C. R. Acad. Sci. Paris, Ser. I 335 (2002) 821–826

Probabilités/*Probability Theory*

Asymptotic behavior of a stochastic growth process associated with a system of interacting branching random walks

Alejandro F. Ramírez [a], Vladas Sidoravicius [b]

[a] Facultad de Matemáticas, Pontificia Universidad Católica de Chile, Casilla 306-Correo 22, Santiago, Chile

[b] IMPA, Estr. Dona Castorina 110, 22460-320 Rio de Janeiro, Brazil

Received 15 November 2001; accepted after revision 3 September 2002

Note presented by Jean-Christophe Yoccoz.

Abstract We study a continuous time growth process on $\mathbb{Z}^d$ ($d \geqslant 1$) associated to the following interacting particle system: initially there is only one simple symmetric continuous time random walk of total jump rate one located at the origin; then, whenever a random walk visits a site still unvisited by any other random walk, it creates a new independent random walk starting from that site. Let us call P_d the law of such a process and $S_d^0(t)$ the set of sites, visited by all walks by time t. We prove that there exists a bounded, non-empty, convex set $C_d \subset \mathbb{R}^d$, such that for every $\varepsilon > 0$, P_d-a.s. eventually in t, the set $S_d^0(t)$ is within an ε neighborhood of the set $[C_d t]$, where for $A \subset \mathbb{R}^d$ we define $[A] := A \cap \mathbb{Z}^d$. Moreover, for d large enough, the set C_d is not a ball under the Euclidean norm. We also show that the empirical density of particles within $S_d^0(t)$ converges weakly to a product Poisson measure of parameter one. ***To cite this article: A.F. Ramírez, V. Sidoravicius, C. R. Acad. Sci. Paris, Ser. I 335 (2002) 821–826.***

Comportement asymptotique d'un processus stochastique de croissance associé à un système de marches aléatoires en interaction

Résumé Nous étudions un modèle de croissance à temps continu sur $\mathbb{Z}^d$ ($d \geqslant 1$) associé au système de particules en interaction suivant : initiallement il y a seulement une marche aléatoire à temps continu simple, symétrique, à taux un et située à l'origine ; ensuite, aussitôt qu'une marche aléatoire visite un site jamais encore visité par aucune autre marche, une nouvelle marche est créée, partant de ce site et indépendante des autres. Nous notons P_d la loi d'un tel processus et $S_d^0(t)$ l'ensemble des sites déjà visités à l'instant t. Nous prouvons qu'il existe un ensemble $C_d \subset \mathbb{R}^d$ convexe, non-vide et borné tel que, pour tout $\varepsilon > 0$, P_d-p.s. et pour t assez grand, l'ensemble $S_d^0(t)$ soit inclus dans un ε-voisinage de $[C_d t]$, où l'on a défini, pour $A \subset \mathbb{R}^d$, $[A] := A \cap \mathbb{Z}^d$. En outre, pour d assez grand, l'ensemble C_d n'est pas une boule pour la norme euclidienne. Enfin, nous montrons que la densité empirique de particules à l'intérieur de $S_d^0(t)$ converge faiblement vers un produit de lois de Poisson de paramétre un. ***Pour citer cet article : A.F. Ramírez, V. Sidoravicius, C. R. Acad. Sci. Paris, Ser. I 335 (2002) 821–826.***

E-mail addresses: aramirez@mat.puc.cl (A.F. Ramírez); vladas@impa.br (V. Sidoravicius).

S1631-073X(02)02568-2/FLA

Version française abrégée

Nous étudions le processus stochastique de croissance suivant qui est associé à un système de marches aléatoires qui se branchent sur $\mathbb{Z}^d$: initiallement quand $t = 0$ le système a seulement une marche aléatoire à temps continu, simple, symétrique à taux un et situé à l'origine. Ensuite, chaque fois qu'une autre marche aléatoire du système visite un site jamais encore visité par aucune marche alétoire, une nouvelle marche est créée partant de ce site et indépendante des autres. Notre intérêt principal est le comportement asymptotique de l'ensemble $S_d^0(t) =$ [sites visités par le système au temps t], qui est un ensemble aléatoire connexe qui croit avec le temps.

Ce processus est lié à la réaction chimique $A + B \to A + A$ associée à un phenomène de combustion décrit par un système composé de deux types de particules – des particules actives A qui représentent de la chaleur qui se diffuse, et des particules passives B qui représentent des molécules inertes et combustibles. Le système démarre avec une particule de chaleur A dans l'origine et une particule passive B dans chaque autre site. Chaque fois qu'une particule active A touche une particule passive B, la particule passive est « brulée » et devient active.

D'autre part ce processus a des éléments similaires aux problèmes de la percolation du premier passage : à chaque site $x \in \mathbb{Z}^d$ on associe la collection de temps de passage $\{t_{x,y}\colon\ y \in \mathbb{Z}^d\}$, où $t_{x,y}$ répresente le première temps quand le site y est touché par une marche aléatoire à temps continu, simple, symétrique qui part du site x. Alors, pour chaque ensemble $x_i \in \mathbb{Z}^d$, $1 \leqslant i \leqslant n$, et son chemin associé $r = (x_1, \ldots, x_n)$ on peut définir le temps de passage par ce chemin comme $T(r) = \sum_{i=1}^{n-1} t_{x_i,x_{i+1}}$, et le temps de passage point à point entre le site 0 et le site x comme $T(0, x) := \inf\{T(r)\colon\ r$ un chemin de 0 à $x\}$. De cette manière l'ensemble de sites visités avant le temps t est representé comme $S_d^0(t) = \{y \in \mathbb{Z}^d\colon\ T(0, y) \leqslant t\}$. D'autre part, ce processus a des similarités avec des problèmes de marches aléatoires dans des potentiels aléatoires et peut être vu comme un modèle contraire à la Diffusion Limitée Agregée Interne (introduit par Diaconis et Fulton [3] et ensuite étudié dans [6,2,1,4]), où les particules meurent quand elles visitent un site qui n'a jamais été visité avant. Pour plus de details voir [9].

Les premièrs résultats s'adressent aux questions relatives à la vitesse de croissance et aux proprietés géometriques de $S_d^0(t)$. On définit par P_d la mesure de probabilité associée au processus dans $\mathbb{Z}^d$, et pour tout ensemble $A \subset \mathbb{R}^d$, on definit $[A] = A \cap \mathbb{Z}^d$.

THÉORÈME 1.1. – *Il existe un ensemble fermé convexe et bornée $C_d \subset R^d$, symétrique sous des permutations des coordonées et avec un interieur non-vide, tel que pour tout $\varepsilon > 0$, P_d-a.s. éventuellement en t on a*

$$\big[C_d t(1-\varepsilon)\big] \subset S_d^0(t) \subset \big[C_d t(1+\varepsilon)\big].$$

THÉORÈME 1.2. – *Pour d assez grande, C_d n'est pas une boule par rapport à la norme Euclidéene.*

La preuve du Théorème 1.1 utilise une méthode d'induction par rapport à la dimension d. D'autre part la preuve du Théorème 1.2 s'insipire de l'idée selon laquelle le réseau $\mathbb{Z}^d$ ressemble à un arbre quand la dimension d est grande.

Le résultat suivant décrit le comportement de la densité empirique des marches aléatoires dans l'ensemble $S_0^d(t)$. On définit la densité empirique de marches aléatoires sur le site $x \in \mathbb{Z}^d$ dans le temps t comme $\eta_x(t) := \sum_{n=0}^{\infty} \mathbf{1}_x(Z_N^0(t))$. On appelle $\mu(t)$ la loi de la densité empirique $\eta(t)$ sous P_d, et on définit $\mathcal{M} := \mathbb{N}^{\mathbb{Z}^d}$ avec la topologie produit, et on appelle $\mathcal{C}$ la σ-algebre de Borel de $\mathcal{M}$.

THÉORÈME 1.3. – *Soit ν un produit de lois de Poisson de paramètre un sur $(\mathcal{M}, \mathcal{C})$. On a,*

$$\lim_{t\to\infty} \mu(t) = \nu,$$

où la convergence est dans le sens de la topologie etroite.

To cite this article: A.F. Ramírez, V. Sidoravicius, C. R. Acad. Sci. Paris, Ser. I 335 (2002) 821–826

Finalement, nous aimerions faire rémarques que indépendemment, et par des techniques différents, un résultat analogue au Théorème 1.1 (existence et forme asymptotique) a été recemment obtenu dans [8], mais dans le contexte d'une dynamique à temps discret. Notre méthode est basée et utilise quelques idées liées à la dépendance de la croissance par rapport à la dimension.

2. Introduction

We study the following stochastic growth process associated with a system of interacting branching random walks on $\mathbb{Z}^d$: at time $t = 0$ the system consists of only one continuous time simple symmetric random walk of jump rate one, which begins to move from the origin. Then, as soon as any random walk of the system visits a site previously unvisited by any other random walk from this system it branches in two, i.e., it creates a new simple symmetric random walk, which becomes part of the system, and begins to move independently of all other random walks. Our main concern is the asymptotic behavior of the set $S_d^0(t) =$ [sites visited by the system by time t], which is a random connected set growing in time.

This process is linked with a chemical reaction $A + B \to A + A$ associated to a combustion phenomena described by a system composed of two types of particles – active particles A representing diffusing heat, and passive particles B representing inert combustible molecules. The system starts with one heat particle A at the origin and one passive particle at each other site, and whenever an active particle A reaches a passive particle B, the passive particle is "burned" becoming active.

From one side this process has the flavor of problems in first-passage percolation: to each site $x \in \mathbb{Z}^d$ we associate the collection of passage times $\{t_{x,y}\colon\ y \in \mathbb{Z}^d\}$, where $t_{x,y}$ represents the first hitting time of site y by a continuous time d-dimensional simple symmetric random walk starting from site x. Then, given $x_i \in \mathbb{Z}^d$, $1 \leqslant i \leqslant n$, and a path $r = (x_1, \dots, x_n)$ (note that $x_i,\ x_{i+1}$ do not ned to be nearest neighbors), we can define passage time along this path as $T(r) = \sum_{i=1}^{n-1} t_{x_i,x_{i+1}}$, and point to point passage time from site 0 to site x as $T(0,x) := \inf\{T(r)\colon\ r$ a path from 0 to $x\}$. In this way the set of visited sites at time t is represented as $S_d^0(t) = \{y \in \mathbb{Z}^d\colon\ T(0,y) \leqslant t\}$. On the other hand, this process presents some similarities to problems of random walks in random potentials and it can be viewed as an opposite of the Internal DLA, (introduced by Diaconis and Fulton [3] and later extensively studied in [6,2,1,4]), where particles are killed when visiting an unvisited site. More detailed comments see in [9].

The first results deal with the spead of growth and geometric properties of $S_d^0(t)$. Let P_d be the probability measure associated to the process on $\mathbb{Z}^d$, and for any subset $A \subset \mathbb{R}^d$, we define $[A] = A \cap \mathbb{Z}^d$.

THEOREM 2.1. – *There is a closed convex bounded subset $C_d \subset R^d$, symmetric under permutations of the coordinate axis and with non-empty interior, such that for every $\varepsilon > 0$, P_d-a.s. eventually in t one has that*

$$\bigl[C_d t(1-\varepsilon)\bigr] \subset S_d^0(t) \subset \bigl[C_d t(1+\varepsilon)\bigr].$$

THEOREM 2.2. – *For d large enough, C_d is not a ball under the Euclidean norm.*

The next theorem concerns the behavior of the empirical density of random walks within the set $S_d^0(t)$. We denote by $\eta_x(t) := \sum_{n=0}^{\infty} \mathbf{1}_x(Z_N^0(t))$, the total number of random walks at site $x \in \mathbb{Z}^d$ at time t and refer to the quantity $\eta(t) = \{\eta_x(t)\colon\ x \in \mathbb{Z}^d\}$ as the empirical density of random walks at time t. Then let $\mu(t)$ be the distribution of the empirical density $\eta(t)$ under P_d, and let us define $\mathcal{M} := \mathbb{N}^{\mathbb{Z}^d}$ with the product topology and endow it with its Borel σ-algebra $\mathcal{C}$.

THEOREM 2.3. – *Let ν be the product Poisson measure of parameter* 1 *on* $(\mathcal{M}, \mathcal{C})$. *We have that,*

$$\lim_{t\to\infty} \mu(t) = \nu,$$

where the convergence is in the sense of the weak topology on $\mathcal{M}$.

To conclude, we would like to remark that independently, and by different means, recently in [8] a result analogous to Theorem 1.1 (existence of the asymptotic shape) was obtained in the context of a discrete time dynamics. Our method is based on, and makes precise some ideas related to the dependence of growth on dimension.

3. Outline of the proofs

Outline of the proof of Theorem 1.1. – For any pair of sites $x, y \in \mathbb{Z}^d$, we define

$$T(x, y) \stackrel{\text{def}}{=} \inf_{t \geqslant 0} \{t\colon\ y \in S_d^x(t)\},$$

i.e., $T(x, y)$ is the first time, when the site $y \in \mathbb{Z}^d$ is visited by some particle of the process which starts at the site x. It is not hard to see that the family $\{T(x, y)\colon\ x, y \in \mathbb{Z}^d\}$ satisfies the following sub-additivity property: for any $x, y, z \in \mathbb{Z}^d$

$$T(x, y) \leqslant T(x, z) + T(z, y), \quad P_d\text{-a.s.}$$

The crucial step of the proof is the verification of the hypothesis $E_d(T(0, z)) < \infty$, where E_d denotes expectation with respect to P_d. Once $E_d(T(0, z)) < \infty$, we define

$$\mu_d(z) = \inf_{n \geqslant 1} \frac{E_d(T(0, nz))}{n} = \lim_{n \to \infty} \frac{E_d(T(0, nz))}{n}.$$

The quantity $\mu_d(z)$ is called time constant and represents the time needed for the set S_d^0 to reach the point z.

For fixed $z \in \mathbb{Z}^d$, $z \neq 0$, consider the family of random variables $\{T(nz, mz)\colon\ n, m \in \mathbb{N}\}$. Using stationarity and ergodic properties of this family, and applying the sub-additive ergodic theorem (see [5,7]), we can conclude that P_d-a.s.,

$$\lim_{n \to \infty} \frac{T(0, nz)}{n} = \mu_d(z),$$

from where it follows that the set $S_d^0(t)$ grows linearly in the direction defined by z. An appropriate pasting and continuity argument enables us to finish the proof of Theorem 1.1.

To prove that $E_d(T(0, z)) < \infty$ one has to control both—the number of active random walks and their distance to z, showing that at time t there are "many" random walks "close enough" to z. An important idea of the proof is the use of induction on dimension to obtain lower bounds on the number of random walks at some given time.

The first step of the induction is the following lemma:

LEMMA 3.1. – *Let $n \in \mathbb{N}$. Assume that there is a constant $c_1(n, d)$ such that for every $t > 0$,*

$$P_d\big(T(0, e_1) \geqslant t\big) \leqslant \frac{c_1}{t^{4(n+d+1)}}.$$

Then, for every r such that $r < 1/(dE_d[T(0, e_1)])$, and $t > 0$ it is true that

$$P_d\big(B(0, rt) \subset S_d^0(t)\big) \geqslant 1 - \frac{c_2}{t^n},$$

where $c_2(n, d) := dw_d \frac{c_1(n,d)}{(1/(dr) - E_d[T(0,e_1)])^{4(n+d)} r^n}$, and $B(x, r) := \{y \in \mathbb{Z}^d\colon\ |y - x| \leqslant r\}$ is the Euclidean ball centered at x of radius r.

Once we know that in dimension d the process grows linearly we obtain estimates in dimension $d + 1$ by "lifting" the process to the next higher dimension via a coupling between the d-dimensional and the

$d+1$-dimensional processes in such a way, that with probability one,

$$S_d^0\left(\frac{d}{d+1}t\right) \subset \pi_d S_{d+1}^0(t), \quad Q_d\text{-a.s.}$$

Thus, if a linear shape theorem is proved in dimension d with a good enough control on the slowdown deviations from this linear growth, we know modulo this slowdown deviation probability, that the number of random walks at time t in dimension $d+1$ must be at least of the order of t^d, which corresponds to the order of the volume of $S_d^0(t)$. A separate argument shows that the distance of these random walks to z at time t cannot be larger than t. Therefore, with a probability tending to one as $t\to\infty$, we know that in dimension $d+1$, at time $t^{1/4}$ we have at least $t^{d/4}$ random walks. Then using the fact that $t^{1/4}=\mathrm{o}(\sqrt{t})$, classical random walk hitting time probability estimates show that in dimension $d+1$, the probability that at time t site z has not been visited by any random walk is smaller than $(1-\mathrm{const}/t^{(d-1)/4})^{t^{d/4}} \approx \exp(-t^{1/4})$, which is integrable. We obtain the second step of the induction:

LEMMA 3.2. – *Let $n\in\mathbb{N}$. Assume that there are constants $r>0$ and $c_3(n,d)$ such that for every $t>0$ we have that*

$$P_d^0\big(B(0,rt)\subset S_d^0(t)\big) \geqslant 1-\frac{c_3}{t^n}.$$

Then for every $z\in\mathbb{Z}^{d+1}$ there is a constant $c_4(n,d+1)$ such that for every $t>0$,

$$P_{d+1}^0\big(T(0,z)\geqslant t\big) \leqslant \frac{c_4}{t^{n/4}}.$$

Finally, to complete the induction we prove one-dimensional estimates:

LEMMA 3.3. – *There is a constant $c_5>0$ such that*

$$P_1^0\big(T(x,x+1)>t\big) \leqslant 2\left(\frac{3}{t^{1/4}}\right)^{t^{1/4}}$$

whenever $t\geqslant c_5$.

Outline of the proof of Theorem 1.2. – To prove that the limiting shape is not an Euclidean ball in high enough dimensions, it is enough to show that to leading order in time, the growth velocity of the set $S_d(t)$ in the axial direction is larger than in the diagonal one. Let $z_1=(1,\dots,1)\in\mathbb{Z}^d$ be the lattice point having all coordinates with the value 1, and $e_1=(1,0,\dots,0)$ be the lattice point having the first coordinate with the values 1 and the others 0.

PROPOSITION 3.4. – (i) *There is a constant $C>0$ such that for d large enough,*

$$\mu_d(e_1)\leqslant Cd^{1/3}.$$

(ii) *For $d\geqslant 1$, we have that,*

$$\frac{1}{|z_1|}\mu_d(z_1)\geqslant d^{1/2}.$$

The idea behind the proof explores the fact that as dimension increases, the hyper-cubic lattice becomes richer in terms of connections and locally it has a structure similar to that of a tree. In case (ii) to upper bound the velocity in the diagonal direction, we essentially use the fact that asymptotically, the maximum number of sites that a rate one random walk can visit at time t is bounded by $t(1+\varepsilon)$, for $\varepsilon>0$. Then, the time it takes for the process to visit the site z_1 is at least of order d. Since the Euclidean distance of z_1 to the origin is $\sqrt{d}$, our process moves at most at a speed of $d^{-1/2}$ Euclidean units per unit time.

We denote H_1 the hyperplane orthogonal to the x_1-axis and containing e_1. To establish the lower bound in the axial direction it is enough to show that the probability that the event

$$\left[\text{the process starting from the origin does not hit the hyperplane } H_1 \text{ by time } d^{1/3-\varepsilon/2}\right]$$

is of order $(1 - d^{1/3-\varepsilon/2}/d)^{d^{2/3+\varepsilon}} \sim \exp\{-d^{\varepsilon/2}\}$, which goes to zero as $d \to \infty$. In order to obtain this, we prove the following claim: at time $d^{1/3}$ there are at least of the order of $d^{2/3+\varepsilon}$ moving random walks, for some $\varepsilon > 0$, at a unit distance from the hyperplane H_1. Since the probability for each of these random walks to hit this hyper-plane by time t is of order t/d when $t \ll d$, we get the above estimate. The proof of the claim consists basically of two steps. First, we show that with a high enough probability, any single random walk visits of the order of t sites by time t, provided the dimension d is large enough (t being fixed). Hence, we know that by some time d^β, with $\beta < \alpha$, there are of the order of d^β random walks. In the second step we show that if we wait an additional time of order d^α, each one of these random walks will in turn visit of the order of d^α new sites, producing a total of $d^{\beta+\alpha}$ random walks by a time of the order of d^α. A key feature of this proof will be the fact that the dimension d is much larger than the order d^α of the times involved. This will complete the proof.

Outline of the proof of Theorem 1.3. – Theorem 1.3 is a corollary of Theorem 1.1. Once it is shown that the cluster of visited sites is growing linearly in time, the problem can be reduced to the study of the empirical density of particles of a set of independent simple random walks on the periodic torus T_N^d, with $N = t^{2/3}$, and having as initial condition one random walk per site. The reduction is achieved by first observing that random walks at a distance larger than $\mathrm{O}(\sqrt{t})$ (as it is the case for $t^{2/3}$), do not affect what happens within some finite set Λ at time t. Thus, the behavior of $\mu_\Lambda(t)$ can be approximated to the behavior of the corresponding marginal of a version of the process defined on the torus $T_{t^{2/3}}^d$. Next, one observes that particles within some small enough central region of $S_d^0(t)$ are born at times which are small enough compared to t so that it is irrelevant if one approximates their birth time to 0. This is a consequence of the fact that the linear growth of $S_d^0(t)$ is much larger than the typical distance $\sqrt{t}$ traveled by a random walk at time t. Once this is proved, the convergence follows via standard methods of approximation to Poisson product measures. At this point we rely on Laplace transform techniques.

Acknowledgements. The authors would like to acknowledge the generous support of Fundación Andes through the *Programa de colaboración científico-académica entre Argentina, Brasil y Chile*. A.F. is supported by the grant 1020686 of Fondecyt, and V.S. is supported by FAPERJ grant E-26/151.905/2000, and CNPq.

References

[1] G. Ben Arous, J. Quastel, A.F. Ramírez, Internal DLA in a random environment, Ann. IHP, to appear.
[2] G. Ben Arous, A.F. Ramírez, Asymptotic survival probabilities in the random saturation process, Ann. Probab. 28 (4) (2000).
[3] P. Diaconis, W. Fulton, A growth model, a game, an algebra, Lagrange inversion, and characteristic classes, in: Commutative Algebra and Algebraic Geometry, II, Turin, 1990, Rend. Sem. Mat. Univ. Politec. Torino 49 (1) (1991) 95–119.
[4] J. Gravner, J. Quastel, Internal DLA and the Stefan problem, Ann. Probab. 28 (4) (2000) 1528–1563.
[5] H. Kesten, Aspects of first passage percolation, École d'été de probabilités de Saint Flour XIV-1984, in: Lecture Notes in Math., Vol. 1180, 1986.
[6] G. Lawler, M. Bramson, D. Griffeath, Internal diffusion limited aggregation, Ann. Probab. 20 (1992) 2117–2140.
[7] T. Liggett, Interacting Particle Systems, Springer, New York, 1985.
[8] O. Alves, F. Machado, S. Popov, The shape theorem for the frog model, Ann. Appl. Probab., to appear.
[9] A.F. Ramirez, V. Sidoravicius, Asymptotic behaviour of a growth process of boundary branching random walks, Preprint.