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Fractional monodromy of resonant classical and quantum oscillators

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Abstract We introduce fractional monodromy for a class of integrable fibrations which naturally arise for classical nonlinear oscillator systems with resonance. We show that the same fractional monodromy characterizes the lattice of quantum states in the joint spectrum of the corresponding quantum systems. Results are presented on the example of a two-dimensional oscillator with resonance 1:(−1) and 1:(−2). ***To cite this article: N.N. Nekhoroshev et al., C. R. Acad. Sci. Paris, Ser. I 335 (2002) 985–988.***

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Monodromie fractionnelle des oscillateurs résonnants classiques et quantiques

Résumé La monodromie fractionnelle est introduite pour une classe de fibrations intégrables, qui apparaissent naturellement en mécanique classique dans le cas d'un oscillateur nonlinéaire avec résonance. On démontre, que la même monodromie fractionnelle caractérise de façon qualitative le réseau des états quantiques dans le spectre conjoint des observables pour les systèmes quantiques correspondants. Les résultats sont présentés en utilisant l'exemple d'un oscillateur à deux degrés de liberté avec la résonance 1:(−1) et 1:(−2). ***Pour citer cet article : N.N. Nekhoroshev et al., C. R. Acad. Sci. Paris, Ser. I 335 (2002) 985–988.***

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Consider a $2n$-dimensional symplectic manifold M with the set $F = \{F_1, \dots, F_n\}$ of functions on it which are mutually in involution. The map $F : M \to \mathbb{R}^n$ foliates the manifold M into connected components of inverse images $F^{-1}(f)$ of points $f \in \mathbb{R}^n$. The so obtained *integrable fibration* is *regular* if differentials $\{\mathrm{d}F_1, \dots, \mathrm{d}F_n\}$ are linearly independent at any point $x \in M$ and *toric* if, additionally, all leafs are compact. Remark that the commonly used *energy momentum map* $\mathcal{EM}$ [5] is a particular case of the map F.

We study certain integrable fibrations which are non-regular but relatively simple and important for applications. Recall that a point x in M is a critical point of F if differentials $\{\mathrm{d}F_1, \dots, \mathrm{d}F_n\}$ are linearly dependent at x. Integrable fibrations with isolated critical points were studied in [6,5,13,12,18] mainly because they provide examples of classical Hamiltonian monodromy, or the simplest obstruction to the

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existence of global action-angle variables [1,14,9,8]. The interest in classical integrable systems with monodromy was increased significantly in the last years by discovering manifestations of monodromy in corresponding quantum systems [6,7,15,16,3,11,17].

In this Note we extend the monodromy analysis to the case of non-isolated critical values and introduce fractional monodromy. We study both classical and quantum systems. We give only a brief summary of the results and use concrete examples of integrable fibrations corresponding to the two-dimensional nonlinear oscillators with $1{:}(-1)$ and $1{:}(-2)$ resonances to explain them. Detailed proofs will be published separately.

Consider a two-degrees-of-freedom Hamiltonian system with the phase space $\mathbb{R}^4_{p,q}$, the standard symplectic structure $\mathrm{d}q_1 \wedge \mathrm{d}p_1 + \mathrm{d}q_2 \wedge \mathrm{d}p_2$, and two first integrals in involution,

$$F_1 = \tfrac{1}{2}m_1\left(p_1^2 + q_1^2\right) - \tfrac{1}{2}m_2\left(p_2^2 + q_2^2\right), \qquad F_2 = \mathrm{Im}\left[(q_1 + \mathrm{i}p_1)^{m_2}(q_2 + \mathrm{i}p_2)^{m_1}\right] + R(q, p), \tag{1}$$

where m_1 and m_2 are positive integers with $\gcd(m_1, m_2) = 1$. The function $R(p, q)$ is in involution with F_1 and is small compared to other terms near the origin $q = p = 0$. In order to guarantee compactness of the combined level sets of (F_1, F_2), this function should be positively (or negatively) definite and dominate far from the origin. We use

$$R(q, p) = \varepsilon\left[\tfrac{1}{2}m_1\left(p_1^2 + q_1^2\right) + \tfrac{1}{2}m_2\left(p_2^2 + q_2^2\right)\right]^{2s}, \quad \text{where } 2s > \tfrac{1}{2}(m_1 + m_2). \tag{2}$$

The lowest order term in the expansion of a generic Hamiltonian $H(F_1, F_2)$ at the origin equals ωF_1 with nonzero "frequency" ω. It follows that our system is a nonlinear oscillator with the resonance $m_1{:}(-m_2)$. We study the integrable fibration defined by (1) for the two simplest cases $1{:}(-1)$ and $1{:}(-2)$ with s in (2) set to 1. As a tribute to the tradition [5], we denote the values (f_1, f_2) as (m, h), thus implying that F_1 and F_2 can be regarded as an "momentum–energy" pair.

The $1{:}(-1)$ case is equivalent to the "champagne bottle" system [5]. The map F in this case has an isolated critical value $m = h = 0$ corresponding to the isolated critical point of rank zero. The singular fiber $F^{-1}(0, 0)$ is a *pinched torus* shown in Fig. 1. The neighborhood of this fiber consists exclusively of regular fibers $\mathbb{T}^2$. One of the basic circles of the regular torus vanishes in the singular fiber. Below we compare this case to the $1{:}(-2)$ system which has a non-isolated critical value at $m = h = 0$.

The topological origin of the classical Hamiltonian monodromy [9] is the presence in the integrable fibration of an isolated singular fiber with one vanishing cycle. After we complete a closed path around the corresponding singular value in the base space, the basis cycles of the regular fibers $\mathbb{T}^2$ are modified. In an appropriately chosen cycle basis, the corresponding monodromy matrix is $\left(\begin{smallmatrix}1 & 1\\ 0 & 1\end{smallmatrix}\right)$. Due to the S_1 symmetry

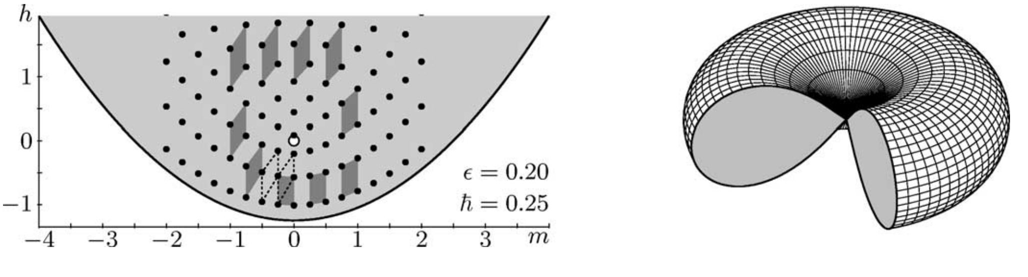

Figure 1. – Lattice of quantum states in the base space (light shaded area) of the integrable fibration in the $1{:}(-1)$ case (left). Dark gray quadrangles show the evolution of elementary lattice cells along the closed path around the singular value $(0, 0)$. The singular fiber $F^{-1}(0, 0)$ is a pinched torus (right) whose part is cut out to show better its geometry.

Figure 1. – Réseau des états quantiques superposé dans la base de la fibration intégrable pour la rśonance $1{:}(-1)$ *(à gauche). Les quadrilatères gris foncés montrent l'évolution de la cellule élémentaire le long d'un contour fermé autour de la valeur singulière* $(0, 0)$*. La fibre singulière* $F^{-1}(0, 0)$ *est un tore pincé (à droite), dont une partie est découpée afin de démontrer sa géométrie.*

of our system, one (vanishing) basis cycle remains invariant. The other cycle transforms into the sum of the two basis cycles. Quantum states form a lattice in $\mathbb{R}^2$ with coordinates (m, h). This lattice is given by the Einstein–Brillouin–Kramers quantization conditions, which are the generalized integrity conditions $(I_k/(2\pi\hbar) - \mu_k) \in \mathbb{Z}$, $k = 1, 2$, for the classical actions (I_1, I_2) computed along the two basic cycles on the regular tori $\mathbb{T}^2$. The 1:(-1) lattice in Fig. 1, left, has the characteristic pattern of quantum states around the isolated critical point. To uncover quantum monodromy [16,6], we take an elementary cell formed by the neighboring points of the lattice of quantum states and move it along a circular path in $\mathbb{R}^2_{m,h}$ around the singular value $(0, 0)$. The transformation of the cell after completing one tour is a clear manifestation of classical monodromy in quantum problems. (Note, that the monodromy matrix for actions is the inverse transposed monodromy matrix for cycles in angles.) Relation of quantum monodromy to the redistribution of energy levels between energy bands is discussed in [16,11,10].

The energy-momentum map F in the case of the 1:(-2) resonance is more complicated. The critical value $m = h = 0$ with rank 0 is now connected in the base space to a one-dimensional stratum of "weaker" singular values. The latter is formed by points which lift to a 2-torus with one cycle covering itself twice. A geometric representation of such "weak" singular fiber in the three-dimensional space is shown in Fig. 2, right. The cycle which covers itself twice coincides with the vanishing cycle. [1]

Any closed path in the base space around the singular value $(0, 0)$ crosses the line of weak singularities. We consider how the basis cycles of the regular fibers $\mathbb{T}^2$ change when their image $F(\mathbb{T}^2)$ moves along such path. The vanishing cycle γ_1 remains again invariant. To find the transformation of the other (nonvanishing) cycle γ we should start with a *double cycle* 2γ. After $F(\mathbb{T}^2)$ completes one tour, we find that 2γ transforms into $2\gamma + \gamma_1$. The transformation of γ itself can be, therefore, formally given by the monodromy matrix $\left(\begin{smallmatrix}1 & 1/2\\ 0 & 1\end{smallmatrix}\right)$ with one half integer element.

The lattice of quantum states (i.e., values of F which obey generalized integrity conditions) around the critical value of the energy-momentum map F for the system with the 1:(-2) resonance is shown in Fig. 2, left. Near the line of weak singular values of F this lattice seems to behave irregularly. In fact, the density of states has an oscillatory contribution along the line. To cross this line unambiguously we must use a *double elementary cell*. Like in the 1:(-1) case, we can define a transformation of the 1:(-2) lattice. This

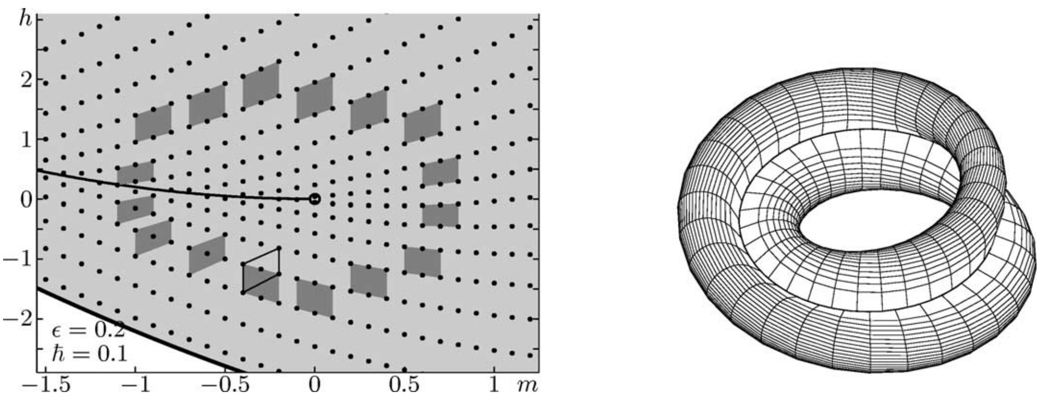

Figure 2. – Lattice of quantum states in the base space (light shaded area) of the integrable fibration in the 1:(-2) case (left). Dark gray quadrangles show the evolution of elementary lattice cells along the closed path around the singular value $(0, 0)$. Solid line represents other "weaker" singular values, the corresponding fiber is a twice curled torus (right).

Figure 2. – Réseau des états quantiques superposé dans la base de la fibration intégrable pour la résonance 1:(-2) (*à gauche*). *Les quadrilatères gris foncés montrent l'évolution de la cellule élémentaire le long d'un contour fermé autour de la valeur singulière* $(0, 0)$. *La ligne représente les autres valeurs « plus faiblement » singulières, la fibre correspondante est un tore deux fois enroulé* (*à droite*).

transformation is properly defined only for the index 2 sublattice of $\mathbb{Z}^2$, but can be formally introduced for the lattice itself. In this latter case, the monodromy matrix for actions is $\begin{pmatrix} 1 & 0 \\ -1/2 & 1 \end{pmatrix}$.

Generalization to the case of arbitrary resonances $m_1{:}(-m_2)$ leads naturally to monodromy matrices with fractional entries. The generalized monodromy occurs in many cases of systems with non-isolated critical values of the energy-momentum map. Such systems are especially common in atomic and molecular physics. Relation of quantum monodromy to the redistribution phenomenon [11] and the topological quantum numbers of energy bands [10] suggests further generalization of these numbers to systems with resonances.

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[1] General description of Hamiltonian systems with one-dimentional singular strata is given in [2]. Three dimensional sections of integrable fibration F near weak singularity can be characterized as Seifert fibrations. Local semiclassical analysis is detailed in [4].

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