace en deux composantes connectées D_1 and D_2. Soit $f \in C_0^\infty(\mathbb{R}^n)$ une fonction à valeurs réelles, supp $f \subset D_1$.

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où δ est la delta-fonction, $y \in S$ et $r > 0$ sont quelconques. Une condition générale, locale à l'infini, est donnée sur S, sous laquelle M est injective, c'est-à-dire, $Mf = 0 \Rightarrow f = 0$. Le résultat d'injectivité est généralisé dans le cas où la transformée de Fourier de f est quasi analytique, de façon à ne pas supposer que f est à support compact.

Une condition suffisante sur S est donnée, sous laquelle M^{-1} peut être construit analytiquement. Deux exemples de formules d'inversion sont donnés : dans le cas où S est plan et dans le cas où S est une sphère. Ces formules peuvent être utilisés dans les applications.

Nos résultats sont formulés dans Théorèmes 1.1 et 1.2.

THÉORÈME 1.1. – *Si S est lipschitzienne, S_R est en forme d'étoile par rapport à un point $O \in D_1$, $f \in C_0(\mathbb{R}^n)$, supp $f := D \subset D_1$, et $Mf = 0$, c'est-à-dire, $m(y,r) = 0$ $\forall y \in S$, $\forall r > 0$, alors $f = 0$.*

THÉORÈME 1.2 (Ramm [8]). – *Sous l'hypothèse sur S dans le Théorème* 1.1, *le problème* :

$$\big(\Delta + k^2\big)u = 0 \quad \textit{dans } D_2, \quad k = \text{const} > 0, \quad u = 0 \quad \textit{sur } S,$$
$$\lim_{R\to\infty} \int_{|z|=R,\, z\in D_2} \left|\frac{\partial u}{\partial |z|} - \mathrm{i}ku\right|^2 \mathrm{d}s = 0,$$

n'a qu'une solution triviale, $u = 0$.

Le Théorème 1.1 *reste valide pour des fonctions f à décroissance suffisamment rapide, par exemple, pour f avec transformée de Fourier $\tilde{f}$ une fonction entière. Un exemple d'une telle fonction f est $|f(z)| \leqslant c_1\,\mathrm{e}^{-c_2|z|^b}$, où $b > 1$ et $c_j > 0$ sont des constantes, $j = 1, 2$. Alors, il n'est pas nécessaire de supposer que f est à support compact pour que M soit injective* : *M est injective si $\tilde{f}$ est quasi analytique.*

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1. Introduction

In some physical problems and problems of mechanics it is of interest to know if the integrals of a function f over a family of spheres of all radii $r > 0$, with centers y running through a given surface S, determine the f uniquely. These integrals are called the spherical means of f. It is also of interest to have inversion formulas, which allow one to calculate f, given its spherical means.

In this paper the injectivityof the operator M of the spherical means is studied for a wide class of surfaces, and inversion formulas are derived. These formulas are obtained by a method different from the one used in [1–4]. The injectivity result we obtain, is based on the results in [6], see also [7–9].

Let S be a Lipschitz surface which divides the space $\mathbb{R}^n$ into two connected components, D_1 and D_2. By S_R we denote the part of S which is lying outside of the ball B_R, centered at a point $O \in D_1$, of radius R, where $R > 0$ is an arbitrary large fixed number.

We assume that: *S_R is star-shaped with respect to O.*

This assumption is local at infinity: it does not restrict S inside B_R, apart of the Lipschitz smoothness of S.

Let f be a real-valued compactly supported continuous function, $f \in C_0(\mathbb{R}^n)$, with $\operatorname{supp} f := D \subset D_1$. *This assumption is relaxed below: it is sufficient to assume that the Fourier transform $\tilde{f}$ of f is quasi-analytic, so that if $\tilde{f} = 0$ on an open set, then $\tilde{f} = 0$ everywhere, and so $f = 0$.*

Let $y \in S$ be an arbitrary point. Define the spherical means of f by the formula:

$$Mf := m(y, r) = \int_{\mathbb{R}^n} f(z)\delta\big(|y - z| - r\big)\,\mathrm{d}z, \quad y \in S,\ r > 0, \tag{1}$$

where δ is the delta function, $r > 0$ is arbitrary, and y runs through the whole S. The injectivity problem we study is:

PROBLEM I. – Does $Mf = 0$ imply $f = 0$?

This problem has been discussed in the literature for S being a plane, a sphere, etc., [1–4,6,7]. Inversion formulas, which allow one to calculate f given $m(y, r)$ for all $y \in S$ and all $r > 0$, were obtained for the above cases. Some of these formulas [3, pp. 754–756], [4, pp. 170–175] are not suitable for practical calculations, and some are [6,7,2]. The goal of this paper is to give a fairly general condition on S sufficient for M to be injective. In Section 2 we give some analytical inversion formulas for M.

The method we use is developed in [6]. In the proof of our main injectivity result, Theorem 1.1, an essential role is played by Theorem 1.2 of Ramm [8,9], [6, pp. 292, 296], which we formulate below. This theorem is our main technical tool. Our main result is Theorem 1.1:

THEOREM 1.1. – *If S is a Lipschitz, S_R is star-shaped with respect to a point $O \in D_1$, $f \in C_0(\mathbb{R}^n)$, $\operatorname{supp} f := D \subset D_1$, and $Mf = 0$, that is, $m(y, r) = 0$ for all $y \in S$ and all $r > 0$, then $f = 0$.*

See also Remark 1.3 below, in which the assumption of compactness of support of f is discarded.

THEOREM 1.2 (Ramm [8]). – *Under the assumptions on S in Theorem* 1.1, *the problem*:

$$\big(\Delta + k^2\big)u = 0 \quad \textit{in } D_2, \quad k = \text{const} > 0, \quad u = 0 \quad \textit{on } S, \tag{2}$$

$$\lim_{R\to\infty} \int_{|z|=R,\ z\in D_2} \left|\frac{\partial u}{\partial |z|} - \mathrm{i}ku\right|^2 \mathrm{d}s = 0, \tag{3}$$

has only the trivial solution $u = 0$.

Let $g = g(x, z, k)$ be the unique solution to the problem:

$$\big(\Delta + k^2\big)g = -\delta(x - z) \quad \text{in } \mathbb{R}^n, \quad k = \text{const} > 0, \tag{4}$$

which satisfies the radiation condition (3). It is known that

$$g(x, z, k) := \frac{\mathrm{i}}{4}\frac{k^{(n-2)/2}H_{(n-2)/2}(k|x - z|)}{(2\pi k|x - z|)^{(n-2)/2}}. \tag{5}$$

We need the following formula, which can be obtained from (5) easily:

$$g(x, z, k) = g(x, 0, k)\,\mathrm{e}^{-\mathrm{i}k(x^0, z)} + \mathrm{o}\left(\frac{1}{|x|^{(n-1)/2}}\right), \tag{6}$$

as $|x| \to \infty$, $x^0 := x/|x|$. A similar formula for the resolvent kernel of the Schrödinger operator was proved in [8] (see also [6, pp. 45–46], where formula (6) was proved, and a similar formula for the resolvent kernel of the Laplacian in domains with boundaries was established, with the plane wave replaced by the scattering solution).

Theorem 1.1 includes many of the known to the author results on the injectivity of M as very particular cases.

Remark 1.3. – Our proof of Theorem 1.1 remains valid for sufficiently rapidly decaying functions f, for example, for such f that their Fourier transform $\tilde{f}$ is an entire function. An example of such f is $|f(z)| \leqslant c_1 \,\mathrm{e}^{-c_2|z|^b}$ where $b > 1$ and $c_j > 0$ are constants, $j = 1, 2$. Thus, it is not necessary to assume f compactly supported for M to be injective. It is sufficient for the injectivity of M that $\tilde{f}$ is quasianalytic.

Often an analytical inversion formula can be constructed for M if the problem:

$$\bigl(\Delta + k^2\bigr)u = 0 \quad \text{in } D_2, \quad k = \text{const} > 0, \quad u = F(y, k) \quad \text{on } S, \tag{7}$$

where u satisfies (3), and $F(y, k)$ is a known rapidly decaying function, can be analytically solved. In [5, p. 655] the coordinate systems in which the operator $\Delta + k^2$ admits separation of variables are listed. If S is a coordinate surface (curve if $n = 2$) of one of these coordinate systems, then problem (7), (3) can be solved analytically, and an analytical inversion formula for M can be obtained. Planes and spheres are particular cases of such coordinate surfaces, and examples of inversion formulas for M are given for these cases in Section 2. The method used in Section 2, is the basis for the argument in Section 3, where Theorem 1.1 is proved.

In [7, pp. 317–320] the following problem of integral geometry is solved analytically: *Given* $\mathcal{H}(x) = \int_{S^2} h(x + |x|\alpha)\,\mathrm{d}\alpha$ *for all* $x \in \mathbb{R}^3$, *find* $h(x)$.

2. Auxiliary results

2.1.

In this section we prove injectivity of M and derive analytical inversion formulas in the case when S is a plane or a sphere. For simplicity of writing we take $n = 3$. Our proofs are valid with trivial modifications for any $n \geqslant 2$. We follow the method developed in [6, p. 244].

THEOREM 2.1. – *Let S be a plane* $\{x : x_3 = 0\}$, *and* supp $f \subset \mathbb{R}^3_- := \{z : z_3 < 0\}$ *be compact. Then M is injective and M^{-1} can be computed by inverting the Fourier transform of f given by formula* (14) *below.*

Proof. – Multiply (1) by $\mathrm{e}^{\mathrm{i}kr}/(4\pi r)$ and integrate over $[0, \infty)$ with respect to r to get:

$$\int_{\mathbb{R}^3} \frac{\mathrm{e}^{\mathrm{i}k|y-z|}}{4\pi|y - z|} f(z)\,\mathrm{d}z = \mu(y, k), \tag{8}$$

where

$$\mu(y, k) := \int_0^\infty \frac{m(y, r)}{4\pi r} \mathrm{e}^{\mathrm{i}kr}\,\mathrm{d}r, \tag{9}$$

is the known function for all $y \in S$ and all $k > 0$. Fourier transform (8) with respect to two-dimensional variable y over the plane $\mathbb{R}^2$ to get:

$$\int_{\mathbb{R}^2} \mathrm{d}y\, \mathrm{e}^{\mathrm{i}\xi^1\cdot y} \int_{\mathbb{R}^3} \mathrm{d}z \frac{\mathrm{e}^{\mathrm{i}k|y-z|}}{4\pi|y - z|} f(z) = F(\xi), \tag{10}$$

where

$$\xi^1 := (\xi_1, \xi_2), \quad \xi := \bigl(\xi^1, -\xi_3\bigr), \quad \xi_3 := \bigl(k^2 - |\xi^1|^2\bigr)^{1/2}, \tag{11}$$

and

$$F\left(\xi^1,k\right) := \int_{\mathbb{R}^2} \mathrm{d}y\, \mathrm{e}^{\mathrm{i}\xi^1\cdot y}\mu(y,k)\,\mathrm{d}y \tag{12}$$

is a known function $\forall \xi^1 \in \mathbb{R}^2$ and $\forall k > 0$. Using the formula [6, p. 244]:

$$\int_{\mathbb{R}^2} \frac{\mathrm{e}^{\mathrm{i}k|y-z|}}{4\pi|y-z|}\mathrm{e}^{\mathrm{i}\xi^1\cdot y}\,\mathrm{d}y = \frac{\mathrm{i}\mathrm{e}^{\mathrm{i}(\xi^1\cdot z^1+|z_3|\xi_3)}}{2\xi_3}, \quad k^2 \geqslant \left|\xi^1\right|^2, \tag{13}$$

and taking into account that $|z_3| = -z_3$, because $\operatorname{supp} f \subset \mathbb{R}^3_- := \{z : z_3 < 0\}$, one calculates the left-hand side of (10) and gets:

$$\tilde{f}(\xi) := \int_{\mathbb{R}^3} \mathrm{e}^{\mathrm{i}\xi\cdot z} f(z)\,\mathrm{d}z = -2\mathrm{i}\xi_3 F\left(\xi^1,k\right), \quad k^2 \geqslant \left|\xi^1\right|^2. \tag{14}$$

Since the right-hand side of (14) is known for all $\xi^1 \in \mathbb{R}^2$ and all $k > 0$, formula (14) defines the Fourier transform of f for all $\xi^1 \in \mathbb{R}^2$ and all $\xi_3 \geqslant 0$. Since f is real-valued, $\tilde{f}(-\xi) = \overline{\tilde{f}(\xi)}$, where the overbar stands for complex conjugate, so $\tilde{f}$ is defined everywhere in $\mathbb{R}^3$. Thus, f can be obtained uniquely by the Fourier inversion of $\tilde{f}$. □

2.2.

Assume now that S is a sphere of radius a, and $\operatorname{supp} f \subset B_a := \{z : |z| \leqslant a\}$. The problem is: given $m(y,r)$ for all $y \in S$ and all $r > 0$, find f.

The method for solving this problem is similar to the one in Section 2.1. Therefore we outline it briefly.

THEOREM 2.2. – *Let S be a sphere of radius a, centered at the origin, and* $\operatorname{supp} f \subset B_a$. *Then M is injective and M^{-1} can be computed by formulas* (2.9) *and* (2.10) *below.*

Proof. – One gets formula (8), as above. Then one uses eigenfunction expansion of the Green function:

$$\frac{\mathrm{e}^{\mathrm{i}k|y-z|}}{4\pi|y-z|} = \mathrm{i}k\sum_{\ell=0}^{\infty} \overline{Y}_\ell\left(z^0\right)Y_\ell\left(y^0\right)j_\ell\left(k|z|\right)h_\ell(ka), \quad z^0 := \frac{z}{|z|},\ |z| < |y| = a, \tag{15}$$

where $Y_\ell = Y_{\ell m}$, $-\ell \leqslant m \leqslant \ell$, are orthonormal spherical harmonics, $\sum_{\ell=0}^{\infty} = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}$, $j_\ell(r) = \sqrt{\frac{\pi}{2r}}J_{\ell+1/2}(r)$ is the spherical Bessel function, and $h_\ell(r) = \sqrt{\frac{\pi}{2r}}H^{(1)}_{\ell+1/2}(r)$ is the spherical Hankel function (see, e.g., [6, p. 140]). Let

$$f(z) = \sum_{\ell=0}^{\infty} Y_\ell\left(z^0\right)f_\ell\left(|z|\right), \tag{16}$$

where $f_\ell(|z|)$ are the Fourier coefficients of $f(z)$ with respect to the spherical harmonics.

Substitute (15) and (16) into (8), use the orthonormality of the spherical harmonics, and get:

$$\int_0^{\infty} f_\ell(s)j_\ell(ks)s^2\,\mathrm{d}s = \left[\mathrm{i}kh_\ell(ka)\right]^{-1}\mu_\ell(k), \quad s := |z|, \tag{17}$$

where

$$\mu_\ell(k) := \int_{S^2} \mu\left(ay^0,k\right)\overline{Y}_\ell\left(y^0\right)\mathrm{d}y^0,$$

and we do not show the dependence of μ_ℓ on a.

Formula (17) is the Bessel transform of $f_\ell(s)$, and it can be inverted analytically for $f_\ell(s)$ for every $\ell = 0, 1, 2, \ldots$. Thus, all the Fourier coefficients $f_\ell(|z|)$ of f are recovered, and f is recovered by formula (16).

Recall that the standard inversion formulas for the Bessel transform with an arbitrary positive index ℓ are:

$$H(k) := \int_0^\infty h(s) J_\ell(ks) s \,\mathrm{d}s, \quad h(s) = \int_0^\infty H(k) J_\ell(ks) k \,\mathrm{d}k. \tag{18}$$

Theorem 2.2 is proved. □

3. Proof of Theorem 1.1.

Multiply (1) by $g(kr, k) := (\mathrm{i}/4) k^{(n-2)/2} / H_{(n-2)/2}(kr)/(2\pi kr)^{(n-2)/2}$, integrate with respect to r over $[0, \infty)$, and get an analog of (8):

$$w(y, k) := \int_D g(y, z, k) f(z) \,\mathrm{d}z = \int_0^\infty m(y, r) g(kr, k) \,\mathrm{d}r := F(y, k), \tag{19}$$

where

$$w(x, k) := \int_D g(x, z, k) f(z) \,\mathrm{d}z, \quad x \in \mathbb{R}^n, \tag{20}$$

solves equation (7) in D_2, satisfies the radiation condition (8) in D_2, and takes the value $F(y, k)$ on S. To prove the injectivity of M, it is sufficient to prove that problem (7), (8) with $F(y, k) = 0$ has only the trivial solution, and this is exactly the claim of Theorem 1.2.

Indeed, if problem (7), (8) with $F(y, k) = 0$ has only the trivial solution, then function (19) solves Eq. (7) everywhere outside the support D of f, vanishes in D_2, and by the unique continuation property for solutions of elliptic equations, it vanishes outside D. Thus, using formula (1.6), one sees that the Fourier transform of f vanishes, because x^0 is arbitrary in S^{n-1}, the unit sphere in $\mathbb{R}^n$, and $k > 0$ is arbitrary. Thus $f = 0$, and the injectivity of M is proved. Theorem 1.1 is proved. □

After this paper was written, Professors Quinto and Agranovsky informed me about their recent works on the injectivity of M. Their interesting results do not contain the results of this paper, and our method of proof is completely different from theirs [1].

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