C. R. Acad. Sci. Paris, Ser. I 335 (2002) 979–984

Problèmes mathématiques de la mécanique/*Mathematical Problems in Mechanics*

# The Rayleigh–Stokes problem for an edge in an Oldroyd-B fluid *

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**Received 2 January 2002; accepted after revision 12 September 2002**

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**Abstract** The velocity fields corresponding to an incompressible fluid of Oldroyd-B type subject to a linear flow within an infinite edge are determined for all values of the relaxation and retardation times. The well known solution for a Navier–Stokes fluid, as well as those corresponding to a Maxwell fluid and a second grade one, appears as a limiting case of our solutions. ***To cite this article: C. Fetecau, C. R. Acad. Sci. Paris, Ser. I 335 (2002) 979–984.***



**Le problème Rayleigh–Stokes pour un dièdre dans un fluide Oldroyd-B**

**Résumé** Les champs de vitesses correspondant à un fluide de type Oldroyd-B qui exécute un mouvement linéaire dans un dièdre infini sont déterminés pour toutes les valeurs des temps de relaxation et de retard. La solution bien connue pour le fluide de Navier–Stokes, les solutions correspondant à un fluide de Maxwell et à un fluide de grade deux apparaissent comme un cas limite de nos solutions. ***Pour citer cet article : C. Fetecau, C. R. Acad. Sci. Paris, Ser. I 335 (2002) 979–984.***



## *Version française abrégée*

Les équations constitutives (1) et (2), représentant la version Jeffreys du modèle Oldroyd, définissent un fluide de type Oldroyd-B. Elles contiennent, en guise de cas particuliers, le fluide de Maxwell (pour $\lambda_r = 0$) et le fluide visqueux linéaire ou de Navier–Stokes (pour $\lambda = \lambda_r = 0$).

En 1979, Zierep [10] a déterminé le champ de vitesses correspondant a problème de Rayleigh–Stokes pour un fluide de Navier–Stokes. Récemment, sa solution a été étendue aux fluides de grade deux [1] et aux fluides de type Maxwell [2].

Dans cette Note le même problème est résolu pour un fluide de type Oldroyd-B. Des calculs directs montrent que $v(y, z, t)$, donnée par les relations (16), (17) et (18), correspondant aux $\lambda_r > \lambda$, $\lambda_r = \lambda$ et $\lambda_r < \lambda$, satisfait à l'équation avec dérivées partielles (6) et à toutes les conditions initiales et sur la frontière

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S1631-073X(02)02577-3/FLA

imposées. De plus, les relations similaires correspondant aux fluides de Maxwell, de grade deux et de Navier–Stokes apparaissent comme un cas limite de nos solutions.

Ainsi, en faisant $\lambda \to 0$ dans (16) nous obtenons le champ de vitesses (24) correspondant à un fluide de grade deux ([1], la relation $(3\cdot 4)$), pour $\alpha$ et donc $\lambda_r \to 0$ dans (18) on obtient la solution (25) correspondant à un fluide de Maxwell [2]. Aussi pour $\alpha \to 0$ dans (24) ou $\lambda \to 0$ dans (17) ou (25) on obtient le champ de vitesses (26) correspondant au problème de Rayleigh–Stokes dans un dièdre pour un fluide de Navier–Stokes [10].

## 1. Introduction

The mechanical behavior of non-Newtonian fluids has been modelled by several constitutive equations [4]. Here, we shall consider the Jeffrey's version of the Oldroyd model [5] whose constitutive equation appends an additional term to Maxwell's equation. The Cauchy stress $\mathbf{T}$ in such a model is given by

$$\mathbf{T} = -p\mathbf{I} + \mathbf{S}, \quad \mathbf{S} + \lambda \frac{\delta \mathbf{S}}{\delta t} = \mu\left(\mathbf{A} + \lambda_r \frac{\delta \mathbf{A}}{\delta t}\right), \tag{1}$$

where $\lambda$ is the relaxation time, $\mu$ the dynamic viscosity, $\lambda_r$ the retardation time, $-p\mathbf{I}$ denotes the indeterminate spherical stress, $\mathbf{A} = \mathbf{L} + \mathbf{L}^{\mathrm{T}}$ where $\mathbf{L}$ is the velocity gradient and $\frac{\delta}{\delta t}$ represents the upper-convected time derivative defined through

$$\frac{\delta \mathbf{B}}{\delta t} = \dot{\mathbf{B}} - \mathbf{L}\mathbf{B} - \mathbf{B}\mathbf{L}^{\mathrm{T}}, \tag{2}$$

the superposed dot indicating the material time derivative.

The model characterized by the constitutive equation (1) is usually referred to as the Oldroyd-B fluid. This model includes as special cases the Maxwell model and the linearly viscous fluid model. Recently, it has received special attention from both the theoreticians and experimentalists in rheology. Existence results for the flow of such a fluid have been established in [3] and a special flow was considered in [5].

Our purpose in this Note is to determine the velocity field and the associated tangential tensions corresponding to the Rayleigh–Stokes problem for a fluid of Oldroyd-B type. For $\lambda_r$ or $\lambda$ tending to zero our solutions reduce to those corresponding to a Maxwell or a second grade fluid, respectively [1,2]. If both $\lambda_r$ and $\lambda$ tend to zero they end up to the well-known solution for a Navier–Stokes fluid [10].

## 2. Formulation of the problem

Let us suppose that an incompressible fluid of Oldroyd-B type, at rest, occupies the space of the first dial of a rectangular edge $(-\infty < x < \infty;\ y, z \geqslant 0)$. At the moment $t = 0^+$ the infinitely extended edge is impulsively brought to the constant velocity $V$ in the $x$ direction. By the influence of shear the fluid is gradually moved. Its velocity field will be of the form

$$\mathbf{v} = v(y, z, t)\mathbf{i}, \tag{3}$$

where $\mathbf{i}$ is the unit vector along the $x$-coordinate direction.

Substituting (3) in (2) and $(1)_2$ we get for the tangential tensions $S_{xy}$ and $S_{xz}$ the differential equations

$$S_{xy} + \lambda \partial_t S_{xy} = \mu(1 + \lambda_r \partial_t)\partial_y v \quad \text{and} \quad S_{xz} + \lambda \partial_t S_{xz} = \mu(1 + \lambda_r \partial_t)\partial_z v. \tag{4}$$

From the balance of linear momentum, in the absence of a pressure gradient in the $x$-direction, we also have

$$\partial_y S_{xy} + \partial_z S_{xz} = \rho \partial_t v, \tag{5}$$

where $\rho$ is the constant density of the fluid.

Eliminating $S_{xy}$ and $S_{xz}$ between Eqs. (4) and (5) we attain to the linear partial differential equation

$$\partial_t v(y,z,t) + \lambda \partial_t^2 v(y,z,t) = \nu(1+\lambda_r \partial_t)\big(\partial_y^2 + \partial_z^2\big) v(y,z,t); \quad y,z,t>0, \tag{6}$$

where $\nu = \mu/\rho$ is the kinematic viscosity of the fluid.

Since the fluid has been at rest for all $t \leqslant 0$ and the edge is maintained at the constant speed $V$ for all $t>0$ we have

$$v(y,z,0) = 0; \quad y,z>0, \tag{7}$$

and

$$v(0,z,t) = v(y,0,t) = V; \quad t>0. \tag{8}$$

Eq. (6) being of higher order than its corresponding for Navier–Stokes fluids, we have need of additional initial and boundary conditions. The appropriate boundary conditions [6,7]

$$v(y,z,t),\ \partial_y v(y,z,t),\ \partial_z v(y,z,t) \to 0 \quad \text{as } y^2+z^2 \to \infty, \tag{9}$$

are consequences of the fact that the fluid is at rest at infinity and there is no shear in the free stream. Moreover, we assume that (cf. [9])

$$\partial_t v(y,z,t) \to 0 \quad \text{as} \quad t \to 0. \tag{10}$$

## 3. Solution of the problem

Multiplying both sides of Eq. (6) by $(2/\pi)\sin(\xi y)\sin(\eta z)$, integrating with respect to $y$ and $z$ from 0 to $\infty$ and having in mind the boundary conditions (8) and (9) we find that

$$\lambda \partial_t^2 v_s(\xi,\eta,t) + \big[1+\alpha\big(\xi^2+\eta^2\big)\big]\partial_t v_s(\xi,\eta,t) + \nu\big(\xi^2+\eta^2\big) v_s(\xi,\eta,t) = \frac{2}{\pi}\nu V \frac{\xi^2+\eta^2}{\xi\eta}, \tag{11}$$

where $\alpha = \nu\lambda_r$ and $v_s(\xi,\eta,t)$ is the double Fourier sine transform of $v(y,z,t)$. In view of (7) and (10) $v_s(\xi,\eta,t)$ has to satisfy the initial conditions

$$v_s(\xi,\eta,0) = \partial_t v_s(\xi,\eta,0) = 0; \quad \xi,\eta>0. \tag{12}$$

The solution of the ordinary differential equation (11) with the initial conditions (12) has one of the following forms

$$v_s(\xi,\eta,t) = \frac{2}{\pi}\frac{V}{\xi\eta}\left[1 - \frac{r_1\exp(r_2 t) - r_2\exp(r_1 t)}{r_1 - r_2}\right] \quad \text{if} \quad \lambda_r > \lambda, \tag{13}$$

$$v_s(\xi,\eta,t) = \frac{2}{\pi}\frac{V}{\xi\eta}\left[1 - \frac{\alpha(\xi^2+\eta^2)\exp(-t/\lambda) - \exp[-\nu(\xi^2+\eta^2)t]}{\alpha(\xi^2+\eta^2) - 1}\right] \quad \text{if} \quad \lambda_r = \lambda, \tag{14}$$

or

$$v_s(\xi,\eta,t)=\begin{cases}\dfrac{2}{\pi}\dfrac{V}{\xi\eta}\left[1-\dfrac{r_1\exp(r_2t)-r_2\exp(r_1t)}{r_1-r_2}\right] & \text{on } \mathcal{D}_1,\\ \dfrac{2}{\pi}\dfrac{V}{\xi\eta}\left\{1-\exp\left(-\dfrac{1+\alpha(\xi^2+\eta^2)}{2\lambda}t\right)\right.\\ \qquad\left.\times\left[\cos\left(\dfrac{\beta t}{2\lambda}\right)+\dfrac{1+\alpha(\xi^2+\eta^2)}{\beta}\sin\left(\dfrac{\beta t}{2\lambda}\right)\right]\right\} & \text{on } \mathcal{D}_2,\end{cases} \tag{15}$$

if $\lambda_r<\lambda$. In (13)–(15) the following notations have been used

$$r_{1,2}=\frac{-[1+\alpha(\xi^2+\eta^2)]\pm\sqrt{[1+\alpha(\xi^2+\eta^2)]^2-4\nu\lambda(\xi^2+\eta^2)}}{2\lambda},$$
$$\beta=\sqrt{4\nu\lambda\left(\xi^2+\eta^2\right)-\left[1+\alpha\left(\xi^2+\eta^2\right)\right]^2},$$
$$\mathcal{D}_1=\left\{(\xi,\eta);\ \xi,\eta>0;\ 0<\xi^2+\eta^2\leqslant a^2\right\}\cup\left\{(\xi,\eta);\ \xi,\eta>0;\ \xi^2+\eta^2\geqslant b^2\right\}\quad\text{and}$$
$$\mathcal{D}_2=\left\{(\xi,\eta);\ \xi,\eta>0;\ a^2<\xi^2+\eta^2<b^2\right\}$$

where

$$a=\frac{1}{\sqrt{\nu}(\sqrt{\lambda}+\sqrt{\lambda-\lambda_r})}\quad\text{and}\quad b=\frac{1}{\sqrt{\nu}(\sqrt{\lambda}-\sqrt{\lambda-\lambda_r})}.$$

Inverting the above results by means of Fourier's sine formula we find that

$$v(y,z,t)=V-\frac{4V}{\pi^2}\int_0^\infty\frac{\sin(y\xi)}{\xi}\int_0^\infty\frac{\sin(z\eta)}{\eta}\frac{r_1\exp(r_2t)-r_2\exp(r_1t)}{r_1-r_2}\,\mathrm{d}\eta\,\mathrm{d}\xi\quad\text{if}\quad\lambda_r>\lambda, \tag{16}$$

$$v(y,z,t)=V-\frac{4V}{\pi^2}\int_0^\infty\frac{\sin(y\xi)}{\xi}\int_0^\infty\frac{\sin(z\eta)}{\eta}\frac{\alpha(\xi^2+\eta^2)\exp(-t/\lambda)-\exp[-\nu(\xi^2+\eta^2)t]}{\alpha(\xi^2+\eta^2)-1}\,\mathrm{d}\eta\,\mathrm{d}\xi, \tag{17}$$

if $\lambda_r=\lambda$, and

$$v(y,z,t)=V-\frac{4V}{\pi^2}\iint_{\mathcal{D}_1}\frac{r_1\exp(r_2t)-r_2\exp(r_1t)}{r_1-r_2}\frac{\sin(y\xi)}{\xi}\frac{\sin(z\eta)}{\eta}\,\mathrm{d}\xi\,\mathrm{d}\eta \tag{18}$$
$$-\frac{4V}{\pi^2}\iint_{\mathcal{D}_2}\exp\left[-\frac{1+\alpha(\xi^2+\eta^2)}{2\lambda}t\right]\left[\cos\left(\frac{\beta t}{2\lambda}\right)+\frac{1+\alpha(\xi^2+\eta^2)}{\beta}\sin\left(\frac{\beta t}{2\lambda}\right)\right]\frac{\sin(y\xi)}{\xi}\frac{\sin(z\eta)}{\eta}\,\mathrm{d}\xi\,\mathrm{d}\eta$$

if $\lambda_r<\lambda$.

By making $t\to\infty$ in anyone of the above expressions, we obtain the steady state flow.

The tangential tensions $S_{xy}$ and $S_{xz}$, corresponding to these velocity fields, are solutions of the ordinary differential equations (4) with the initial conditions

$$S_{xy}(y,z,0)=S_{xz}(y,z,0)=0. \tag{19}$$

One of them, for instance, will be given by

$$S_{xy}(y,z,t)=\frac{\mu}{\lambda}\int_0^t\exp\left(\frac{\tau-t}{\lambda}\right)(1+\lambda_r\partial_\tau)\partial_y v(y,z,\tau)\,\mathrm{d}\tau, \tag{20}$$

from which, in view of (16), (17) and (18), it results

$$S_{xy}(y,z,t)=\frac{-4\mu V}{\pi^2}\int_0^\infty \cos(y\xi)\int_0^\infty \frac{\sin(z\eta)}{\eta}\frac{(1+\lambda_r r_1)(1+\lambda_r r_2)}{\lambda-\lambda_r}\frac{\exp(r_2t)-\exp(r_1t)}{r_2-r_1}\,\mathrm{d}\eta\,\mathrm{d}\xi, \quad (21)$$

$$S_{xy}(y,z,t)=\frac{4\mu V}{\pi^2}\int_0^\infty \cos(y\xi)\int_0^\infty \frac{\sin(z\eta)}{\eta}\frac{\exp[-\nu(\xi^2+\eta^2)t]-\exp(-t/\lambda)}{\alpha(\xi^2+\eta^2)-1}\,\mathrm{d}\eta\,\mathrm{d}\xi, \quad (22)$$

or

$$S_{xy}(y,z,t)=\frac{-4\mu V}{\pi^2}\iint_{\mathcal{D}_1}\frac{(1+\lambda_r r_1)(1+\lambda_r r_2)}{\lambda-\lambda_r}\frac{\exp(r_2t)-\exp(r_1t)}{r_2-r_1}\frac{\cos(y\xi)\sin(z\eta)}{\eta}\,\mathrm{d}\xi\,\mathrm{d}\eta$$
$$-\frac{8\mu V}{\pi^2}\iint_{\mathcal{D}_2}\frac{1}{\beta}\sin\left(\frac{\beta t}{2\lambda}\right)\exp\left(-\frac{1+\alpha(\xi^2+\eta^2)}{2\lambda}t\right)\frac{\cos(y\xi)\sin(z\eta)}{\eta}\,\mathrm{d}\xi\,\mathrm{d}\eta, \quad (23)$$

if $\lambda_r>\lambda$, $\lambda_r=\lambda$, respectively, $\lambda_r<\lambda$.

## 4. Limiting cases

1. Taking the limit of Eq. (16) as $\lambda\to 0$, we obtain

$$v(y,z,t)=V\left\{1-\frac{4}{\pi^2}\int_0^\infty\frac{\sin(y\xi)}{\xi}\int_0^\infty\frac{\sin(z\eta)}{\eta}\exp\left[-\frac{\nu(\xi^2+\eta^2)}{1+\alpha(\xi^2+\eta^2)}t\right]\mathrm{d}\eta\,\mathrm{d}\xi\right\}, \quad (24)$$

which coincides with the relation $(3\cdot 4)$ of [1] and gives the velocity field corresponding to the Rayleigh–Stokes problem for an edge in a second grade fluid.

2. In the special case when $\alpha$ and then $\lambda_r\to 0$, corresponding to a Maxwell fluid, our solution (18) reduces to that obtained in [2], i.e.,

$$v(y,z,t)=V-\frac{4V}{\pi^2}\iint_{\mathcal{D}_3}\frac{r_5\exp(r_6t)-r_6\exp(r_5t)}{r_5-r_6}\frac{\sin(y\xi)}{\xi}\frac{\sin(z\eta)}{\eta}\,\mathrm{d}\xi\,\mathrm{d}\eta$$
$$-\frac{4V}{\pi^2}\exp\left(-\frac{t}{2\lambda}\right)\iint_{\mathcal{D}_4}\left[\cos\left(\frac{\gamma t}{2\lambda}\right)+\frac{1}{\gamma}\sin\left(\frac{\gamma t}{2\lambda}\right)\right]\frac{\sin(y\xi)}{\xi}\frac{\sin(z\eta)}{\eta}\,\mathrm{d}\xi\,\mathrm{d}\eta, \quad (25)$$

where

$$r_5=\frac{-1+\sqrt{1-4\nu\lambda(\xi^2+\eta^2)}}{2\lambda},\quad r_6=\frac{-1-\sqrt{1-4\nu\lambda(\xi^2+\eta^2)}}{2\lambda},\quad \gamma=\sqrt{4\nu\lambda(\xi^2+\eta^2)-1},$$

$\mathcal{D}_3=\{(\xi,\eta);\ \xi,\eta>0;\ 0<\xi^2+\eta^2\leqslant 1/(2\sqrt{\nu\lambda})\}$ and

$\mathcal{D}_4=\{(\xi,\eta);\ \xi,\eta>0;\ \xi^2+\eta^2>1/(2\sqrt{\nu\lambda})\}$.

3. By letting now $\alpha\to 0$ in (24) or $\lambda\to 0$ in (17) or (25) and taking into account the entry 6 of Table 5 of [8] we attain to the solution of the Rayleigh–Stokes problem for a Navier–Stokes fluid [10]

$$v(y,z,t)=V\left[1-Erf\left(\frac{y}{2\sqrt{\nu t}}\right)Erf\left(\frac{z}{2\sqrt{\nu t}}\right)\right], \quad (26)$$

where $Erf(\cdot)$ is the error function of Gauss.

The tangential tensions corresponding to (24), (25) and (26) can be also obtained as limiting cases of (21), (22) and (23). Thus, making $\lambda$ and then $\alpha$ to tend at zero in (22), it results (cf. [8], Tables 4 and 5)

$$S_{xy}(y,z,t)=-\frac{\mu V}{\sqrt{\pi\nu t}}Erf\left(\frac{z}{2\sqrt{\nu t}}\right)\exp\left(-\frac{y^2}{4\nu t}\right), \quad (27)$$

which is the tangential tension corresponding to the velocity field (26). Finally, by letting $t\to\infty$ in anyone of the above expressions, we get the corresponding solutions for the steady state flow.

**Acknowledgement.** I am grateful to the referee for his assessment and constructive comments on the first version of this paper.

* Dedicated to the memory of Caius Iacob and his Professor Henri Villat.

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