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Invertibility of functional Galois connections

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Abstract We consider equations of the form $Bf = g$, where B is a Galois connection between lattices of functions. This includes the case where B is the Fenchel transform, or more generally a Moreau conjugacy. We characterize the existence and uniqueness of a solution f in terms of generalized subdifferentials, which extends K. Zimmermann's covering theorem for max-plus linear equations. ***To cite this article: M. Akian et al., C. R. Acad. Sci. Paris, Ser. I 335 (2002) 883–888.***

Inversibilité des correspondances de Galois fonctionnelles

Résumé On considère des équations de la forme $Bf = g$, où B est une correspondance de Galois entre des treillis de fonctions, ce qui inclut le cas où B est la transformation de Fenchel, ou plus généralement une conjugaison de Moreau. Nous caractérisons l'existence et l'unicité d'une solution f, en termes de sous-différentiels généralisés, et étendons ainsi le théorème de couverture de K. Zimmermann pour les équations linéaires max-plus. ***Pour citer cet article : M. Akian et al., C. R. Acad. Sci. Paris, Ser. I 335 (2002) 883–888.***

Version française abrégée

Soient $(\mathcal{F}, \leqslant_{\mathcal{F}})$ et $(\mathcal{G}, \leqslant_{\mathcal{G}})$ deux ensembles partiellement ordonnés, et $B : \mathcal{F} \to \mathcal{G}$, $C : \mathcal{G} \to \mathcal{F}$. On dit que (B, C) est une correspondance de Galois si la propriété (7b) ci-dessous est satisfaite. L'application C, qui est unique, est notée B°. On dit aussi que B et C sont des correspondances de Galois. On s'intéresse au cas où $\mathcal{F} = \mathrm{sci}(Y, \overline{\mathbb{R}})$ est l'ensemble des fonctions semi-continues inférieurement d'un espace topologique séparé Y dans $\overline{\mathbb{R}}$, et où $\mathcal{G} = \overline{\mathbb{R}}^X$, pour un espace topologique séparé X. On montre en particulier que B et B° s'écrivent sous la forme (8), où b et b° sont des applications $X \times Y \times \overline{\mathbb{R}} \to \overline{\mathbb{R}}$, et où pour tout $x \in X,\ y \in Y$, $(b(x, y, \cdot), b^\circ(x, y, \cdot))$ est une correspondance de Galois. Quand $b(x, y, \lambda) = b^\circ(x, y, \lambda) = b(x, y) - \lambda$, pour une application $b : X \times Y \to \overline{\mathbb{R}}$ (avec la convention $(-\infty) + (+\infty) = (+\infty) + (-\infty) = -\infty$), on retrouve la conjuguaison de Moreau (6), dont la transformée de Fenchel est un cas spécial. Étant donné un

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sous-ensemble $X' \subset X$ et une application $g \in \mathcal{G}$, on considère le problème

$$(\mathcal{P}'): \qquad \text{Trouver } f \in \mathcal{F} \text{ tel que } Bf \leqslant g,\ Bf(x) = g(x),\ \forall x \in X'.$$

On supposera qu'il existe un ensemble $\mathcal{S} \subset X \times Y$ tel que : (A1) $\mathcal{S}_x = \{y \in Y \mid (x, y) \in \mathcal{S}\} \neq \emptyset$, pour tout $x \in X$; (A2) $\mathcal{S}^y = \{x \in X \mid (x, y) \in \mathcal{S}\} \neq \emptyset$, pour tout $y \in Y$; (A3) $b(x, y, \cdot)$ est une bijection décroissante $\overline{\mathbb{R}} \to \overline{\mathbb{R}}$, pour tout $(x, y) \in \mathcal{S}$; et (A4) $b(x, y, \cdot) \equiv -\infty$, pour $(x, y) \in X \times Y \setminus \mathcal{S}$. Pour toute application g d'un espace topologique Z dans $\overline{\mathbb{R}}$, on utilise les notations (9). On définit aussi les sous-différentiels (10), pour $f \in \mathcal{F}$ et $g \in \mathcal{G}$. On dit que b est coercive si pour tout $x \in X$ et $\alpha \in \mathbb{R}$, $b(x, \cdot, \alpha)$ est semi-continue supérieurement, et si pour tout $x \in X$, pour tout voisinage V de x dans X, et pour tout $\alpha \in \mathbb{R}$, la fonction $b^{\alpha}_{x,V}$ de (11) est telle que tous les ensembles de sous-niveau $\{y \in Y \mid b^{\alpha}_{x,V}(y) \leqslant \beta\}$, avec $\beta \in \mathbb{R}$, sont relativement compacts. Lorsque F est une application de Y dans l'ensemble $\mathcal{P}(X)$ des parties de X, on dit que $\{F(y)\}_{y \in Y}$ est un recouvrement de $X' \subset X$ si $\bigcup_{y \in Y} F(y) \supset X'$. Ce recouvrement est dit algébriquement minimal si pour tout $z \in Y$, $\bigcup_{y \in Y \setminus \{z\}} F(y) \not\supset X'$. Il est dit topologiquement minimal si pour tout ouvert non vide $U \subset Y$, $\bigcup_{y \in Y \setminus U} F(y) \not\supset X'$. Si G est une application de X dans $\mathcal{P}(Y)$, on note G^{-1} l'application de Y dans $\mathcal{P}(X)$, donnée par $G^{-1}(y) = \{x \in X \mid y \in G(x)\}$. Nous énonçons maintenant les résultats principaux de cette note, en suivant la numérotation de la version anglaise.

THÉORÈME 2. – *Soit $X' \subset X$, et $g \in \mathcal{G}$, et considérons les assertions suivantes* :

$$\textit{Le problème } (\mathcal{P}') \textit{ a une solution,} \tag{1}$$

$$\{(\partial^\circ g)^{-1}(y)\}_{y \in Y} \textit{ est un recouvrement de } X', \tag{2}$$

$$\{(\partial^\circ g)^{-1}(y)\}_{y \in \operatorname{ldom}(B^\circ g)} \textit{ est un recouvrement de } X' \cap \operatorname{udom}(g). \tag{3}$$

Alors, (3)⇔(2)⇒(1). *En outre, l'implication* (1)⇒(3) *a lieu dans les cas suivants* : (i) *si Y est fini, ou* (ii) *si $X' \subset \operatorname{idom}(g)$, b est coercive, et si Y est discret ou bien $B^\circ g(y) > -\infty$ pour tout $y \in Y$.*

On dit qu'une application $h : X \to Y$ est quasi-continue si pour tout ouvert G de Y, l'ensemble $h^{-1}(G)$ est semi-ouvert, ce qui signifie que $h^{-1}(G)$ est inclus dans la cloture de son interieur. Toute application s.c.i. convexe propre sur $\mathbb{R}^n$ est quasi-continue sur son domaine $\operatorname{dom}(f)$. On définit $\operatorname{dom}(\partial^\circ g) = \{x \in X \mid \partial^\circ g(x) \neq \emptyset\} = \bigcup_{y \in Y} (\partial^\circ g)^{-1}(y)$.

THÉORÈME 3. – *Soit $X' \subset X$, et $g \in \mathcal{G}$, et considérons les assertions suivantes* :

$$\textit{Le problème } (\mathcal{P}') \textit{ a une unique solution,} \tag{4}$$

$$\{(\partial^\circ g)^{-1}(y)\}_{y \in \operatorname{ldom}(B^\circ g)} \textit{ est un recouvrement topologiquement minimal de } X' \cap \operatorname{udom}(g). \tag{5}$$

L'équivalence (4)⇔(5) *a lieu lorsque Y est fini et est muni de la topologie discrète. Supposons en outre que b est coercive et que $B^\circ g(y) > -\infty$ pour tout $y \in Y$. Alors, l'implication* (5)⇒(4) *a lieu si $X' \subset \operatorname{idom}(g)$ et $B^\circ g$ est quasi-continu sur son domaine. L'implication* (4)⇒(5) *a lieu si $X' \subset \operatorname{dom}(\partial^\circ g) \cap \operatorname{dom}(g)$.*

Le Théorème 3 fournit en particulier une condition suffisante pour l'unicité de $f \in \mathcal{F}$ tel que $Bf = g$.

Une fonction convexe s.c.i. propre g sur $\mathbb{R}^n$ est dite essentiellement régulière si $\operatorname{idom}(g) \neq \emptyset$, si g est différentiable dans $\operatorname{idom}(g)$, et si la norme de la différentielle de g au point x tend vers l'infini, lorsque x s'approche de la frontière de $\operatorname{dom}(g)$.

COROLLAIRE 5. – *Soit g une fonction convexe essentiellement régulière* (*s.c.i. propre*) *sur $\mathbb{R}^n$. Si f est une fonction s.c.i. propre telle que $f^\star \leqslant g$ et $f^\star(x) = g(x)$ pour tout $x \in \operatorname{idom}(g)$, alors $f = g^\star$. En particulier, g a une unique préimage par la transformation de Fenchel.*

La condition suffisante du Corollaire 5 n'est pas nécessaire.

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1. Introduction

We call *functional Galois connection* a (dual) Galois connection between a sublattice $\mathcal{F}$ of $\overline{\mathbb{R}}^Y$ and a sublattice $\mathcal{G}$ of $\overline{\mathbb{R}}^X$, where X, Y are two sets and $\overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$, see Section 2 for definitions. An important example of functional Galois connection is the Fenchel transform, and more generally, the Moreau conjugacy associated to a kernel $b : X \times Y \to \overline{\mathbb{R}}$,

$$B : \overline{\mathbb{R}}^Y \to \overline{\mathbb{R}}^X, \qquad Bf(x) = \sup\{b(x, y) - f(y) \mid y \in Y\}, \tag{6}$$

with the convention $(-\infty) + (+\infty) = (+\infty) + (-\infty) = -\infty$. Moreau conjugacies are instrumental in nonconvex duality, see [16, Chapter 11, §E], [17]. They are equivalent to max-plus linear operators with kernel. Such operators, which are of the form $f \mapsto B(-f)$, arise in deterministic optimal control and asymptotics, and have been widely studied, see in particular [7,12,3,11,1,9]. Given a map $g \in \mathcal{G}$ and a functional Galois connection $B : \mathcal{F} \to \mathcal{G}$, we consider the problem:

$$(\mathcal{P}): \qquad \text{Find } f \in \mathcal{F} \text{ such that } Bf = g.$$

In particular, we look for effective conditions on g for the solution f to exist and be unique. This problem arises in large deviations: as shown in [2], the classical Gärtner–Ellis theorem (see, e.g., [8, Theorem 2.3.6(c)]) is essentially a uniqueness result for the preimage of the Fenchel transform, and the results of this Note yield a new proof, as well as generalizations, of this theorem. Problem $(\mathcal{P})$ is also intimately related to the Monge–Kantorovitch mass transfer problem (see, e.g., [14, §3.3]). When X, Y are finite sets, $\mathcal{F} = \mathbb{R}^Y$ and $\mathcal{G} = \mathbb{R}^X$, when B is as in (6) with $b(x, y) \in \mathbb{R}$, the existence and uniqueness of the solutions of $(\mathcal{P})$ was characterized by Zimmermann [19] (see [5]), who gave conditions involving coverings of X by sets. In this Note, we extend Zimmermann's theorem to the case of functional Galois connections: the existence and uniqueness of the solutions of $(\mathcal{P})$ are characterized in terms of generalized subdifferentials of g.

2. Representation of functional Galois connections

Let $(\mathcal{F}, \leqslant_\mathcal{F})$ and $(\mathcal{G}, \leqslant_\mathcal{G})$ be two partially ordered sets, and let $B : \mathcal{F} \to \mathcal{G}$ and $C : \mathcal{G} \to \mathcal{F}$. We say that B is *antitone* if $f \leqslant_\mathcal{F} f' \Rightarrow Bf' \leqslant_\mathcal{G} Bf$. The pair (B, C) is a *dual Galois connection* (see, e.g., [4, Chapter V, Section 8]) between $\mathcal{F}$ and $\mathcal{G}$ if it satisfies one of the following equivalent conditions:

$$I_\mathcal{F} \geqslant_\mathcal{F} CB, \quad I_\mathcal{G} \geqslant_\mathcal{G} BC, \quad \text{and} \quad B,\ C \text{ are antitone maps,} \tag{7a}$$

$$(g \geqslant_\mathcal{G} Bf \iff f \geqslant_\mathcal{F} Cg) \quad \forall f \in \mathcal{F},\ g \in \mathcal{G}, \tag{7b}$$

$$Cg = \min{}_\mathcal{F}\{f \mid g \geqslant_\mathcal{G} Bf\} \quad \forall g \in \mathcal{G}, \tag{7c}$$

$$Bf = \min{}_\mathcal{G}\{g \mid f \geqslant_\mathcal{F} Cg\} \quad \forall f \in \mathcal{F}, \tag{7d}$$

where $\min_\mathcal{F}$ and $\min_\mathcal{G}$ denote the minimal elements for the orders $\leqslant_\mathcal{F}$ and $\leqslant_\mathcal{G}$, respectively, and where $I_\mathcal{A}$ denotes the identity on a set $\mathcal{A}$. Since there is at most one map C such that (B, C) is a dual Galois connection, we shall denote this C by B°, and call B or B° a dual Galois connection.

Ordinary (nondual) *Galois connections* are defined by reversing the order of $\mathcal{F}$ and $\mathcal{G}$ in (7). In the sequel, we shall only consider dual Galois connections and omit the term "dual".

We call *lattice of functions* a sublattice $\mathcal{F}$ of S^Y, where $(S, \leqslant)$ is a lattice, Y is a set, and S^Y is equipped with the product ordering. When S has a maximum element $\top_S$, we define the *Dirac function* at point $y \in Y$ with value $s \in S$: $\delta_y^s \in S^Y$, $\delta_y^s(y') = s$ if $y' = y$, and $\delta_y^s(y') = \top_S$ otherwise.

THEOREM 1. – *Let S, T be two lattices that have a maximum element, let X, Y be arbitrary nonempty sets and let $\mathcal{F} \subset S^Y$ (resp. $\mathcal{G} \subset T^X$) be a lattice of functions containing all the Dirac functions of S^Y*

(resp. T^X). Then, (B, B°) is a Galois connection between $\mathcal{F}$ and $\mathcal{G}$ if, and only if, there exists two maps $b : X \times Y \times S \to T$ and $b^\circ : X \times Y \times T \to S$ such that: for all $(x, y) \in X \times Y$, $(b(x, y, \cdot), b^\circ(x, y, \cdot))$ is a Galois connection between S and T; for all $(x, t) \in X \times T$, $b^\circ(x, \cdot, t) \in \mathcal{F}$; for all $(y, s) \in Y \times S$, $b(\cdot, y, s) \in \mathcal{G}$; and

$$Bf = \sup_{\mathcal{G}}\bigl\{b\bigl(\cdot, y, f(y)\bigr) \mid y \in Y\bigr\}, \quad \forall f \in \mathcal{F}, \tag{8a}$$

$$B^\circ g = \sup_{\mathcal{F}}\bigl\{b^\circ\bigl(x, \cdot, g(x)\bigr) \mid x \in X\bigr\}, \quad \forall g \in \mathcal{G}. \tag{8b}$$

When Y is a T_1 topological space and S has a maximum element, Theorem 1 can be applied to the set $\mathcal{F} = \mathrm{lsc}(Y, S)$ of lower semicontinuous maps $Y \to S$ (we say that a map $f : Y \to S$ is *lower semicontinuous*, or l.s.c., if for all $s \in S$, the sublevel set $\{y \in Y \mid f(y) \leqslant s\}$ is closed). In this case, $\sup_{\mathcal{F}} = \sup$ since the sup of l.s.c. maps is l.s.c.

Kolokoltsov already proved [10] (see also [11, Theorem 1.4]) a "Riesz representation theorem" similar to Theorem 1, for a continuous map B between (non-complete) lattices of *continuous* functions $\mathcal{F}$ and $\mathcal{G}$, assuming that B preserves finite sups. Singer [17, Theorem 7.1] also proved a theorem similar to Theorem 1 when $\mathcal{F} = A^Y$ and A is a complete lattice included in $\overline{\mathbb{R}}$.

3. Existence and uniqueness of solutions of $Bf = g$

In this section, we take $S = T = \overline{\mathbb{R}}$, we assume that X and Y are Hausdorff topological spaces, and take $\mathcal{F} = \mathrm{lsc}(Y, \overline{\mathbb{R}})$, $\mathcal{G} = \overline{\mathbb{R}}^X$, together with $B,\ B^\circ,\ b,\ b^\circ$ as in Theorem 1. The case when $\mathcal{F} = \overline{\mathbb{R}}^Y$ and X, Y are arbitrary sets can be obtained by taking discrete topologies on X and Y. The property that $b(x, y, \cdot)$, or $b^\circ(x, y, \cdot)$, is a Galois connection can be made explicit by noting that a map $h : \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ is a Galois connection if, and only if, h is nonincreasing, right-continuous, and $h(+\infty) = -\infty$.

We shall assume that there is a subset $\mathcal{S} \subset X \times Y$ such that the following properties hold: (A1) $\mathcal{S}_x = \{y \in Y \mid (x, y) \in \mathcal{S}\} \neq \emptyset$, for all $x \in X$; (A2) $\mathcal{S}^y = \{x \in X \mid (x, y) \in \mathcal{S}\} \neq \emptyset$, for all $y \in Y$; (A3) $b(x, y, \cdot)$ is a bijection $\overline{\mathbb{R}} \to \overline{\mathbb{R}}$ for all $(x, y) \in \mathcal{S}$; and (A4) $b(x, y, \cdot) \equiv -\infty$, for $(x, y) \in X \times Y \setminus \mathcal{S}$. When B is the Moreau conjugacy given by (6), $b(x, y, \lambda) = b^\circ(x, y, \lambda) = b(x, y) - \lambda$, Assumptions (A3), (A4) are satisfied when $b(x, y) \in \mathbb{R} \cup \{-\infty\}$, for all $x \in X$, $y \in Y$, and Assumptions (A1), (A2) are satisfied when for all $x \in X$, $y \in Y$, $b(x, \cdot)$ and $b(\cdot, y)$ are not identically $-\infty$.

Rather than Problem $(\mathcal{P})$, we will consider the more general problem:

$$(\mathcal{P}'): \qquad \text{Find } f \in \mathcal{F} \text{ such that } Bf \leqslant g,\ Bf(x) = g(x)\ \forall x \in X',$$

where $g \in \mathcal{G}$ and $X' \subset X$ are given.

To state our results, we need some definitions. First, for any map g from a topological space Z to $\overline{\mathbb{R}}$, we define the lower domain, upper domain, domain, and inner domain:

$$\begin{aligned} &\mathrm{ldom}(g) = \bigl\{z \in Z \mid g(z) < +\infty\bigr\}, && \mathrm{udom}(g) = \bigl\{z \in Z \mid g(z) > -\infty\bigr\}, \\ &\mathrm{dom}(g) = \mathrm{ldom}(g) \cap \mathrm{udom}(g), && \mathrm{idom}(g) = \Bigl\{z \in \mathrm{dom}(g) \mid \limsup_{z' \to z} g(z') < +\infty\Bigr\}. \end{aligned} \tag{9}$$

Next, given $f \in \mathcal{F}$, $y \in Y$, $g \in \mathcal{G}$ and $x \in X$, we define the subdifferentials:

$$\partial f(y) = \bigl\{x \in X \mid (x, y) \in \mathcal{S},\ b\bigl(x, y', f(y')\bigr) \leqslant b\bigl(x, y, f(y)\bigr)\ \forall y' \in Y\bigr\}, \tag{10a}$$

$$\partial^\circ g(x) = \bigl\{y \in Y \mid (x, y) \in \mathcal{S},\ b^\circ\bigl(x', y, g(x')\bigr) \leqslant b^\circ\bigl(x, y, g(x)\bigr)\ \forall x' \in X\bigr\}. \tag{10b}$$

We say that b is *coercive* if for all $x \in X$ and $\alpha \in \mathbb{R}$, $b(x, \cdot, \alpha)$ is upper semicontinuous (u.s.c.) and if for all $x \in X$, all neighborhoods V of x in X, and all $\alpha \in \mathbb{R}$, the function

$$y \in Y \mapsto b^\alpha_{x,V}(y) = \sup_{z \in V} b\bigl(z, y, b^\circ(x, y, \alpha)\bigr), \tag{11}$$

has relatively compact finite sublevel sets, which means that $\{y \in Y \mid b^{\alpha}_{x,V}(y) \leqslant \beta\}$ is relatively compact for all $\beta \in \mathbb{R}$. The latest property holds trivially, and independently of the topology on X, when Y is compact (and in particular when Y is finite). If $X = Y = \mathbb{R}^n$ and $b(x, y, \alpha) = \langle x, y\rangle - \alpha$ then for any neighborhood V of x, and $\alpha \in \mathbb{R}$, $b^{\alpha}_{x,V}(y) \geqslant \varepsilon\|y\| + \alpha$, for some $\varepsilon > 0$, so that b is coercive. Similarly, if $b(x, y, \alpha) = a\|x - y\|^2 - \alpha$, where $a \in \mathbb{R} \setminus \{0\}$ and $\|\cdot\|$ is the Euclidean norm, then $b^{\alpha}_{x,V}(y) \geqslant \varepsilon\|y - x\| - 1 + \alpha$, for some $\varepsilon > 0$, so that b is coercive.

When F is a map from Y to the set $\mathcal{P}(X)$ of all subsets of X, we say that $\{F(y)\}_{y\in Y}$ is a *covering* of $X' \subset X$ if $\bigcup_{y\in Y} F(y) \supset X'$. This covering is *algebraically minimal* if for all $z \in Y$, $\bigcup_{y\in Y\setminus\{z\}} F(y) \not\supset X'$. It is *topologically minimal* if for all non-empty open sets $U \subset Y$, $\bigcup_{y\in Y\setminus U} F(y) \not\supset X'$. Algebraic minimality implies topological minimality. Both notions coincide if Y is a discrete topological space. If G is a map from X to $\mathcal{P}(Y)$, we denote by G^{-1} the map from Y to $\mathcal{P}(X)$ given by $G^{-1}(y) = \{x \in X \mid y \in G(x)\}$.

THEOREM 2. – *Let $X' \subset X$, and $g \in \mathcal{G}$. Consider the following statements*:

$$\textit{Problem } (\mathcal{P}') \textit{ has a solution,} \tag{12}$$

$$\big\{(\partial^\circ g)^{-1}(y)\big\}_{y\in Y} \textit{ is a covering of } X', \tag{13}$$

$$\big\{(\partial^\circ g)^{-1}(y)\big\}_{y\in \mathrm{ldom}(B^\circ g)} \textit{ is a covering of } X' \cap \mathrm{udom}(g). \tag{14}$$

Then, (14)⇔(13)⇒(12)*. Moreover, the implication* (12)⇒(14) *holds in the following cases:* (i) *when Y is finite, or* (ii) *when $X' \subset \mathrm{idom}(g)$, b is coercive and either Y is discrete or $B^\circ g(y) > -\infty$ for all $y \in Y$.*

To give an uniqueness result, we need some additional definitions. We say that a map $h : X \to Y$ is *quasi-continuous* if for all open sets G of Y, the set $h^{-1}(G)$ is *semi-open*, which means that $h^{-1}(G)$ is included in the closure of its interior (see for instance [13] for definitions and properties of quasi-continuous functions and multi-applications). If $h : X \to \overline{\mathbb{R}}$ is l.s.c., then h is quasi-continuous if, and only if, $h = \mathrm{lsc}(\mathrm{usc}(h))$, where lsc (resp. usc) means the l.s.c. (resp. u.s.c.) hull. We say that B is *regular* if for all $f \in \mathcal{F}$, Bf is l.s.c. on X and quasi-continuous on its domain, which means that the restriction of Bf to its domain is quasi-continuous for the induced topology. The notion of regularity for B° is defined in the symmetric way. When X is endowed with the discrete topology, B is always regular. When $\mathcal{S} = X \times Y$ and $\{b(\cdot, y, \alpha)\}_{y\in Y,\, \alpha\in\mathbb{R}}$ is an equicontinuous family of functions, then Bf is continuous on X for any $f \in \mathcal{F}$, so B is regular. When $X = Y = \mathbb{R}^n$, the Fenchel transform is regular because any l.s.c. proper convex function f is quasi-continuous on its domain $\mathrm{dom}(f)$. We define $\mathrm{dom}(\partial^\circ g) = \{x \in X \mid \partial^\circ g(x) \neq \emptyset\} = \bigcup_{y\in Y} (\partial^\circ g)^{-1}(y)$.

THEOREM 3. – *Let $X' \subset X$, and $g \in \mathcal{G}$. Consider the following statements*:

$$\textit{Problem } (\mathcal{P}') \textit{ has a unique solution,} \tag{15}$$

$$\big\{(\partial^\circ g)^{-1}(y)\big\}_{y\in \mathrm{ldom}(B^\circ g)} \textit{ is a topologically minimal covering of } X' \cap \mathrm{udom}(g). \tag{16}$$

The equivalence (15)⇔(16) *holds when Y is finite* (*with the discrete topology*)*. Moreover, assume that b is coercive and $B^\circ g(y) > -\infty$ for all $y \in Y$. Then, the implication* (16)⇒(15) *holds if $X' \subset \mathrm{idom}(g)$ and $B^\circ g$ is quasi-continuous on its domain. The implication* (15)⇒(16) *holds if $X' \subset \mathrm{dom}(\partial^\circ g) \cap \mathrm{dom}(g)$.*

Since (15) implies that Problem $(\mathcal{P})$ has at most one solution, Theorem 3 yields a sufficient uniqueness condition for the solution of Problem $(\mathcal{P})$. However, for Problem $(\mathcal{P})$, the necessary uniqueness condition implied by Theorem 3 only holds when Y is finite, or when $\mathrm{dom}(\partial^\circ g) \cap \mathrm{dom}(g) = X$. To give a more specific uniqueness result, we shall use the following condition:

$$(\mathcal{C}):\quad \begin{array}{l} \text{there exists a basis } \mathcal{B} \text{ of neighborhoods such that} \\ \forall U \in \mathcal{B},\ \exists \varepsilon > 0,\ \forall x \in X, \quad \displaystyle\sup_{y\in U\cap \mathcal{S}_x,\ \alpha\in\mathbb{R}} \big(b(x, y, \alpha) - b(x, y, \alpha + \varepsilon)\big) < +\infty. \end{array}$$

Condition $(\mathcal{C})$ is satisfied in particular when for all $x \in X$, $\{b(x, y, \cdot)\}_{y\in Y}$ is a family of α-Hölder continuous functions (for $0 < \alpha \leqslant 1$), uniformly in $y \in U$, for all small enough open sets U, or if $\{b(x, y, \cdot)\}_{x\in X,\, y\in U}$ is an equicontinuous family, for all small enough open sets U. In particular, condition $(\mathcal{C})$ is satisfied when $b(x, y, \alpha) = b(x, y) - \alpha$ or when $b(x, y, \alpha) = -(|\langle x, y\rangle| + 1)\alpha$.

THEOREM 4. – *Let $g \in \mathcal{G}$. Assume that b is coercive, and that $B^\circ g(y) > -\infty$ for all $y \in Y$. Then, the uniqueness of the solution of Problem $(\mathcal{P})$ implies* (16)*, when any of the following assumptions holds*:

(1) *$X' = \mathrm{dom}(\partial^\circ g) \cap \mathrm{dom}(g)$, and Y is locally compact.*

(2) *$X' = \mathrm{idom}(g)$, B is regular, $\mathrm{dom}(g)$ is included in the closure of $\mathrm{idom}(g)$, and either Y is locally compact or condition $(\mathcal{C})$ holds.*

The topological minimality in (16) is a relaxation of algebraic minimality, which is a generalized differentiability condition. Indeed, if $\{(\partial^\circ g)^{-1}(y)\}_{y\in \mathrm{ldom}(B^\circ g)}$ is a covering of $X' \cap \mathrm{udom}(g)$, this covering is algebraically minimal if, and only if, for all $y \in \mathrm{ldom}(B^\circ g)$, there is an $x \in X' \cap \mathrm{udom}(g)$ such that $\partial^\circ g(x) = \{y\}$. When B is the Fenchel transform, the following classical notion is intermediate between algebraic and topological minimality: a l.s.c. proper convex function g on $\mathbb{R}^n$ is *essentially smooth* if $\mathrm{idom}(g) \neq \emptyset$ (the interior of its domain is nonempty), g is differentiable in $\mathrm{idom}(g)$, and the norm of the differential of g at x tends to infinity, when x goes to the boundary of $\mathrm{dom}(g)$, see [15, §26].

COROLLARY 5. – *Let g be an essentially smooth* (*l.s.c. proper*) *convex function on $\mathbb{R}^n$. If f is a l.s.c. proper function such that $f^\star \leqslant g$ and $f^\star(x) = g(x)$ for all $x \in \mathrm{idom}(g)$, then $f = g^\star$. In particular, g has a unique preimage by the Fenchel transform.*

The following function g satisfies (16) but is not essentially smooth: consider $X = Y = \mathbb{R}^2$, $g = Bf = f^*$ where $f : \mathbb{R}^2 \to \mathbb{R} \cup \{+\infty\}$, with $f(y) = y_1^2(y_2^2 + 3)$ if $|y_2| \leqslant 1$ and $f(y) = +\infty$ elsewhere.

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