C. R. Acad. Sci. Paris, Ser. I 335 (2002) 683–688

Équations aux derivées partielles/*Partial Differential Equations*
(Physique mathématique/*Mathematical Physics*)

Spectral asymptotics for magnetic Schrödinger operators with rapidly decreasing electric potentials

Georgi D. Raikov [a], Simone Warzel [b]

[a] Departamento de Matemáticas, Universidad de Chile, Las Palmeras 3425, Casilla 653, Santiago, Chile

[b] Institut für Theoretische Physik, Universität Erlangen-Nürnberg, Staudtstrasse 7, 91058 Erlangen, Germany

Received 4 September 2002; accepted 13 September 2002

Note presented by Jean-Michel Bony.

Abstract We consider the Schrödinger operator $H(V)$ on $\mathrm{L}^2(\mathbb{R}^2)$ or $\mathrm{L}^2(\mathbb{R}^3)$ with constant magnetic field, and a class of electric potentials V which typically decay at infinity exponentially fast or have a compact support. We investigate the asymptotic behaviour of the discrete spectrum of $H(V)$ near the boundary points of its essential spectrum. If V decays like a Gaussian or faster, this behaviour is non-classical in the sense that it is not described by the quasi-classical formulas known for the case where V admits a power-like decay. ***To cite this article: G.D. Raikov, S. Warzel, C. R. Acad. Sci. Paris, Ser. I 335 (2002) 683–688.***

Asymptotiques spectrales pour des opérateurs magnétiques de Schrödinger avec des potentiels électriques qui décroissent rapidement à l'infini

Résumé On considère l'opérateur de Schrödinger $H(V)$ agissant dans $\mathrm{L}^2(\mathbb{R}^2)$ ou $\mathrm{L}^2(\mathbb{R}^3)$ avec un champ magnétique constant et un potentiel électrique V qui génériquement décroît à l'infini exponentiellement vite ou est à un support compact. On étudie le comportement asymptotique du spectre discret de $H(V)$ en voisinage des points de la frontière de son spectre essentiel. Si la décroissance de V est gaussienne ou plus rapide ce comportement ne se décrit pas par les formules semi-classiques connues dans le cas où V décroît comme une puissance. ***Pour citer cet article : G.D. Raikov, S. Warzel, C. R. Acad. Sci. Paris, Ser. I 335 (2002) 683–688.***

Version française abrégée

On considère l'opérateur de Schrödinger $H(V) := (-\mathrm{i}\nabla - \mathbf{A})^2 + V$ agissant dans $\mathrm{L}^2(\mathbb{R}^d)$, $d = 2, 3$. On suppose que le champ magnétique engendré par le potentiel vectoriel $\mathbf{A} : \mathbb{R}^d \to \mathbb{R}^d$ est constant et on note $b > 0$ son intensité scalaire. De plus, on suppose que le potentiel électrique $V : \mathbb{R}^d \to \mathbb{R}$ décroît á l'infini et a un signe fixé. Le spectre de l'opérateur non perturbé $H(0)$ est donné par (1.1) oû $E_q := (2q + 1)b$, $q \in \mathbb{Z}_+ := \{0, 1, \ldots\}$, désignent les niveaux de Landau. Les hypothéses sur V impliquent que l'opérateur $|V|^{1/2}H(0)^{-1/2}$ est compact. Donc, le spectre essentiel de $H(V)$ coïncide avec celui de $H(0)$, c'est-á-dire la première égalité dans (1.1) est vraie. Cependant, le spectre discret de $H(V)$ n'est plus vide. Le but de cette Note est de présenter des résultats sur le comportement du spectre discret de $H(V)$ en voisinage des points de la frontière de $\sigma_{\mathrm{ess}}(H(V))$. Si V décroît à l'infini comme une puissance (ou moins rapidement)

E-mail addresses: graykov@uchile.cl (G.D. Raikov); simone.warzel@physik.uni-erlangen.de (S. Warzel).

S1631-073X(02)02554-2/FLA

ce comportement a été analysé en détail dans pluiseurs travaux (voir [11,13,8,9,6]). La nouveauté dans la présente Note est que l'on considère des potentiels V qui décroissent à l'infini exponentiellement vite ou sont à support compact. Si la décroissance de V est gaussienne ou plus rapide on démontre que le comportement du spectre discret de $H(V)$ ne se décrit pas par les formules semi-classiques connues.

Introduisons les notations nécessaires. Soient T un opérateur auto-adjoint dans un espace hilbertien et $\mathbb{P}_I(T)$ son projecteur spectral associé à l'intervalle $I \subset \mathbb{R}$. On pose $N(\lambda_1, \lambda_2; T) := \operatorname{rank} \mathbb{P}_{]\lambda_1,\lambda_2[}(T)$, $\lambda_1, \lambda_2 \in \mathbb{R}$, et $N_-(\lambda; T) := \operatorname{rank} \mathbb{P}_{]-\infty,\lambda[}(T)$, $N_+(\lambda; T) := \operatorname{rank} \mathbb{P}_{]\lambda,\infty[}(T)$, $\lambda \in \mathbb{R}$. De plus, la notation $f(\tau) \sim g(\tau)$, $\tau \to \tau_0$, signifie que $\lim_{\tau\to\tau_0} f(\tau)/g(\tau) = 1$.

Les premiers deux théorèmes concernent le cas bidimensionnel $d = 2$.

THÉORÈME 1. – *Soit V borné et positif sur $\mathbb{R}^2$. On suppose qu'il existe $0 < \mu < \infty$ et $0 < \beta < \infty$ tels que $\lim_{|\mathbf{x}|\to\infty} |\mathbf{x}|^{-2\beta} \ln V(\mathbf{x}) = -\mu$. On fixe le niveau de Landau E_q, $q \in \mathbb{Z}_+$, et l'énergie $E' \in]E_q, E_{q+1}[$. Alors on a* (2.1).

THÉORÈME 2. – *Soit V borné et positif sur $\mathbb{R}^2$. On suppose que le support de V est compact et qu'il existe une constante $C_- > 0$ telle que $V \geqslant C_-$ sur un ouvert non vide de $\mathbb{R}^2$. On fixe $q \in \mathbb{Z}_+$ et $E' \in]E_q, E_{q+1}[$. Alors on a* (2.2).

Remarques. – 1. Supposons que V soit négatif et que $-V$ satisfasse aux hypothèses du Théorème 1 (respectivement, Théorème 2). Alors (2.1) (respectivement, (2.2)) reste valide si on remplace $N(E_q + E, E'; H(V))$ par $N(E'', E_q - E; H(V))$, $E'' \in]E_{q-1}, E_q[$, si $q \geqslant 1$, ou par $N_-(E_0 - E; H(V))$ si $q = 0$.

2. On introduit la quantité semi-classique (2.3) ou $|\cdot|$ désigne la mesure de Lebesgue. On suppose que $V \geqslant 0$ et $V(\mathbf{x}) = \mathbf{v}(\mathbf{x}/|\mathbf{x}|)|\mathbf{x}|^{-\alpha}(1 + o(1))$ lorsque $|\mathbf{x}| \to \infty$ avec $\mathbf{v} \in C(\mathbb{S}^1)$, $\mathbf{v} > 0$, et $0 < \alpha < \infty$. Si le comportement asymptotique de $\mathcal{N}_{\mathrm{cl}}(E)$ lorsque $E \downarrow 0$ est assez régulier, on a (2.4) (voir [8, Théorème 2.6], [6, Chapitre 11]). D'autre part, si V satisfait aux hypothèses du Théorème 1, on a (2.5). Donc, les termes principaux asymptotiques quand $E \downarrow 0$ des quantités $N(E_q + E, E'; H(V))$ et $\mathcal{N}_{\mathrm{cl}}(E)$ sont pareils si et seulement si $0 < \beta < 1$. Dans le cas $\beta = 1$ les ordres asymptotiques de (2.1) et de (2.5) coïncident mais les coefficients correspondants sont différents. Autrement dit, la relation (2.1) est semi-classique pour des potentiels V dont la décroissance est moins rapide que gaussienne $(0 < \beta < 1)$ et non classique pour des potentiels V dont la décroissance est plus rapide que gaussienne $(1 < \beta < \infty)$, alors que la décroissance gaussienne $(\beta = 1)$ est le cas intermédiaire.

3. La relation (2.2) pourrait être considérée comme le cas limite quand $\beta \to \infty$ de (2.1). Cette relation est non classique comme sous les hypothèses du Théorème 2 la quantité $\mathcal{N}_{\mathrm{cl}}(E)$ reste bornée lorsque $E \downarrow 0$.

Les deux derniers théorèmes concernent le cas $d = 3$. Dans ce cas on suppose que le champ magnétique $\nabla \times \mathbf{A} = (0, 0, b)$ est parallèle à l'axe Oz et on note $X_\perp = (x, y) \in \mathbb{R}^2$ les variables sur le plan orthogonal à lui. On note $\chi_{r,X'_\perp}$ la fonction caractéristique du disque $\{X_\perp \in \mathbb{R}^2 \mid |X_\perp - X'_\perp| < r\}$.

THÉORÈME 3. – *Soit V négatif sur $\mathbb{R}^3$. On suppose qu'il existe une constante $C > 0$ et une fonction $0 \leqslant v \in L^1(\mathbb{R}; (1 + |z|)\,dz)$ pas identiquement égale à zéro telles que $|V(\mathbf{x})| \leqslant Cv(z)$, $\mathbf{x} = (X_\perp, z) \in \mathbb{R}^3$. De plus, on fixe $0 < \mu < \infty$ et $0 < \beta < \infty$ et on suppose que pour tout $\delta > 0$ il existe $r_\delta > 0$ et deux fonctions positives $v_\delta^\pm \in L^1(\mathbb{R}; (1 + |z|)\,dz)$ pas identiquement égales à zéro, tels que l'inégalité $|X_\perp| \geqslant r_\delta$ implique $e^{-\delta|X_\perp|^{2\beta}} v_\delta^-(z) \leqslant e^{\mu|X_\perp|^{2\beta}} |V(X_\perp, z)| \leqslant e^{\delta|X_\perp|^{2\beta}} v_\delta^+(z)$, $z \in \mathbb{R}$. Alors on a* (3.1).

THÉORÈME 4. – *Soit V négatif sur $\mathbb{R}^3$. On suppose qu'il existe $r_\pm > 0$, $X_\perp^\pm \in \mathbb{R}^2$, et deux fonctions positives $v^\pm \in L^1(\mathbb{R}; (1 + |z|)\,dz)$ pas identiquement égales à zéro, tels que $\chi_{r_-,X_\perp^-}(X_\perp)v^-(z) \leqslant |V(\mathbf{x})| \leqslant \chi_{r_+,X_\perp^+}(X_\perp)v^+(z)$, $\mathbf{x} = (X_\perp, z) \in \mathbb{R}^3$. Alors on a* (3.2).

Remarque. – Supposons que $V \leqslant 0$ et $V(\mathbf{x}) = -\mathbf{v}(\mathbf{x}/|\mathbf{x}|)|\mathbf{x}|^{-\alpha}(1 + o(1))$ quand $|\mathbf{x}| \to \infty$ avec $\mathbf{v} \in C(\mathbb{S}^2)$, $\mathbf{v} > 0$, et $2 < \alpha < \infty$. Pour $E > 0$ on introduit la quantité (3.3). Sous certaines hypothèses sur la régularité de $\widetilde{\mathcal{N}}_{\mathrm{cl}}(E)$ lorsque $E \downarrow 0$ on a (3.4) (voir [11], [13, Théorème 1(ii)], [8, Théorème 2.4(i)] et [6, Chapitre 12]).

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Le Théorème 3 implique que (3.4) reste valide si la décroissance de V est moins rapide que gaussienne dans les directions orthogonales au champ magnétique. D'autre part, si cette décroissance est gaussienne ou plus rapide, le terme principal asymptotique quand $E \downarrow 0$ de $N(E_0 - E;\, H(V))$ est différent de celui de $\widetilde{\mathcal{N}}_{\mathrm{cl}}(E)$.

Les démonstrations seront publiées ailleurs mais une version préliminaire est déjà disponible (voir [10]). On note que le Théorème 2 est contenu aussi dans la prépublication plus récente [7] où on le géneralise dans le cas des champs magnétiques constants non-dégénérés en dimension paire quelconque.

1. Introduction

We consider the magnetic Schrödinger operator $H(V) := (-\mathrm{i}\boldsymbol{\nabla} - \mathbf{A})^2 + V$, self-adjoint in $\mathrm{L}^2(\mathbb{R}^d)$, $d = 2, 3$, which describes a non-relativistic spinless quantum particle subject to a constant magnetic field of strength $b > 0$, and a decaying electric potential $V : \mathbb{R}^d \to \mathbb{R}$ of a fixed sign. The magnetic field is generated by the vector potential $\mathbf{A} = (A_1, \ldots, A_d) : \mathbb{R}^d \to \mathbb{R}^d$, i.e., it is identified with $\partial A_2/\partial x - \partial A_1/\partial y$ if $d = 2$, where $(x, y) \in \mathbb{R}^2$, or with $\nabla \times \mathbf{A}$ if $d = 3$. The spectrum of the unperturbed operator $H(0)$ is well-known and notably different for the two considered cases: if $d = 2$ it consists of a sequence of infinitely degenerate eigenvalues, the so-called Landau levels, given by $E_q := (2q + 1)b$, $q \in \mathbb{Z}_+ := \{0, 1, \ldots\}$, and if $d = 3$ the spectrum of $H(0)$ is purely absolutely continuous and coincides with $[E_0, \infty[$ (see, e.g., [1]). Our assumptions on V will always imply that the operator $|V|^{1/2} H(0)^{-1/2}$ is compact. Hence, Weyl's theorem ensures that the essential spectrum of $H(V)$ remains unchanged, i.e.,

$$\sigma_{\mathrm{ess}}\big(H(V)\big) = \sigma_{\mathrm{ess}}\big(H(0)\big) = \sigma\big(H(0)\big) = \begin{cases} \bigcup_{q=0}^{\infty}\{E_q\} & \text{if } d = 2, \\ [E_0, \infty[& \text{if } d = 3. \end{cases} \tag{1.1}$$

However, in contrast to $H(0)$, the perturbed operator $H(V)$ has a non-empty discrete spectrum. The aim of this Note is to present results on the behaviour of the discrete spectrum of $H(V)$ (or, in physical terms, the bound-state levels of the quantum particle) near the boundary points of the essential spectrum. This behaviour has been extensively studied in the literature in case where V admits power-like or slower decay at infinity (see [11,13,8,9,6], as well as the preceding works [1,12] where the special case of axisymmetric potentials V in three dimensions has been considered). The novelty in the present paper is that we consider V's which decay exponentially fast or have compact support. If the decay of V is Gaussian or faster, we show that the discrete-spectrum behaviour of $H(V)$ is not quasi-classical.

In order to formulate our results we need the following notations. Let T be a self-adjoint operator in a Hilbert space and $\mathbb{P}_I(T)$ be its spectral projection associated with the interval $I \subset \mathbb{R}$. Set $N(\lambda_1, \lambda_2; T) := \operatorname{rank} \mathbb{P}_{]\lambda_1, \lambda_2[}(T)$, $\lambda_1, \lambda_2 \in \mathbb{R}$, and $N_-(\lambda; T) := \operatorname{rank} \mathbb{P}_{]-\infty, \lambda[}(T)$, $N_+(\lambda; T) := \operatorname{rank} \mathbb{P}_{]\lambda, \infty[}(T)$, $\lambda \in \mathbb{R}$. Moreover, we use the notation $f(\tau) \sim g(\tau)$, $\tau \to \tau_0$, to indicate that $\lim_{\tau \to \tau_0} f(\tau)/g(\tau) = 1$.

2. Spectral asymptotics in two dimensions

For definiteness we choose V to be non-negative, so that discrete eigenvalues of $H(V)$ may only accumulate on the right of each Landau level E_q.

Our first theorem concerns the case where V decays exponentially fast.

THEOREM 1. – *Let V be bounded and non-negative on $\mathbb{R}^2$. Assume that there exist two constants $0 < \mu < \infty$ and $0 < \beta < \infty$ such that $\lim_{|\mathbf{x}| \to \infty} |\mathbf{x}|^{-2\beta} \ln V(\mathbf{x}) = -\mu$. Moreover, fix a Landau level E_q, $q \in \mathbb{Z}_+$, and some energy $E' \in]E_q, E_{q+1}[$. Then the following asymptotics holds*:

$$N\big(E_q + E, E'; H(V)\big) \sim a_\mu^{(\beta)}\big(|\ln E|\big) := \begin{cases} \dfrac{b}{2}\left(\dfrac{|\ln E|}{\mu}\right)^{1/\beta}, & 0 < \beta < 1, \\ \dfrac{|\ln E|}{\ln(1 + 2\mu/b)}, & \beta = 1, \\ \dfrac{\beta}{\beta - 1}\,\dfrac{|\ln E|}{\ln|\ln E|}, & 1 < \beta < \infty, \end{cases} \qquad E \downarrow 0. \tag{2.1}$$

The next theorem deals with the case where V has a compact support.

THEOREM 2. – *Let V be bounded and non-negative on $\mathbb{R}^2$. Assume that the support of V is compact, and there exists a constant $C_- > 0$ such that $V \geqslant C_-$ on a non-empty open subset of $\mathbb{R}^2$. Moreover, let $q \in \mathbb{Z}_+$ and $E' \in]E_q, E_{q+1}[$. Then we have*

$$N\big(E_q + E, E'; H(V)\big) \sim \frac{|\ln E|}{\ln|\ln E|}, \quad E \downarrow 0. \tag{2.2}$$

Remarks. – 1. Assume that V is non-positive and $-V$ satisfies the hypotheses of Theorem 1 (respectively, Theorem 2). Then (2.1) (respectively, (2.2)) remains valid if we replace $N(E_q + E, E'; H(V))$ by $N(E'', E_q - E; H(V))$, $E'' \in]E_{q-1}, E_q[$, if $q \geqslant 1$, or by $N_-(E_0 - E; H(V))$ if $q = 0$.

2. Introduce the quasi-classical quantity

$$\mathcal{N}_{\mathrm{cl}}(E) := \frac{b}{2\pi}\big|\{\mathbf{x} \in \mathbb{R}^2 \mid V(\mathbf{x}) > E\}\big|, \quad E > 0, \tag{2.3}$$

where $|\cdot|$ denotes the Lebesgue measure. If $V \geqslant 0$ and $V(\mathbf{x}) = \mathbf{v}(\mathbf{x}/|\mathbf{x}|)|\mathbf{x}|^{-\alpha}(1 + \mathrm{o}(1))$ as $|\mathbf{x}| \to \infty$ with some $\mathbf{v} \in \mathrm{C}(\mathbb{S}^1)$, $\mathbf{v} > 0$, and $0 < \alpha < \infty$, and if $\mathcal{N}_{\mathrm{cl}}(E)$ is regular enough as $E \downarrow 0$, we have

$$N\big(E_q + E, E'; H(V)\big) \sim \mathcal{N}_{\mathrm{cl}}(E) \sim E^{-2/\alpha}\frac{b}{4\pi}\int_{\mathbb{S}^1}\mathbf{v}(s)^{2/\alpha}\,\mathrm{d}s, \quad E \downarrow 0, \tag{2.4}$$

(see [8, Theorem 2.6], [6, Chapter 11]). On the other hand, if V satisfies the assumptions of Theorem 1, then

$$\mathcal{N}_{\mathrm{cl}}(E) \sim \frac{b}{2}\left(\frac{|\ln E|}{\mu}\right)^{1/\beta}, \quad 0 < \beta < \infty,\ E \downarrow 0. \tag{2.5}$$

Hence, the leading asymptotic terms of $N(E_q + E, E'; H(V))$ and $\mathcal{N}_{\mathrm{cl}}(E)$ are the same if and only if $0 < \beta < 1$. In case $\beta = 1$ the asymptotic orders of (2.1) and (2.5) coincide, but their coefficients differ although they have the same main asymptotic term in the strong magnetic field regime $b \to \infty$. In other words, asymptotic relation (2.1) is quasi-classical for potentials V whose decay is slower than Gaussian ($0 < \beta < 1$), and it is non-classical if the decay of V is faster than Gaussian ($1 < \beta < \infty$), while Gaussian decay ($\beta = 1$) is the border-line case.

3. Asymptotic relation (2.2) can be regarded as the limiting case of (2.1) as $\beta \to \infty$. Since under the hypotheses of Theorem 2 the quantity $\mathcal{N}_{\mathrm{cl}}(E)$ remains bounded as $E \downarrow 0$, (2.2) is non-classical. The prime-number theorem (see, e.g., [4, Section 1.8, Theorem 6]) implies that (2.2) is equivalent to $N(E_q + E, E'; H(V)) \sim \pi(|\ln E|)$, $E \downarrow 0$, where $\pi(\lambda)$ is the number of positive primes less than $\lambda > 0$.

3. Spectral asymptotics in three dimensions

In case $d = 3$ the only boundary point of $\sigma_{\mathrm{ess}}(H(V))$ is its infimum E_0. We therefore choose the perturbing electric potential V to be non-positive so that the discrete eigenvalues of $H(V)$ may accumulate on the left of E_0. We will write $\mathbf{x} = (X_\perp, z) \in \mathbb{R}^3$ where z is chosen to be the variable along the magnetic field vector $\nabla \times \mathbf{A} = (0, 0, b)$, while $X_\perp = (x, y) \in \mathbb{R}^2$ are the co-ordinates for the plane perpendicular to it.

The next two theorems are the three-dimensional analogues of Theorems 1 and 2, respectively. The first one treats the case where V decays exponentially fast in the directions perpendicular to the magnetic field.

THEOREM 3. – *Let V be non-positive on $\mathbb{R}^3$. Assume that there exist a constant $C > 0$ and a function $0 \leqslant v \in \mathrm{L}^1(\mathbb{R}; (1 + |z|)\,\mathrm{d}z)$, which does not vanish identically, such that $|V(\mathbf{x})| \leqslant Cv(z)$, $\mathbf{x} = (X_\perp, z) \in \mathbb{R}^3$. Moreover, let $0 < \mu < \infty$ and $0 < \beta < \infty$, and suppose that for each $\delta > 0$ there exist $r_\delta > 0$ and two non-negative functions $v_\delta^\pm \in \mathrm{L}^1(\mathbb{R}; (1 + |z|)\,\mathrm{d}z)$, which do not vanish identically, such that $|X_\perp| \geqslant r_\delta$ implies $\mathrm{e}^{-\delta|X_\perp|^{2\beta}}v_\delta^-(z) \leqslant \mathrm{e}^{\mu|X_\perp|^{2\beta}}|V(X_\perp, z)| \leqslant \mathrm{e}^{\delta|X_\perp|^{2\beta}}v_\delta^+(z)$, $z \in \mathbb{R}$. Then we have*

$$N\big(E_0 - E; H(V)\big) \sim a_\mu^{(\beta)}\big(\big|\ln\sqrt{E}\,\big|\big), \quad E \downarrow 0, \tag{3.1}$$

where $a_\mu^{(\beta)}$ was defined in (2.1).

Our last theorem concerns the case where the projection of the support of V onto the plane perpendicular to the magnetic field is compact. Let $\chi_{r,X'_\perp}$ be the characteristic function of $\{X_\perp \in \mathbb{R}^2 \mid |X_\perp - X'_\perp| < r\}$.

THEOREM 4. – *Let V be non-positive on $\mathbb{R}^3$. Assume that there exist constants $r_\pm > 0$, $X_\perp^\pm \in \mathbb{R}^2$, and two non-negative functions $v^\pm \in \mathrm{L}^1(\mathbb{R}; (1+|z|)\,\mathrm{d}z)$, which do not vanish identically, such that $\chi_{r_-,X_\perp^-}(X_\perp)v^-(z) \leqslant |V(\mathbf{x})| \leqslant \chi_{r_+,X_\perp^+}(X_\perp)v^+(z)$, $\mathbf{x} = (X_\perp, z) \in \mathbb{R}^3$. Then we have*

$$N\big(E_0 - E; H(V)\big) \sim \frac{|\ln\sqrt{E}|}{\ln|\ln\sqrt{E}|}, \quad E \downarrow 0. \tag{3.2}$$

Remark. – Assume that $V \leqslant 0$, and $V(\mathbf{x}) = -\mathbf{v}(\mathbf{x}/|\mathbf{x}|)|\mathbf{x}|^{-\alpha}(1+\mathrm{o}(1))$ as $|\mathbf{x}| \to \infty$ with some $\mathbf{v} \in \mathrm{C}(\mathbb{S}^2)$, $\mathbf{v} > 0$, and $2 < \alpha < \infty$. For $E > 0$ set

$$\widetilde{\mathcal{N}}_{\mathrm{cl}}(E) := \frac{b}{2\pi}\left|\left\{X_\perp \in \mathbb{R}^2 \;\Big|\; \int_{\mathbb{R}} \big|V(X_\perp, z)\big|\,\mathrm{d}z > 2\sqrt{E}\right\}\right|. \tag{3.3}$$

Under some supplementary regularity assumptions of $\widetilde{\mathcal{N}}_{\mathrm{cl}}(E)$ as $E \downarrow 0$, we have

$$N\big(E_0 - E; H(V)\big) \sim \widetilde{\mathcal{N}}_{\mathrm{cl}}(E), \quad E \downarrow 0, \tag{3.4}$$

(see [11], [13, Theorem 1(ii)], [8, Theorem 2.4(i)] and [6, Chapter 12]). Theorem 3 shows that (3.4) remains valid if the decay of V is slower than Gaussian in the directions perpendicular to the magnetic field. On the other hand, if this decay is Gaussian or faster, the leading asymptotics of $N(E_0 - E; H(V))$ as $E \downarrow 0$ differs from (3.4).

4. Sketch of the proofs of the main results

The proofs of Theorems 1 and 2 consist of three steps. The first step is to reduce the asymptotic analysis of the discrete spectrum of $H(V)$ near the Landau level E_q, $q \in \mathbb{Z}_+$, to the study of the asymptotic eigenvalue distribution of the compact Toeplitz operators $P_q V P_q$, where P_q is the spectral projection of $H(0)$ corresponding to its eigenvalue E_q. Namely, we show that under the hypotheses of Theorem 1 or 2 the estimates

$$N_+\big(E; (1-\varepsilon)P_q V P_q\big) + \mathrm{O}(1) \leqslant N\big(E_q + E, E'; H(V)\big) \leqslant N_+\big(E; (1+\varepsilon)P_q V P_q\big) + \mathrm{O}(1), \quad E \downarrow 0,$$

hold for each $\varepsilon \in]0, 1[$.

The second step is to calculate explicitly the eigenvalues of some model Toeplitz operators. Here we exploit the fact that if V is radially symmetric, then the eigenvalues of the operator $P_q V P_q$ with domain $P_q\mathrm{L}^2(\mathbb{R}^2)$, coincide with the numbers $\langle V\varphi_{q,k}, \varphi_{q,k}\rangle$, $k \in \mathbb{Z}_+ - q$, where $\langle\cdot,\cdot\rangle$ denotes the scalar product in $\mathrm{L}^2(\mathbb{R}^2)$, and $\{\varphi_{q,k}\}$ is the orthonormal angular-momentum eigenbasis of $P_q\mathrm{L}^2(\mathbb{R}^2)$ given by

$$\varphi_{q,k}(\mathbf{x}) := \sqrt{\frac{q!}{(k+q)!}}\left[\sqrt{\frac{b}{2}}(x + \mathrm{i}y)\right]^k \mathrm{L}_q^{(k)}\left(\frac{b|\mathbf{x}|^2}{2}\right)\sqrt{\frac{b}{2\pi}}\exp\left(-\frac{b|\mathbf{x}|^2}{4}\right), \quad \mathbf{x} = (x, y) \in \mathbb{R}^2,$$

in terms of the generalized Laguerre polynomials $\mathrm{L}_q^{(k)}$ (see, e.g., [5]). We consider two radially symmetric model potentials. The first one is $G_\mu^{(\beta)}(\mathbf{x}) := \exp(-\mu|\mathbf{x}|^{2\beta})$, $\mathbf{x} \in \mathbb{R}^2$, $\mu > 0$, $\beta > 0$; this model potential is useful in the proof of Theorem 1. In particular, we obtain the asymptotics

$$\big|\ln\big\langle G_\mu^{(\beta)}\varphi_{q,k}, \varphi_{q,k}\big\rangle\big| \sim \big(a_\mu^{(\beta)}\big)^{-1}(k), \quad k \to \infty, \tag{4.1}$$

where $(a_\mu^{(\beta)})^{-1}$ denotes the function inverse to $a_\mu^{(\beta)}$ (see (2.1)). The second model potential coincides with the characteristic function $\chi_{r,\mathbf{x}_0}$ of the disk of radius $r > 0$ centered at $\mathbf{x}_0 \in \mathbb{R}^2$. In this case we get

$$\big|\ln\langle\chi_{r,\mathbf{x}_0}\varphi_{q,k}, \varphi_{q,k}\rangle\big| \sim k\ln k, \quad k \to \infty. \tag{4.2}$$

The final step in our proof of asymptotic formulas (2.1) and (2.2) is a suitable approximation of the potential V satisfying the assumptions of Theorem 1 or 2 by the model potentials considered in the previous step.

The proofs of Theorems 3 and 4 are based on relations (4.1) and (4.2), and the following two lemmas. Let $v = \overline{v} \in \mathrm{L}^1(\mathbb{R})$. Denote by $h(v)$ the self-adjoint operator generated in $\mathrm{L}^2(\mathbb{R})$ by the closed lower-bounded quadratic form $\int_{\mathbb{R}} \{|u'(z)|^2 - v(z)|u(z)|^2\} \, \mathrm{d}z$, $u \in \mathrm{W}_2^1(\mathbb{R})$.

LEMMA 1. – *Let $V \leqslant 0$. Suppose that there exist four non-negative functions $v^{\pm} \in \mathrm{L}^1(\mathbb{R})$ and $U^{\pm} \in \mathrm{L}^1(\mathbb{R}^2) \cap \mathrm{L}^{\infty}(\mathbb{R}^2)$ such that $U^-(X_\perp)v^-(z) \leqslant |V(\mathbf{x})| \leqslant U^+(X_\perp)v^+(z)$, $\mathbf{x} = (X_\perp, z) \in \mathbb{R}^3$. Then for every $\varepsilon \in]0, 1[$ we have*

$$\sum_{k \in \mathbb{Z}_+} N_-\big(-E; h(\varkappa_k^- v^-)\big) \leqslant N\big(E_0 - E; H(V)\big) \leqslant \sum_{k \in \mathbb{Z}_+} N_-\big(-E; h\big((1+\varepsilon)\varkappa_k^+ v^+\big)\big) + \mathrm{O}(1), \quad E \downarrow 0,$$

where $\varkappa_k^{\pm}$, $k \in \mathbb{Z}_+$, are the eigenvalues of the compact operators $P_0 U^{\pm} P_0$ on $P_0 \mathrm{L}^2(\mathbb{R}^2)$.

LEMMA 2 ([2,3,12]). – *Let $0 \leqslant v \in \mathrm{L}^1(\mathbb{R}; (1+|z|)\,\mathrm{d}z)$, $v \neq 0$. Then we have $1 \leqslant N_-(0; h(gv)) \leqslant g \int_{\mathbb{R}} |z| v(z) \, \mathrm{d}z + 1$. Assume in addition that $g \int_{\mathbb{R}} |z| v(z) \, \mathrm{d}z < 1$, and denote by $-\mathcal{E}(gv)$ the unique negative eigenvalue of $h(gv)$. Then $\sqrt{\mathcal{E}(gv)} \sim \frac{g}{2} \int_{\mathbb{R}} v(z) \, \mathrm{d}z$ as $g \downarrow 0$.*

The detailed proofs of the results announced in the present Note will be published elsewhere. A preliminary version is already available (see [10]). Note that Theorem 2 is also contained in the more recent preprint [7] where it is extended to the case of constant magnetic-field tensors of full rank in arbitrary even dimensions, as well as to a relativistic setting.

Acknowledgements. The authors thank Hajo Leschke and Grigori Rozenblum for stimulating discussions. The financial support of the Chilean Science Foundation *Fondecyt* under Grants 1020737 and 7020737 is gratefully acknowledged.

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