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On the Lagrange problem about the strongest column

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Abstract A new approach to the classical Lagrange problem about the form of the strongest clamped column of fixed volume and height is proposed. The existence of the optimal column is proved and a method to find its design is given. ***To cite this article: Yu.V. Egorov, C. R. Acad. Sci. Paris, Ser. I 335 (2002) 997–1002.***

Sur le problème de Lagrange de la forme optimale d'une colonne

Résumé On propose une nouvelle approche au problème classique de Lagrange de la forme d'une colonne encastrée la plus solide à volume et hauteur fixés. On montre l'existence d'une telle colonne et on donne un algorithme pour la calculer. ***Pour citer cet article : Yu.V. Egorov, C. R. Acad. Sci. Paris, Ser. I 335 (2002) 997–1002.***

Version française abrégée

On considère le problème classique de Lagrange qui consiste à chercher la forme d'une colonne encastrée la plus solide à volume et hauteur fixés. On montre l'existence d'une telle colonne et on donne un algorithme pour la calculer. Le problème considéré est ramené au problème suivant : trouver une fonction positive $Q(x) \in \mathrm{C}[0,1]$ telle que

$$\int_0^1 Q(x)^{1/2}\,\mathrm{d}x = 1 \tag{1}$$

et pour laquelle la valeur minimale λ de la fonctionnelle

$$L_1[Q,y] \equiv \frac{\int_0^1 Q(x)y'(x)^2\,\mathrm{d}x}{\int_0^1 y^2(x)\,\mathrm{d}x} \tag{2}$$

dans la classe des fonctions $y \in \mathrm{C}^1(0,1)$, satisfaisant les conditions

$$y(0)=0, \quad y(1)=0, \quad \int_0^1 y(x)\,\mathrm{d}x = 0, \tag{3}$$

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est maximale. Notre résultat principal est le théorème suivant :

THÉORÈME 1. – *Il existe une solution du problème de Lagrange. La fonction optimale Q_0 est donnée par la relation* :

$$Q_0(x) = \big(u'(x)^2 + v'(x)^2\big)^{-2},$$

où $u(x),\ v(x)$ sont des solutions linéairement indépendantes des équations :

$$m\left(\frac{u'(x)}{(u'(x)^2 + v'(x)^2)^2}\right)' + u = 0, \quad u(0) = u(1) = 0, \tag{4}$$

$$m\left(\frac{v'(x)}{(u'(x)^2 + v'(x)^2)^2}\right)' + v = C, \quad v(0) = v(1) = 0, \quad \int_0^1 v(x)\,\mathrm{d}x = 0, \tag{5}$$

$$u(x) = -u(1-x), \qquad v(x) = v(1-x).$$

La fonction Q_0 est symétrique, i.e. $Q_0(x) = Q_0(1-x)$, et peut être définie aussi par

$$Q_0(x) = r^2(x)/9m^2,$$

où r est la solution du problème de Cauchy suivant :

$$r^4(x) r'(x)^2 = 12m\big[c_1 r(x)^3 - r(x)^4 - c_2(3m)^3\big], \quad r(0) = 4m,$$

qui n'est constante sur aucun intervalle, où

$$c_1 = 4m + (32mk/9)^2, \qquad c_2 = (512mk/81)^2\big(3/4 - k^2\big), \quad k = v'(0).$$

Donc, Q_0 est bien déterminée si les deux constantes k et m sont connues. Pour les trouver on peut résoudre numériquement le système suivant :

$$2\int_{r_1}^{4m} \frac{r^2\,\mathrm{d}r}{\sqrt{P(r)}} + \int_{4m}^{r_2} \frac{r^2\,\mathrm{d}r}{\sqrt{P(r)}} = \sqrt{3m},$$

$$2\int_{r_1}^{4m} \frac{\sqrt{P(r)}\,\mathrm{d}r}{c_1 - r} + \int_{4m}^{r_2} \frac{\sqrt{P(r)}}{c_1 - r} = 3m\sqrt{3m}\big(1 - 2\pi\sqrt{c_2}\big),$$

où $P(r) = c_1 r^3 - r^4 - c_2(3m)^3$, et $r_1,\ r_2$ sont les racines réelles de P, $0 < r_1 < 4m < r_2 < c_1$.

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1. Introduction

The problem of design of the strongest elastic column of fixed volume with pinned ends was stated by Lagrange in 1773 on the base of some works by Euler and Bernoulli (see [6]). The recent interest to this problem was initiated by the articles of Keller [4] and Keller, Tadjbakhsh [5]. In spite of many efforts and publications (see [1–3,5,7–10] and the bibliography in [1,2,10]) the existence of the optimal column with clamped ends has not been proved. Actually the optimal form has been found numerically in [8] by Olhoff and Rasmussen. Their calculations were verified later by Seyranian and Mazur.

Let h be the height of the column, V be its volume, E its Young's module. If $z(x)$ is the lateral deflection of the column at a point x, $0 \leqslant x \leqslant h$, then the potential energy is:

$$T = \int_0^h EI(x) z''(x)^2\,\mathrm{d}x - \lambda \int_0^h z'(x)^2\,\mathrm{d}x,$$

where $I(x)$ is the second moment of area of the column's cross section, and λ is the magnitude of the axial load. Since $z(0)=0,\ z'(0)=0,\ z(1)=0,\ z'(1)=0$, the potential energy is positive for small values of λ for all $z\neq 0$. The buckling load of the column λ_0 is the supremum of the values λ such that $T\geqslant 0$ for any z. The problem considered is to find the form of the column, i.e., the function $I(x)$, for which the value λ_0 is maximal and $\int_0^h I(x)^{1/2}\,\mathrm{d}x = V$. After rescaling and the passage from z to $y=z'$ the mathematical problem takes the following form:

PROBLEM L. – To find a positive function $Q(x)\in \mathrm{C}[0,1]$ such that

$$\int_0^1 Q(x)^{1/2}\,\mathrm{d}x = 1 \tag{1}$$

and for which the minimal value λ of the functional

$$L_1[Q,y]\equiv \frac{\int_0^1 Q(x)y'(x)^2\,\mathrm{d}x}{\int_0^1 y^2(x)\,\mathrm{d}x} \tag{2}$$

in the class of functions $y\in \mathrm{C}^1(0,1)$, satisfying the conditions

$$y(0)=0,\quad y(1)=0,\quad \int_0^1 y(x)\,\mathrm{d}x = 0, \tag{3}$$

is maximal.

We propose here a new approach based on two-dimensional variation of the functional L_1 and on the study of the nonlinear functional

$$F[u,v]\equiv \int_0^1 \left[u'(x)^2+v'(x)^2\right]^{-1}\mathrm{d}x \int_0^1 \left[u(x)^2+v(x)^2\right]\mathrm{d}x,$$

allowing to prove the existence of the solution and to find the optimal form of the column. We are giving also an algorithm allowing to find the shape of the optimal column.

DEFINITION 1. – Let $\mathcal{A}$ be the set of positive continuous functions Q satisfying (1); S is the set of functions y of the class $\mathrm{C}^1(0,1)$ satisfying (3); S_0 is the set of the pairs of functions u,v of the class S such that the function $\theta(x)=\arctan(v'(x)/u'(x))$ is monotone decreasing with $\theta(0)-\theta(1)\leqslant 3\pi$ and the function $r(x)=u'(x)^2+v'(x)^2$ has in $]0,1[$ not more than three points of extremum and is monotone on the intervals between these points and the points $x=0$ and $x=1$.

The principal idea of our method is to find a pair (u,v) giving the minimal value of the functional $F[u,v]$ in some subclass of $S\times S$ and to show that the optimal functions u,v define the optimal shape of the most solid column as $Q(x)=(u'(x)^2+v'(x)^2)^{-2}$. The functional F has actually an infinite set of the points of local minimum (u_k,v_k) in the space $S\times S$ on the unit sphere in $\mathrm{H}^1(0,1)$ and $F[u_k,v_k]\to 0$ as $k\to\infty$. The most interesting for us here is the first point (u_1,v_1) such that $F[u_1,v_1]=\max F[u_k,v_k]$. However, the other points are also important for the Lagrange problem (see [7]). These points are characterized by the number of rotation of the vector $(u_k'(x),v_k'(x))$ when x is moving from 0 to 1. In particular, the pair (u_1,v_1) belongs to S_0 and the other points (u_k,v_k) do not.

THEOREM 1. – *There exists a solution of the Problem* L. *The optimal function Q_0 can be found from the relation $Q_0(x)=(u'(x)^2+v'(x)^2)^{-2}$, where $u(x),v(x)$ are linearly independent solutions to the equations*:

$$m\left(\frac{u'(x)}{(u'(x)^2+v'(x)^2)^2}\right)'+u=0,\quad u(0)=u(1)=0, \tag{4}$$

$$m\left(\frac{v'(x)}{(u'(x)^2+v'(x)^2)^2}\right)'+v=C,\quad v(0)=v(1)=0,\quad \int_0^1 v(x)\,\mathrm{d}x=0, \tag{5}$$

$$u(x)=-u(1-x),\qquad v(x)=v(1-x).$$

The function Q_0 is symmetric, $Q_0(x)=Q_0(1-x)$, and can be found also as $Q_0(x)=r^2(x)/9m^2$, where r is the solution to the Cauchy problem

$$r^4(x)r'(x)^2=12m\left[c_1r(x)^3-r(x)^4-c_2(3m)^3\right],\quad r(0)=4m,$$

which is not constant on any subinterval, and

$$c_1=4m+(32mk/9)^2,\quad c_2=(512mk/81)^2\left(3/4-k^2\right),\quad k=v'(0).$$

Therefore, Q_0 is defined if the two constants k and m are known. In order to find them it suffices to solve numerically the following system:

$$2\int_{r_1}^{4m}\frac{r^2\,\mathrm{d}r}{\sqrt{P(r)}}+\int_{4m}^{r_2}\frac{r^2\,\mathrm{d}r}{\sqrt{P(r)}}=\sqrt{3m},$$

$$2\int_{r_1}^{4m}\frac{\sqrt{P(r)}\,\mathrm{d}r}{c_1-r}+\int_{4m}^{r_2}\frac{\sqrt{P(r)}}{c_1-r}=3m\sqrt{3m}\left(1-2\pi\sqrt{c_2}\right),$$

where $P(r)=c_1r^3-r^4-c_2(3m)^3$, and r_1,r_2 are the real roots of P, $0<r_1<4m<r_2<c_1$.

Our calculation using Matlab shows that $m=0.019100$, $0.01234\leqslant Q(t)\leqslant 0.07643$. The critical load $M=1/m=52.3562$ is close to the value found in [8] by N. Olhoff and S.N. Rasmussen, but calculations are much simpler than theirs. The optimal column with circular sections is formed by rotation of the curve $y=R(x)=Q_0(x)^{1/4}/\sqrt{\pi}=\sqrt{\frac{r(x)}{3\pi m}}$. We have $R(0)=R(1)=0.65147$, the minimal value of R is 0.26811 and is attained at $x=0.2466$ and at $x=0.7534$, the maximal value of R is 0.651962.

Remark 1. – Our method allows also to find the column for which the k-th eigenvalue λ_k is maximal. Such the problem was considered by N. Olhoff in [7].

Remark 2. – Actually similar equations were found by A.P. Seyranian and N.M. Gura in [5] and by E.F. Masur, but meaning of these equations here is very different, they came from the solving an auxiliary problem of minimizing of a nonlinear functional. S.J. Cox and M.L. Overton proved in [2] the existence theorem under the supplementary condition $0<a\leqslant Q(x)\leqslant b<\infty$. It does not imply the existence theorem for the Lagrange problem.

2. Auxiliary constructions

We start with the study of the functional:

$$F_\varepsilon[u,v]=\int_0^1\left[u(x)^2+v(x)^2+\varepsilon\left(u'(x)^2+v'(x)^2\right)\right]\mathrm{d}x\int_0^1\left[u'(x)^2+v'(x)^2\right]^{-1}\mathrm{d}x,$$

where ε is a positive number.

LEMMA 1. – *There exist two functions $u_\varepsilon, v_\varepsilon\in S_0$ such that*

$$m_\varepsilon\equiv\inf_{u,v\in S_0}F_\varepsilon[u,v]=F_\varepsilon[u_\varepsilon,v_\varepsilon].$$

Analyzing the obtained solution we show that $m_\varepsilon \geqslant \mu$, where $\mu > 0$ does not depend on ε and that $u_\varepsilon, v_\varepsilon$ satisfy the Euler–Lagrange equations:

$$m_\varepsilon \frac{\mathrm{d}}{\mathrm{d}x}\big(Q_\varepsilon(x)u_\varepsilon'\big) + u_\varepsilon = C_{1\varepsilon}, \qquad m_\varepsilon \frac{\mathrm{d}}{\mathrm{d}x}\big(Q_\varepsilon(x)v_\varepsilon'\big) + v_\varepsilon = C_{2\varepsilon}, \tag{6}$$

where $C_{1\varepsilon}, C_{2\varepsilon}$ are constants, and

$$Q_\varepsilon(x) = -\varepsilon/m_\varepsilon + \big(u_\varepsilon'(x)^2 + v_\varepsilon'(x)^2\big)^{-2}.$$

Multiplying the first equation in (6) by u_ε' and the second one by v_ε', adding and integrating, we obtain that

$$-\varepsilon\big(u_\varepsilon'(x)^2 + v_\varepsilon'(x)^2\big) + \frac{3m_\varepsilon}{u_\varepsilon'(x)^2 + v_\varepsilon'(x)^2} + (u_\varepsilon - C_{1\varepsilon})^2 + (v_\varepsilon - C_{2\varepsilon})^2 = C_{3\varepsilon}, \tag{7}$$

with a constant $C_{3\varepsilon}$. If we multiply the first equation in (6) by $v_\varepsilon - C_{2\varepsilon}$, the second one by $u_\varepsilon - C_{1\varepsilon}$ and subtract, we obtain, integrating, that

$$Q_\varepsilon(x)\big[u_\varepsilon'(x)\big(v_\varepsilon(x) - C_{2\varepsilon}\big) - v_\varepsilon'(x)\big(u_\varepsilon(x) - C_{1\varepsilon}\big)\big] = C_{4\varepsilon} \tag{8}$$

with some constant $C_{4\varepsilon}$.

LEMMA 2. – *The function Q_ε is symmetric, i.e., $Q_\varepsilon(x) = Q_\varepsilon(1-x)$. Moreover, u_ε is odd and v_ε is even.*

LEMMA 3. – *The functional*

$$F[u,v] = \int_0^1 \big[u(x)^2 + v(x)^2\big]\,\mathrm{d}x \int_0^1 \big[u'(x)^2 + v'(x)^2\big]^{-1}\,\mathrm{d}x$$

takes its minimal value m in the class S_0 at the point u, v, which is the limit in $\mathrm{C}^1(I)$ of some sequence u_{ε_k}, v_{ε_k} of solutions of the points of minimum of the functional F_{ε_k} with $\varepsilon_k \to 0$. The functions u, v are linearly independent, they belong to $\mathrm{C}^\infty(0,1)$, are analytic on this interval and satisfy Eqs. (4), (5). *Moreover, u is odd and v is even.*

LEMMA 4. – *Let Q be a continuous positive function defined in $[0,1/2]$, $Q(0) > 1$. Consider the following Sturm–Liouville problem*:

$$\big(Q(x)y'\big)' + \lambda y = 0 \quad \textit{on } [0,1/2], \quad 2Q(0)y'(0) = \lambda y(0), \quad y'(1/2) = 0.$$

Its spectrum is discrete, $\lambda_j \to \infty$ as $j \to \infty$, $\lambda_0 = 0 < \lambda_1 < \cdots$. The eigenvalues are simple. For positive j the number of zeroes of the eigenfunction φ_j, corresponding to λ_j, is exactly j and the number of zeroes of φ_j' in $[0,1/2[$ is also j.

LEMMA 5. – *Let u, v be the functions found in Lemma* 3. *Let $k = v'(0)$, $c_1 = 4m + (32mk/9)^2$, $c_2 = (512mk/81)^2(3/4 - k^2)$. Put $r(x) = c_1 - u(x)^2 - (v(x) + C)^2$, where $C = 32mk/9$. Then*

$$r'(x)^2 r(x)^4 = 12m\big[c_1 r(x)^3 - r(x)^4 - c_2(3m)^3\big].$$

The function r cannot take a constant value on any nonempty subinterval.

3. Proof of Theorem 1

Set

$$L[Q,u,v]=\frac{\int_0^1 Q(x)[u'(x)^2+v'(x)^2]\,\mathrm{d}x}{\int_0^1[u(x)^2+v(x)^2]\,\mathrm{d}x}.$$

Let $u_0=u, v_0=v$ be the functions found in Lemma 3 such that $\int_0^1([u_0'(x)^2+v_0'(x)^2]^{-2}\,\mathrm{d}x=1$. Put $Q_0(x)=[u_0'(x)^2+v_0'(x)^2]^{-2}$. We know that $Q_0\in \mathrm{C}^\infty[0,1]$ and that $Q_0(x)=Q_0(1-x)$. We show that for any $Q\in\mathcal{A}$,

$$\inf_{u\in S,\, v\in S} L[Q,u,v]\leqslant \inf_{u,v\in S_0} L[Q,u,v]\leqslant \frac{1}{m}\equiv M$$

and that

$$\inf_{y\in S} L_1[Q_0,y]=\inf_{u\in S,\ v\in S} L[Q_0,u,v]=\inf_{u,v\in S_0} L[Q_0,u,v]=M.$$

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