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Équations aux dérivées partielles/*Partial Differential Equations*

Uniqueness of solutions of some elliptic equations without condition at infinity

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Abstract We prove existence and uniqueness of solutions of the equation:

$$-u'' + u + u|u'|^2 = f \quad \text{in } \mathbf{R},\ f \geqslant 0,$$

without any condition at infinity on f. The result is generalized to other absorbing equations of the same type and to the radial case in $\mathbf{R}^N$. ***To cite this article: A. Porretta, C. R. Acad. Sci. Paris, Ser. I 335 (2002) 739–744.***

Unicité de la solution de certaines équations elliptiques sans conditions à l'infini

Résumé On montre l'existence et l'unicité de la solution de l'équation :

$$-u'' + u + u|u'|^2 = f \quad \text{dans } \mathbf{R},\ f \geqslant 0,$$

sans conditions à l'infini sur f. Le résultat admet des généralisations à d'autres equations avec termes d'absorption du même type et au cas radial dans $\mathbf{R}^N$. ***Pour citer cet article : A. Porretta, C. R. Acad. Sci. Paris, Ser. I 335 (2002) 739–744.***

Version française abrégée

Nous étudions ici l'unicité de la solution de l'équation (3), qui est un cas modèle d'une classe plus générale de problèmes elliptiques avec termes d'absorption dependant du gradient, recemment etudiés dans [4]. La question de l'unicité pour (3), sans supposer conditions à l'infini sur f, était posée par H. Brezis dans le but de généraliser le résultat de [1] concernant le problème semilinéaire :

$$-\Delta u + |u|^{p-1}u = f \quad \text{dans } \mathbf{R}^N, \text{ avec } p > 1. \tag{1}$$

Notre résultat principal ici est le suivant :

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THÉORÈME 0.1. – *Soit f une fonction localement bornée et $f \geqslant 0$. Alors il existe une solution unique $u \in C^{1,1}_{\mathrm{loc}}(\mathbf{R})$ de l'équation* (3).

Pour demontrer ce théorème on construit d'abord une solution maximale $\overline{u}$ comme limite des solutions u_n de l'equation sur l'intervalle $(-n, n)$ tels que $\lim_{x\to\pm n} u_n = +\infty$. La suite $\{u_n\}$ est une suite ponctuellement décroissante qui satisfait des estimations a priori convenables pour que u_n, localement, converge vers une solution de (3). Dans ce raisonnement on utilise de manière essentielle le principe de comparaison valable pour le même problème dans un domaine borné. De façon analogue on construit une solution $\underline{u}$ qui est minimale parmi les solutions $u \geqslant 0$. Après, on raisonne par l'absurde en supposant que $\overline{u} > \underline{u}$, ce qui permet de construire une troisième solution v telle que $v \geqslant \gamma > 0$ où γ est une constante positive. En étudiant alors l'équation pour $\overline{u} - v$ on arrive a deduire que $\overline{u}$ pourrait ne pas exister sur $\mathbf{R}$ tout entier, ce qui conduit à une contradiction. On déduit que $\overline{u} = \underline{u}$ et donc l'unicité des solutions $u \geqslant 0$. La démonstration se termine par le fait que $f \geqslant 0$ implique $u \geqslant 0$ pour toute solution de (3).

La même méthode de démonstration permet d'obtenir le résultat plus général suivant :

THÉORÈME 0.2. – *Supposons $h(s)$ et $g(s)$ fonctions continues satisfaisant les hypothèses* (12) *et* (13). *Soit f une fonction localement bornée et supposons qu'il existe $m > 0$ tel que $f(x) \geqslant -m$ pour tout $x \in \mathbf{R}$. Alors il existe une solution unique $u \in C^{1,1}_{\mathrm{loc}}(\mathbf{R})$ de l'équation* (11).

Remarque 1. – Avec de petites modifications, on peut aussi adapter la démonstration du Théorème 0.1 pour traiter le cas d'une donnée f mesure de Radon non negative sur $\mathbf{R}$. Dans ce cas, on obtient l'unicité des solutions $u \in H^1_{\mathrm{loc}}(\mathbf{R})$ qui satisfont (3) au sens faible.

Finalment, on peut utiliser la méthode du Théorème 0.1 pour traiter le cas radial sur $\mathbf{R}^N$, $N > 1$. En fait, si f est une fonction radiale et $f \geqslant 0$, on démontre l'unicité dans la classe de toutes les solutions possibles. Grâce au fait que les solutions maximales et minimales sont radiales, l'argument précédent reste encore valable, sauf que la démonstration nécessite de quelques estimations supplémentaires à cause des termes singuliers qui apparaissent dans les derivées radiales. On arrive au résultat suivant concernant le problème modèle :

$$-\Delta u + |u|^{\alpha-1}u + g(u)|\nabla u|^2 = f \quad \text{dans } \mathbf{R}^N, \tag{2}$$

pour $\alpha > 0$.

THÉORÈME 0.3. – *Soit $\alpha > 0$ et soit g verifiant* (13). *Soit f une fonction radiale, localement bornée, $f \geqslant 0$. Alors il existe une solution unique $u \in W^{2,p}_{\mathrm{loc}}$, pour tout $p < +\infty$, de l'équation* (2) *et u est radiale.*

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1. Introduction, statements and proof of results

We study here the problem of uniqueness of solutions of the equation:

$$-u'' + u + u|u'|^2 = f \quad \text{in } \mathbf{R}. \tag{3}$$

Eq. (3) is a model case for a larger class of elliptic problems with gradient dependent absorption terms, recently studied in [4], for which it is possible to prove local estimates of same type as in [3,2] for semilinear equations. The question concerning uniqueness of solutions in the whole space without any condition at infinity was addressed by H. Brezis in order to possibly extend the result proved in [1] about the equation:

$$-\Delta u + |u|^{p-1}u = f \quad \text{in } \mathbf{R}^N, \text{ with } p > 1. \tag{4}$$

We give here some partial answers to this question. Our first result is the following:

THEOREM 1.1. – *Let f be a nonnegative locally bounded function. Then there exists a unique solution $u \in C^{1,1}_{\mathrm{loc}}(\mathbf{R})$ of* (3).

Proof. – As a preliminary fact, we observe that a comparison principle holds for bounded domains, that is if u_1 and u_2 are solutions of the equation in (a,b) and if $(u_1 - u_2) \leqslant 0$ on $\partial(a,b)$, then $(u_1 - u_2) \leqslant 0$ in (a,b) from the maximum principle.

We start by proving that, for any $n \in \mathbf{N}$, there exists a solution $\overline{u}_n$ of the problem:

$$\begin{cases} -\overline{u}_n'' + \overline{u}_n + \overline{u}_n|\overline{u}_n'|^2 = f & \text{in } (-n,n), \\ \lim\limits_{x\to -n} \overline{u}_n(x) = \lim\limits_{x\to n} \overline{u}_n(x) = +\infty. \end{cases} \tag{5}$$

The existence of $\overline{u}_n$ is proved by monotone approximations with solutions taking constant increasing values at $x = \pm n$. The fundamental estimate is given by the existence of a supersolution y_n which solves:

$$\begin{cases} -y_n'' + y_n + y_n|y_n'|^2 = M & \text{in } (-n,n), \\ y_n'(0) = 0, \quad y_n(0) = \lambda, \\ y_n(x) = y_n(-x), \quad \lim\limits_{x\to \pm n} y_n(x) = +\infty, \end{cases} \tag{6}$$

with $M = \sup_{[-n,n]} f$. Indeed, in order to show the existence of y_n, it is enough to observe that y_n locally exists near $x = 0$ and is implicitly given by:

$$\int_\lambda^{y_n} \frac{\mathrm{d}s}{(2\int_\lambda^s (\xi - M)\,\mathrm{e}^{(s^2-\xi^2)}\,\mathrm{d}\xi)^{1/2}} = |x|.$$

Standard computations on the left side allow one to deduce that there is exactly one value of λ for which $x = n$ corresponds to $y = +\infty$, and this is how one gets a solution of (6), which in turn implies the existence of $\overline{u}_n$ solution of (5). Again from comparison principle, we have that the sequence $\overline{u}_n$ is decreasing, that is $\overline{u}_n(x) \geqslant \overline{u}_{n+1}(x)$ for any $x \in (-n,n)$. Then $\overline{u}_n$ pointwise converges towards a function $\overline{u}$; by local estimates (proved as before and using Sobolev embeddings) we also have that the convergence takes place in $C^{1,1}_{\mathrm{loc}}(\mathbf{R})$ and then $\overline{u}$ is a solution of (3).

The existence of a solution is then established. In fact, observe that $\overline{u}$ is the maximal solution of (3), since if u is any other solution, then using again the comparison principle, we have $u \leqslant \overline{u}_n$ in $(-n,n)$ and thus $u \leqslant \overline{u}$ everywhere. Similarly, by taking limits of Dirichlet problems in $(-n,n)$ (i.e., with zero boundary conditions) one constructs a solution $\underline{u}$ which is the minimal solution in the class of nonnegative solutions. Now our goal is to prove that $\overline{u} = \underline{u}$. Reasoning by contradiction, assume that $\underline{u} < \overline{u}$. We investigate the equation of the difference of any two solutions of (3), say u_1 and u_2. Subtracting the two equations and setting $w = u_1 - u_2$ we get:

$$-w'' + w\big(1 + |u_1'|^2\big) + u_2|w'|^2 + 2u_2u_2'w' = 0, \tag{7}$$

which also reads as

$$\big(\exp\big(-u_2^2\big)w'\big)' = \exp\big(-u_2^2\big)\big(1 + |u_1'|^2\big)w + u_2\exp\big(-u_2^2\big)|w'|^2. \tag{8}$$

From (8) we deduce that for any two solutions u_1, u_2, the function $\exp(-u_2^2)w'$ is an increasing function. Applying this with $u_1 = \overline{u}$ and $u_2 = \underline{u}$ we get that $\exp(-\underline{u}^2)(\overline{u} - \underline{u})'$ is increasing. Let us assume now that

$$\exists x_0 \in \mathbf{R}: \quad \big(\exp\big(-\underline{u}^2\big)(\overline{u} - \underline{u})'\big)(x_0) > 0. \tag{9}$$

If (9) is true, since $\exp(-\underline{u}^2)(\overline{u}-\underline{u})'$ is increasing and since $\exp(-\underline{u}^2)\leqslant 1$, we deduce that there exists a positive constant c_0 such that $(\overline{u}-\underline{u})'(x)>c_0$ for every $x\geqslant x_0$, which in particular yields $\lim_{x\to+\infty}(\overline{u}-\underline{u})(x)=+\infty$. Actually, there was no loss of generality assuming (9) since, if such a point x_0 did not exist, then $\exp(-\underline{u}^2)(\overline{u}-\underline{u})'$ would be always negative and we could work similarly in an interval $(-\infty,x_1)$ to get that $\lim_{x\to-\infty}(\overline{u}-\underline{u})(x)=+\infty$. Thus hereafter we assume (9) and we will work in the interval $(x_0,+\infty)$. Consider now the solution of the ordinary differential equation:

$$\begin{cases} v''=v+v|v'|^2-f, & x\geqslant x_0,\\ v(x_0)=(\overline{u}(x_0)+\underline{u}(x_0))/2, \\ v'(x_0)=(\overline{u}'(x_0)+\underline{u}'(x_0))/2, \end{cases} \tag{10}$$

which exists up to infinity, since by comparison $\underline{u}\leqslant v\leqslant\overline{u}$. Working on Eq. (8) for the difference of two positive solutions, using that $\underline{u}(x_0)<v(x_0)<\overline{u}(x_0)$ and $\underline{u}'(x_0)<v'(x_0)<\overline{u}'(x_0)$ we deduce that both $\overline{u}-v$ and $v-\underline{u}$ are positive increasing functions. In particular, there exists a constant $\gamma>0$ such that $v(x)\geqslant\gamma$ for every $x\geqslant x_0$. Taking Eq. (8) with $u_1=\overline{u}$ and $u_2=v$, we get (with $w=\overline{u}-v$):

$$\bigl(\exp\bigl(-v^2\bigr)w'\bigr)'\geqslant\exp\bigl(-v^2\bigr)\bigl(1+|\overline{u}'|^2\bigr)w+\gamma\exp\bigl(-v^2\bigr)|w'|^2.$$

Since $w>0$ and $(\exp(-v^2))^2\leqslant\exp(-v^2)$, we deduce that the function $z=\exp(-v^2)w'$ satisfies:

$$z'\geqslant\gamma z^2\quad\text{for } x>x_0,\qquad z(x_0)>0,$$

and by construction, z is defined in the whole interval $(x_0,+\infty)$, since such are $\overline{u}$ and v. This is a contradiction, which proves that $\underline{u}=\overline{u}$.

We have proved so far that there is exactly one nonnegative solution. We are only left to show that, if $f\geqslant 0$, then $u\geqslant 0$ for any solution u of (3). To this goal, define $\varphi(s)=\int_0^s\max(\min(t,0),-1)\,\mathrm{d}t$; then the function $\varphi(u)$ satisfies:

$$\bigl(\varphi(u)\bigr)''=u\varphi'(u)+u\varphi'(u)|u'|^2+\varphi''(u)|u'|^2-\varphi'(u)f.$$

Since $\varphi'(s)\leqslant 0$, $\varphi'(s)s\geqslant 0$, $\varphi''(s)\geqslant 0$, and since $f\geqslant 0$, we get that $\varphi(u)$ is a convex function. Without loss of generality, assume that $\varphi(u)'(x)\geqslant\gamma_0>0$ for $x\geqslant x_0$; we deduce that $\varphi(u)\to+\infty$ as $x\to+\infty$, hence for large x we have $u(x)\varphi'(u)(x)\geqslant c_1|\varphi'(u)|^2$ and then $(\varphi(u))''\geqslant c_1|(\varphi(u))'|^2$. We conclude that $\varphi(u)\equiv 0$, hence $u\geqslant 0$. □

Remark 1. – The same proof of Theorem 1.1 works for more general equations of the type:

$$-u''+h(u)+g(u)|u'|^2=f\quad\text{in } \mathbf{R}. \tag{11}$$

Moreover, it is enough to assume that there exists a constant $m>0$ such that $f\geqslant -m$, so that we have the following more general result:

THEOREM 1.2. – *Assume that $h(s)$ and $g(s)$ are continuous functions satisfying*:

$$h \text{ is increasing},\quad h(0)=0,\quad \lim_{s\to\pm\infty}h(s)=\pm\infty, \tag{12}$$

$$g \text{ is nondecreasing},\quad g(s)s>0\quad \text{for every } s\neq 0. \tag{13}$$

Assume that f is a locally bounded function and there exists a constant $m>0$ such that $f(x)\geqslant -m$ for any $x\in\mathbf{R}$. Then there exists a unique solution $u\in C^{1,1}_{\mathrm{loc}}(\mathbf{R})$ of Eq. (11).

Observe that the most general result, with no assumptions on f, allowing for instance f with $\liminf_{x\to+\infty}f=-\infty$ and $\limsup_{x\to+\infty}f=+\infty$, is technically harder to handle and remains open.

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Remark 2. – Reasoning as in Theorem 1.2, one can deal with the initial value problem:

$$\begin{cases} -u'' + h(u) + g(u)|u'|^2 = f & \text{in } (0, +\infty), \\ u(0) = a. \end{cases} \tag{14}$$

THEOREM 1.3. – *Let* $f \in \mathrm{L}^\infty_{\mathrm{loc}}([0, +\infty))$*,* $f \geqslant 0$*, and let* (12) *and* (13) *hold true. For any given* $a \in \mathbf{R}$*, there exists a unique solution* $u \in \mathrm{C}^{1,1}_{\mathrm{loc}}([0, +\infty))$ *of problem* (14).

Remark 3. – The result of Theorem 1.1 can be extended to the case that f is a nonnegative Radon measure on $\mathbf{R}$. In this case one obtains uniqueness of weak solutions. The proof remains mostly unchanged but for minor modifications in order to give sense to previous arguments in a context of measure data (see [5]).

THEOREM 1.4. – *Given a nonnegative Radon measure* f *on* $\mathbf{R}$*, then there exists a unique solution* u *of* (3) *in the sense that* $u \in \mathrm{H}^1_{\mathrm{loc}}(\mathbf{R})$ *and satisfies*

$$\int_{\mathbf{R}} u'\varphi' \,\mathrm{d}x + \int_{\mathbf{R}} \big(u + u|u'|^2\big)\varphi \,\mathrm{d}x = \int_{\mathbf{R}} \varphi \,\mathrm{d}f, \quad \forall \varphi \in \mathrm{C}^\infty_c(\mathbf{R}). \tag{15}$$

The method developed in Theorem 1.1 also allows to handle the same problem in any dimension $N \geqslant 1$ provided f is a nonnegative radial function. Consider the equation:

$$-\Delta u + u + u|\nabla u|^2 = f \quad \text{in } \mathbf{R}^N, \tag{16}$$

where f is radial and nonnegative. We can prove uniqueness in the class of all classical solutions, that is without restricting to radial nor to nonnegative solutions. Indeed, since f is radial, one proves that the maximal and the minimal solutions $\overline{u}$ and $\underline{u}$ (constructed from problems on balls) are radial; uniqueness in the class of all solutions follows showing that $\overline{u} = \underline{u}$. To this purpose, one is reduced to study an ordinary differential equation which is weighted due to radial derivatives. Most of the arguments used before can be applied; if $\underline{u} < \overline{u}$, one proves that it is possible to construct a third radial solution v of (16) which satisfies $\overline{u}(r) > v(r) \geqslant \gamma > 0$ for any $r \in \mathbf{R}$, and $(\overline{u} - v)(r)$ is increasing. The difference $w = \overline{u} - v$ satisfies:

$$\big(\exp\big(-v^2\big)w'\big)' \geqslant \exp\big(-v^2\big)w + \gamma \exp\big(-v^2\big)|w'|^2 - \frac{N-1}{r}\exp\big(-v^2\big)w'. \tag{17}$$

Using Young's inequality, and since $w(r)$ is increasing, one still obtains that there exists $C > 0$ such that $(\exp(-v^2)w')' \geqslant C\,(\exp(-v^2)w')^2$ for every $r \geqslant r_1$, which contradicts the fact that there exist at least two global solutions.

The same proof works for more general equations, although the last estimate on (17) is more delicate in other examples. We obtain the following result (see [5]):

THEOREM 1.5. – *Assume that* $\alpha > 0$ *and that* g *satisfies* (13). *Let* f *be a nonnegative locally bounded radial function. Then there is exactly one solution* $u \in \mathrm{W}^{2,p}_{\mathrm{loc}}$*, for every* $p < +\infty$*, for*:

$$-\Delta u + |u|^{\alpha-1}u + g(u)|\nabla u|^2 = f \quad \textit{in } \mathbf{R}^N,$$

and u *is a radial function.* $\square$

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