C. R. Acad. Sci. Paris, Ser. I 335 (2002) 797–800

Équations aux dérivées partielles/*Partial Differential Equations*

Exact periodic traveling water waves with vorticity

Adrian Constantin [a], **Walter Strauss** [b]

[a] Department of Mathematics, Lund University, PO Box 118, 22100 Lund, Sweden

[b] Department of Mathematics and Lefschetz Center for Dynamical Systems, Brown University, Providence, RI 02912, USA

Received 18 June 2002; accepted 13 September 2002

Note presented by Haïm Brèzis.

Abstract For the classical inviscid water wave problem under the influence of gravity, described by the Euler equation with a free surface over a flat bottom, we construct periodic traveling waves with vorticity. They are symmetric waves whose profiles are monotone between each crest and trough. We use global bifurcation theory to construct a connected set of such solutions. This set contains flat waves as well as waves that approach flows with stagnation points. ***To cite this article: A. Constantin, W. Strauss, C. R. Acad. Sci. Paris, Ser. I 335 (2002) 797–800.***

Ondes d'eau avec tourbillons

Résumé Pour le problème classique des ondes d'eau sans viscosité sous l'influence de la gravité, décrit par l'équation d'Euler avec une surface libre à fond plat, nous construisons des houles aux tourbillons. Ce sont des ondes symmétriques dont les profils sont monotones entre chaque sommet et creux. Nous employons la théorie de la bifurcation globale pour construire un ensemble connexe de telles solutions. Cet ensemble contient des ondes au profil non-oscillatoire et aussi des ondes qui approchent des flots avec des points de stagnation. ***Pour citer cet article : A. Constantin, W. Strauss, C. R. Acad. Sci. Paris, Ser. I 335 (2002) 797–800.***

We consider the propagation of two-dimensional inviscid gravity waves at the surface of a layer of water with a flat bottom. In its undisturbed state the equation of the flat surface is $y = 0$ and the flat bottom is given by $y = -d$ for some $d > 0$. In the presence of waves, let $y = \eta(t, x)$ be the free surface and let $(u(t, x, y), v(t, x, y))$ be the velocity field. If $P(t, x, y)$ denotes the pressure, P_0 the constant atmospheric pressure, and g the gravitational constant of acceleration, the governing equations [10] are

$$u_x + v_y = 0, \qquad u_t + uu_x + vu_y = -P_x, \qquad v_t + uv_x + vv_y = -P_y - g. \tag{1}$$

The boundary conditions for the water wave problem are

$$P = P_0 \quad \text{on } y = \eta(t, x), \qquad v = \eta_t + u\eta_x \quad \text{on } y = \eta(t, x), \qquad v = 0 \quad \text{on } y = -d. \tag{2}$$

Given $c > 0$, we are looking for periodic waves traveling at speed c. The profile η oscillates around the flat surface $y = 0$ and the horizontal fluid velocity u is less than c at every point. For convenience we shall

E-mail addresses: adrian.constantin@math.lu.se (A. Constantin); wstrauss@math.brown.edu (W. Strauss).

S1631-073X(02)02565-7/FLA

take the length scale to be 2π. Define the (relative) *stream function* $\psi(x,y)$ by $\psi_x = -v$, $\psi_y = u - c$, with $\psi = 0$ on the free surface, and let $\omega = v_x - u_y$ be the vorticity. Then $\Delta\psi = -\omega$. At least locally, away from a stagnation point (a point where $u = c$, $v = 0$), ω is a function of ψ. We will assume that there is a function γ, called the *vorticity function*, such that $\omega = \gamma(\psi)$ throughout the fluid. Thus $\Delta\psi = -\gamma(\psi)$. We define the *relative mass flux* as $p_0 = \int_{-d}^{\eta(x)} [u(x,y) - c]\,dy$, which is independent of x by (2). Since $u < c$, $p_0 < 0$. Let $\Gamma(p) = \int_0^p \gamma(-s)\,ds$ have minimum value $\Gamma_{\min}$ for $p_0 \leqslant p \leqslant 0$. Let $\overline{D_\eta}$ be the closure of the open fluid domain $D_\eta = \{(x,y) \in \mathbb{R}^2\colon\ x \in \mathbb{R},\ -d < y < \eta(x)\}$. Given a set E with a smooth boundary, define for $m \in \mathbb{N}$ and $\alpha \in (0,1)$ the space $C^{m+\alpha}_{\mathrm{per}}(E)$ of functions $f : E \to \mathbb{R}$ with Hölder continuous derivatives (of index α) up to order m and of period 2π in the x-variable.

Our main results are as follows.

THEOREM 1. – *Let the wave speed $c > 0$, the relative mass flux $p_0 < 0$, and an arbitrary $\alpha \in (0,1)$ be given. Let $\gamma(s)$ be a $C^{1+\alpha}$-function defined on $[0, |p_0|]$ such that*

$$\int_{p_0}^{0} \Big[\big(2\Gamma(p) - 2\Gamma_{\min}\big)^{3/2} + (p - p_0)^2 \big(2\Gamma(p) - 2\Gamma_{\min}\big)^{1/2}\Big]\,dp < g p_0^2, \tag{3}$$

and

$$\gamma(s) \geqslant 0, \quad \gamma'(s) \leqslant 0 \quad \text{in } \big[0, |p_0|\big]. \tag{4}$$

Consider traveling solutions of speed c of the water wave problem (1)–(2) *with vorticity function γ such that $u < c$ throughout the fluid. There exists a connected set $\mathcal{C}$ of solutions (u, v, η) in the space $C^{2+\alpha}_{\mathrm{per}}(\overline{D_\eta}) \times C^{2+\alpha}_{\mathrm{per}}(\overline{D_\eta}) \times C^{3+\alpha}_{\mathrm{per}}(\mathbb{R})$ with the following properties.*

(A) $\mathcal{C}$ *contains a trivial flow* (*with a flat surface* $\eta = 0$);

(B) *there is a sequence of solutions* $(u_n, v_n, \eta_n) \in \mathcal{C}$ *for which* $\max_{\overline{D_{\eta_n}}} u_n \uparrow c$.

Furthermore, each solution $(u, v, \eta) \in \mathcal{C}$ *satisfies*

(i) u, v, η *have period* 2π *in* x;

(ii) *within each period the wave profile* η *has a single maximum* (*crest*) *at* $x = a$, *say, and a single minimum* (*trough*);

(iii) u *and* η *are symmetric while* v *is antisymmetric around the line* $x = a$;

(iv) $\eta'(x) < 0$ *on* $(a, a + \pi)$, *i.e., the profile of the wave is strictly decreasing from crest to trough.*

THEOREM 2. – *We make the same assumptions as in Theorem* 1, *except we do not assume* (4). *Then there exists a connected set $\mathcal{C}$ with the same properties as in Theorem* 1, *except that* (B) *is replaced by*:

(B*) *there is a sequence* $(u_n, v_n, \eta_n) \in \mathcal{C}$ *such that either* $\max_{\overline{D_{\eta_n}}} u_n \uparrow c$ *or* $\min_{\overline{D_{\eta_n}}} u_n \downarrow -\infty$.

Now we discuss these results in relation to the existing literature. In 1847 Stokes [20] studied irrotational periodic traveling water waves and some of their nonlinear approximations. The first rigorous constructions by power series of such waves did not occur until the 1920s in the work of Nekrasov [17], Levi-Civita [14], and Struik [21]. These constructions were local in the sense that the wave profiles were almost flat. Constructions of large amplitude irrotational waves were begun by Krasovskii [13] in 1961. These results were refined by Keady and Norbury [11] in 1978, using the methods of global bifurcation theory. Shortly thereafter it was shown by Toland [22] and McLeod [16] that in the closure of the continuum of solutions found in [11] there exist waves with stagnation points at their crests.

Much less is known about waves with vorticity. In 1802 Gerstner [7,3] constructed an explicit example of a periodic traveling wave in water of infinite depth with nonzero vorticity. In 1934 Dubreil-Jacotin [6] considered the problem of the existence of steady periodic water waves with small vorticity. By using power series she constructed solutions that are close to a flat surface. The existence of such waves which are regular (two-dimensional periodic surface waves traveling at constant speed, having one crest per period and a profile decreasing from crest to trough) is however still a matter of some controversy [2]. Nevertheless in the present paper we construct a global continuum of such regular solutions with general vorticity. After the paper of Dubreil-Jacotin [6], improvements of her method for showing the existence of small-amplitude rotational waves were obtained by Goyon [8] and Zeidler [24]. On the other hand, we construct waves of large amplitude. Our solution set in Theorem 1 contains waves with flow speeds u arbitrarily close to the speed of the wave profile c. We do not know whether there is a limiting wave having a stagnation point. However, in the particular case of irrotational flow ($\gamma \equiv 0$) the existence of such a limiting wave with stagnation and angle $2\pi/3$ at its crest, was already conjectured by Stokes [20] in 1847 and proved by Amick, Fraenkel and Toland [1] in 1982. See [23] for a nice review of the irrotational case. Note that in the irrotational case both (3) and (4) are trivially satisfied and therefore we recover the global results of [11]. Let us also remark that for an arbitrary vorticity function γ there are waves of our type for all sufficiently large periods L with a mass flow p_0 sufficiently close to zero. This follows immediately from relation (3) modified to change 2π to the length scale L.

Sketch of the proof. – We first transform the water wave problem into a problem in a fixed domain. It should be noted that because of the vorticity there is no way to convert the problem to an integral equation on the boundary as in [23]. Since $\psi(x, y)$ is a constant on the free surface and on the bottom, it is natural to introduce new independent variables $q = x,\ p = -\psi(x, y)$. The new domain will be the rectangle $R = [0, 2\pi] \times [p_0, 0]$. We can also replace the dependent variables by a single function $h(q, p)$, periodic in q, according to $h_q = \frac{v}{u-c}$, $h_p = \frac{1}{c-u}$. So long as $u < c$, the height $h(q, p) = y + d$ satisfies the quasilinear elliptic equation $(1 + h_q^2)h_{pp} - 2h_p h_q h_{pq} + h_p^2 h_{qq} = -h_p^3\gamma(-p)$ in R with the boundary conditions

$$1 + h_q^2 + (2gh - Q)h_p^2 = 0 \quad \text{on } p = 0, \qquad \text{and} \qquad h = 0 \quad \text{on } p = p_0,$$

and h is even and periodic in q, where $Q - 2gd$ is twice the difference between the hydraulic head at the surface and the atmospheric pressure P_0.

The trivial solutions $H(p) = \int_0^p \frac{ds}{\sqrt{\lambda + 2\Gamma(s)}} + \frac{Q-\lambda}{2g}$, which represent parallel shear flows with a flat surface $\eta \equiv 0$, depend on the free parameter λ, which is the square of the relative speed at the surface. The linearized problem around H is

$$\begin{cases} [\lambda + 2\Gamma(p)]w_{pp} + w_{qq} + 3\gamma(-p)w_p = 0 & \text{in } R, \\ gw = \lambda^{3/2} w_p & \text{at } p = 0, \\ w = 0 & \text{at } p = p_0, \end{cases}$$

where $w(q, p)$ is even and periodic in q. The condition (3) is sufficient to ensure that bifurcation from a simple eigenvalue occurs at some value λ_0. Thus, by the method of Crandall and Rabinowitz [5], there is a smooth local curve of nontrivial solutions (λ, h) in $\mathbb{R} \times C^{3+\alpha}_{\text{per}}(R)$.

We use the global bifurcation theory of Rabinowitz [18], based on Leray–Schauder degree, to extend this curve to a global continuum $\mathcal{C}$. Because of the awkward position of λ in the linearized problem, as well as the fully nonlinear character of the boundary condition, we employ the recent version of global continuation theory of Healey and Simpson [9], who studied general problems in nonlinear elastostatics. Some aspects of their method depend on ideas of Kielhöfer [12]. We work in a subset O_δ of the space $\mathbb{R} \times X$ where $X = \{w \in C^{3+\alpha}_{\text{per}}(R)/w = 0 \text{ at } p = p_0,\ w \text{ is even in } q,\ w_q < 0 \text{ in } (0, \pi) \times (p_0, 0)\}$ and $w = h - H$. For any $\delta > 0$, the subset O_δ is defined by the conditions $H_p + w_p > 0$, $\lambda > 2\Gamma_{\min}$. We define the mapping $G : O_\delta \mapsto C^{1+\alpha}_{\text{per}}(R) \times C^{2+\alpha}_{\text{per}}(\mathbb{R})$ by $G = (G_1, G_2)$ where the quasilinear partial differential equation is $G_1(w) = 0$ and the nonlinear boundary condition is $G_2(w) = 0$. For such mappings Healey and Simpson developed a variant of Leray–Schauder degree. This degree is exactly what we need for the water wave problem. We show that there is a continuum of solutions which is (a) either unbounded in $C^{3+\alpha}_{\text{per}}(R)$ or (b) contains a different trivial solution (λ^*, H^*) or (c) contains in its closure a solution where h_p vanishes.

The next step is to exclude the alternative (b) that there are extraneous trivial solutions in $\mathcal{C}$. Here we strongly use the symmetry and monotonicity properties of our solutions, i.e., their nodal pattern. We prove that no other bifurcation point λ^* can have this nodal pattern and that the pattern persists all along $\mathcal{C}$. To prove the latter, we use Serrin's sharp form of the maximum principle [19] in conjunction with our nonlinear boundary condition.

We reduce the alternative (a) that $\mathcal{C}$ is unbounded in $C^{3+\alpha}_{\text{per}}(R)$ to the condition that h_p is unbounded in $L^\infty(R)$. First we use the maximum principle to prove that h and h_q are *a priori* bounded functions. If h_p were also bounded in $L^\infty(R)$, then we prove that h is successively bounded in $C^{1+\alpha}_{\text{per}}(R)$, in $C^2_{\text{per}}(R)$, in $C^{2+\alpha}_{\text{per}}(R)$, and finally in $C^{3+\alpha}_{\text{per}}(R)$. Here we use the Schauder estimates and several basic *a priori* estimates of Lieberman and Trudinger [15] for nonlinear elliptic equations with nonlinear oblique boundary conditions.

Thus the alternatives are reduced to (a*) either h_p is unbounded in $L^\infty(R)$, or (c) $\mathcal{C}$ contains in its closure a solution where h_p vanishes. Then we return to the original problem in the form of the Euler equation. Under assumption (4) we eliminate the last possibility (c). However, (a*) means that $\max u_n \uparrow c$, while (c) means that $\min u_n \downarrow -\infty$ for some sequence of solutions.

We refer the reader to [4] for more details and for the complete proofs.

Acknowledgement. We are grateful to J.T. Beale for many enlightening conversations.

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