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Exact and asymptotic inverse of the Toeplitz matrix with polynomial singular symbol

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Abstract From a previous work and an application of predictive polynomials we obtain two types of results. In a first part exact entries of the Toeplitz matrix are computed in the case where the symbol is $|P|^2 f$ and where f is a non negative regular function and P a polynomial with all its zeros on $\mathbb{T}$. In a second part we give an asymptotic expansion for symbols $(1-\cos\theta)^p f$ when f is always a nonnegative regular function. These formulas use Green kernels associated to differential operators of order $2p$. Finally, we propose some applications to the computation of traces and determinants. ***To cite this article: P. Rambour, A. Seghier, C. R. Acad. Sci. Paris, Ser. I 335 (2002) 705–710.***

Inverses exacts et asymptotiques des matrices de Toeplitz à symbole singulier polynomial

Résumé À partir de travaux antérieurs et d'une application des polynômes prédicteurs nous obtenons deux types de résultats. Dans une première partie nous proposons une méthode pour obtenir un développement exact des coefficients de matrices de Toeplitz ayant un symbole du type $|P|^2 f$ où f est une fonction régulière positive et P un polynôme ayant ses zéros sur le tore. Dans une deuxième partie nous obtenons des formules asymptotiques pour les coefficients de l'inverse des matrices de Toeplitz dont le symbole est le produit de $(1-\cos\theta)^p$ par une fonction strictement positive. Ces développements utilisent des noyaux de Green associés à des opérateurs différentiels d'ordre $2p$. Enfin nous proposons quelques applications aux calculs des traces et des déterminants. ***Pour citer cet article : P. Rambour, A. Seghier, C. R. Acad. Sci. Paris, Ser. I 335 (2002) 705–710.***

Version française abrégée

Un problème important de l'étude des matrices de Toeplitz (voir le texte anglais pour une définition précise) est l'obtention de l'inverse lorsque le symbole est du type :

$$\prod_{n=1}^{r} |\chi - \chi_n|^{2\alpha_n} f,$$

où $\chi = \mathrm{e}^{\mathrm{i}\theta}$, $\chi_n = \mathrm{e}^{\mathrm{i}\theta_n}$, $\theta_n \in \mathbb{R}$, $\alpha_n \in \mathbb{C}$, $\Re\alpha_n > -1/2$ et f une fonction régulière. Cette Note répond de deux manières différentes à ce problème quand les α_n sont des entiers positifs et f une fonction strictement positive dans $\mathrm{L}^1(\mathbb{T})$. Le premier résultat est une méthode récursive pour calculer exactement ces inverses. Dans un premier temps, en affinant les méthodes développées dans un précédent travail [5], nous obtenons

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l'inverse exact de $T_{N,|1-\chi|^2/|P_{N+1}|^2}$ où P_{N+1} est un polynôme de degré $N+1$, les coefficients de cette matrice ne dépendant que de ceux de P_{N+1} (alors que dans le travail cité plus haut intervenaient également les coefficients du développement de Fourier de $P_{N+1}/\overline{P_{N+1}}$). Nous utilisons ensuite le résultat fondamental de la théorie des polynômes prédicteurs (voir [4] et le texte anglais pour une définition) à savoir :

$$\widehat{\frac{1}{|P_M|^2}}(s) = \widehat{g(s)}, \quad -M \leqslant s \leqslant M, \tag{1}$$

si P_M est le polynôme prédicteur d'ordre M de g. Le fait que les coefficients du polynôme prédicteur d'une fonction g sont donnés par la première colonne de la matrice inverse de $T_{N,g}$ nous permet alors de calculer les coefficients de $T^{-1}_{N,h}$ en fonction de ceux de $T^{-1}_{N,\tilde h}$ où $\tilde h$ est telle que $h = |\chi - \chi_r|^2 \tilde h$ en remplaçant $\tilde h$ par $1/|P_{N+1}|^2$ où P_{N+1} est le polynôme prédicteur d'ordre N de $\tilde h$ et en utilisant la formule (1).

Dans une deuxième partie nous obtenons notre principal résultat qui est une extension importante d'un résultat dû à Spitzer et Stone [6] que nous rappelons ici :

$$\left(T_{N,|1-\chi|^2 f}\right)^{-1}_{[Nx]+1,\,[Ny]+1} = \frac{2}{\sigma^2} N R(x,y) + \mathrm{o}(N),$$

où R est l'opposée de la fonction de Green correspondant à l'opérateur différentiel $\partial^2/\partial x^2$ associé aux conditions initiales $h(0) = h(1) = 0$ et f une fonction définie sur le tore par les conditions :

$$f = 1 - \Phi, \quad \Phi\left(\mathrm{e}^{\mathrm{i}\theta}\right) = \sum_{k \in \mathbb{Z}} c_k \,\mathrm{e}^{\mathrm{i}\theta},$$

$$\sum_{k\in\mathbb{Z}} c_k = 1, \quad c_k = c_{-k}, \quad \forall k \in \mathbb{Z}, \quad c_k \geqslant 0, \quad \text{pgcd}\{k;\ k > 0 \text{ et } c_k > 0\} = 1$$

et où : $\sigma^2 = \sum_{k\in\mathbb{Z}} k^2 c_k < +\infty$. En utilisant les techniques établies dans la première partie de l'article nous obtenons le théorème suivant :

THÉORÈME PRINCIPAL. – *Si* $f_1 \in \mathrm{C}^1(\mathbb{T})$ *et* $f_1' \in A(\mathbb{T})$ *nous avons* :

$$\left(T_{N,|1-\chi|^{2p} f_1}\right)^{-1}_{[Nx]+1,\,[Ny]+1} = (-1)^p \frac{2}{\sigma^2} N^{2p-1} G_p(x,y) + \mathrm{o}\left(N^{2p-1}\right).$$

Avec, si $\widehat{\alpha(k)}$ désigne le k-ième coefficient de Fourier de la fonction $\hat\alpha$:

$$A(\mathbb{T}) = \left\{\hat\alpha \in \mathrm{L}^1(\mathbb{T}) : \sum_{k\in\mathbb{Z}} \left|\widehat{\alpha(k)}\right| < \infty\right\}$$

et où G_p est le noyau de Green correspondant à l'opérateur différentiel $\partial^{2p}/\partial x^{2p}$ associé aux conditions initiales :

$$f^{(0)}(0) = \cdots = f^{(p)}(0) = 0, \qquad f^{(0)}(1) = \cdots = f^{(p)}(1) = 0.$$

1. Recursive formula

1.1. Toeplitz matrix and polynomial prediction

Let h be a nonnegative function, belonging to $\mathrm{L}^1(\mathbb{T})$. We recall that for $h \in \mathrm{L}^1(\mathbb{T})$ ($\mathbb{T} \subset \mathbb{C}$ the unit circle), the Toeplitz operator $T_{N,h}$ acts on the space $\mathcal{P}_N \subset \mathrm{L}^2(\mathbb{T})$ of polynomial of degree less or equal to N and is defined by $\langle T_{N,h} P, Q\rangle$ (i.e., $T_{N,h} P$ is the orthogonal projection of hP on $\mathcal{P}_N$). If h is a nonnegative function, $T_{N,h}$ is $\gg 0$ hence invertible. The N-th predictor polynomial of h is defined as $P_N = (T_{N,h})^{-1}(1)$. It is known [4] that h and $\Phi = \overline{P_N(0)}|P_N|^{-2}$ have the same Fourier coefficients of order less than or equal to N that gives the equality: $T_{N,h} = T_{N,\Phi}$. Hence to compute $\left(T_{N,h}\right)^{-1}$ we begin to compute the inverse of $T_{N,|1-\chi|^2/|P_{N+1}|^2}$ where P_{N+1} is the $N+1$-th predictor polynomial of h.

1.2. Exact inverse of $T_{N,|1-\chi|^2/|P_{N+1}|^2}$

In a previous work we obtained formulas giving exact entries of $(T_{N,|1-\chi|^2/|P|^2})^{-1}$ with P a polynomial without zeros outside the unit closed disk [5] and having degree m with: $m < N+1$.

If $P(\chi) = \sum_{u=0}^{m} \beta_u \chi^u$ and $\overline{P(\chi)}/P(\chi) = \sum_{u \leqslant -m} \gamma_u \chi^u$ then the expression of $(T_{N,|1-\chi|^2/|P|^2|})^{-1}_{k+1,l+1}$ obtained depends on the β_u's and γ_u's. The method is based on a parametrisation $|1 - r\chi|$ with $r \to 1$ and the well known inversion formula.

A more precise calculation allows us to get an expression depending only on β_u for a polynomial P_{N+1} with degree $N+1$.

Putting $P_{N+1}(\chi) = \sum_{u=0}^{N+1} \beta_u \chi^u$ we obtain the following theorem.

THEOREM 1 (inversion formula). – *For* $0 \leqslant k \leqslant l \leqslant N$ *we have*:

$$\left(T_{N,|1-\chi|^2/|P|^2|}\right)^{-1}_{k+1,l+1} = a_{k,l} + r_{k,l} - \frac{1}{N+1+A(P_{N+1})} A_{N,k}\bar{A}_{N,l}, \tag{1}$$

where:

$$a_{k,l} = \sum_{s=0}^{k}\sum_{s'=0}^{l} \overline{\beta_{k-s}}\beta_{l-s'} \min(s+1, s'+1), \quad r_{k,l} = -\frac{1}{2}\left(Q_3 - \widetilde{Q}'_{2,k}(1)\overline{\widetilde{Q}_{2,l}}(1) - \widetilde{Q}_{2,k}(1)\overline{\widetilde{Q}'_{2,l}}(1)\right),$$

$$A_{N,k} = \frac{1}{2}\left(\widetilde{Q}'_{1,k}(1) + \widetilde{Q}'_{2,k}(1) - \tilde{\Phi}'_N(1)\overline{P}_{N+1}(1) - \tilde{\Phi}_N(1)\left(Q'_{1,k}(1) + Q'_{2,k}(1)\right)\right),$$

$$A(P_{N+1}) = -2\Re\left(\overline{P}'_{N+1}(1)P_{N+1}(1)\right)/\left|P_{N+1}(1)\right|^2,$$

and with:

$$\widetilde{Q}_{1,k}(r) = \sum_{u=0}^{N+1-k} \beta_u r^{N+1-k-u}, \qquad \widetilde{Q}_{2,k}(r) = \sum_{u=N+2-k}^{N+1} \beta_u r^{u-(N+1)+k},$$

$$Q_{1,k}(r) = \sum_{u=0}^{k} \bar{\beta}_u r^{k-u+2}, \qquad Q_{2,k}(r) = \sum_{u=k+1}^{N+1} \bar{\beta}_u r^{u-k},$$

$$Q_3 = \sum_{u=N+2-k}^{N+1}\sum_{u'=N+2-k}^{N+1} \beta_u \bar{\beta}'_u |u+k-u'-l|, \quad \tilde{\Phi}_N(\chi) = \frac{P_{N+1}}{\overline{P}_{N+1}} \chi^{-(N+1)}.$$

To compute this inverse with a symbol $|\chi_0 - \chi|^2/|P_{N+1}(\chi)|^2$, where $\chi_0 \in \mathbb{T}$ we use the same calculation with the additional remark that $|\chi_0 - r\chi|^2 = |1 - r\chi_0^{-1}\chi|^2$. Then we obtain another formula:

COROLLARY 1 (shifted formula). – *For* $0 \leqslant k \leqslant l \leqslant N$ *we have*:

$$\left(T_{N,|\chi_0-\chi|^2/|P|^2}\right)^{-1}_{k+1,l+1} = \left(\left(a_{k,l,\chi_0} + r_{k,l,\chi_0} - \frac{1}{N+1+A(\widetilde{P}_{N+1})} A_{N,k,\chi_0}\bar{A}_{N,l,\chi_0}\right)\chi_0^{(l-k)}\right).$$

$\widetilde{P}_{N+1}(\chi) = \sum_{u=0}^{N+1} \chi^u \chi_0^u \beta_u$ and $a_{k,l,\chi_0}, r_{k,l,\chi_0}, A(\widetilde{P}_{N+1}), A_{N,k,\chi_0}$ are the same constants as in the previous theorem but computed with the polynomial $\widetilde{P}_{N+1}$.

1.3. Corollary: recursive principle

If $f = |\chi_0 - \chi|^2 h$, $h > 0$ and $h \in \mathrm{L}^1(\mathbb{T})$, then if P_{N+1} is the predictor polynomial of $(|\chi_0 - \chi|^2 h)$ as defined in a previous paragraph it is known that for all $k,\ l,\ 0 \leqslant k,\ l \leqslant N$,

$$\left(T_{N,|\chi_0-\chi|^2/|P_{N+1}|^2}\right)^{-1}_{k+1,l+1} = \left(T_{N,|\chi_0-\chi|^4 h}\right)^{-1}_{k+1,l+1}.$$

Our recursive principle is based on this last result.

1.4. Recursive theorems

We can now easily prove the two following theorems:

THEOREM 2 (first recursive formula). – *Let h be a nonnegative function, such that $h \in \mathrm{L}^1(\mathbb{T})$. Then*: $\forall p \in \mathbb{N}\ \forall N \in \mathbb{N}\ \forall \chi_0 \in \mathbb{T}\ \exists P_{N+1,p}$ *such that*:

$$\widehat{\frac{1}{|P_{N+1,p}|^2}}(s) = \big(\widehat{|\chi_0 - \chi|^{2p} h}(s)\big), \quad -(N+1) \leqslant s \leqslant (N+1).$$

THEOREM 3 (second recursive formula). – *With the same hypotheses as in the previous theorem we have*: $\forall p \in \mathbb{N}\ \forall N \in \mathbb{N}\ \forall \chi_0 \in \mathbb{T}\ \exists P_{N+1,p}$ *such that*:

$$\forall k, l,\ 0 \leqslant k, l \leqslant N, \quad \big(T_{N,|\chi_0-\chi|^2/|P_{N+1,p}|^2}\big)^{-1}_{k+1,l+1} = \big(T_{N,|\chi_0-\chi|^{2p+2}h}\big)^{-1}_{k+1,l+1}.$$

It is now important to observe that the two last theorems allow us to obtain exact entries of matrices $(T_{N,|\chi_0-\chi|^{2p}h})^{-1}$ for all $p \in \mathbb{N}$.

2. Asymptotics results

2.1. Inverse of Toeplitz matrices with symbol $|1-\chi|^{2p}$

Using previous ideas, the result of Section 1.1 and Euler–Mac-Laurin formula we can get two families of functions. The first family is a polynomial one, it gives an asymptotic expansion of the coefficients of the polynomials P_{N+1} which appear in the recursive method. Hence the knowledge of this family gives an asymptotic order of the first column of the inverse Toeplitz matrix. The second family gives an asymptotic expansion for central entries of these inverses. These functions are Green kernels different from $R^{(2)}$ introduced by Spitzer and Stone [6].

THEOREM 4 (asymptotics of predictor polynomial). – *For all nonnegative integer p there exists a real polynomial $\widetilde{G}_p$ with degree p such that*:

$$\big(T_{N,|1-\chi|^{2p}}\big)^{-1}_{[Ny]+1,1} = \widetilde{G}_p(y)\ N^{p-1} + \mathrm{o}\big(N^{p-1}\big)$$

with $y \in [0,1]$ where $[t]$ is the biggest integer smaller than t.

By using Euler–Mac-Laurin and the expansion given in the first paragraph we obtain the recursive relation between $\widetilde{G}_{p-1}$ and $\widetilde{G}_p$. This relation is:

$$\widetilde{G}_p(x) = \int_0^x \widetilde{G}_{p-1}(t)\,\mathrm{d}t - (2p-1)\int_{1-x}^1 \widetilde{G}_{p-1}(t)(t+x-1)\,\mathrm{d}t - (2p-1)\int_0^x \widetilde{G}_{p-1}(t)(t-x)\,\mathrm{d}t.$$

For p an integer let us now define $G_p(x,y)$ the Green kernel associated to the differential operator $\partial^{2p}/\partial x^{2p}$ associated to the initial conditions:

$$f^{(0)}(0) = \cdots = f^{(p)}(0) = 0, \qquad f^{(0)}(1) = \cdots = f^{(p)}(1) = 0.$$

This kernel is different from those introduced by Spitzer and Stone [6]: the initial conditions are different. Then we can write:

THEOREM 5 (Green kernels). – $\forall p \in \mathbb{N}\ \forall x \in [0,1]\ \forall y \in [0,1]$ *we have the asymptotic equality*:

$$\big(T_{N,|1-\chi|^{2p}}\big)^{-1}_{[Nx]+1,\,[Ny]+1} = N^{2p-1} G_p(x,y) + \mathrm{o}\big(N^{2p-1}\big).$$

It is a remarkable fact that we can deduce an expression of the kernel G_p using our recursive principle and the polynomial $\widetilde{G}_{p-1}$. For instance:

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$$G_2(x,y) = yx^2/2 - x^2y^2 - x^3/6 - x^3y^3/3 + x^2y^3/2 + x^3y^2/2, \quad \text{if } x \leqslant y,$$
$$G_2(x,y) = xy^2/2 - y^2x^2 - y^3/6 - y^3x^3/3 + y^2x^3/2 + y^3x^2/2, \quad \text{if } y \leqslant x.$$

We can compute $G_p(x,y)$ using $\widetilde{G}_{p-1}$ with the following formula:

$$\begin{aligned}
G_p(x,y) = {} & \int_0^x \widetilde{G}_{p-1}(x-t)\int_0^t t'\widetilde{G}_{p-1}(y-t')\,\mathrm{d}t\,\mathrm{d}t' + \int_0^x \widetilde{G}_{p-1}(x-t)t\int_t^y \widetilde{G}_{p-1}(y-t')\,\mathrm{d}t\,\mathrm{d}t' \\
& - (2p-1)\left(\int_{1-x}^1 \widetilde{G}_{p-1}(t)(t+x-1)\,\mathrm{d}t + \int_0^x \widetilde{G}_{p-1}(t)(x-t)\,\mathrm{d}t\right) \\
& \times \left(\int_{1-y}^1 \widetilde{G}_{p-1}(t)(t+y-1)\,\mathrm{d}t + \int_0^y \widetilde{G}_{p-1}(t)(y-t)\,\mathrm{d}t\right) \\
& - \left(\int_1^x \widetilde{G}_{p-1}(t+1-x)t\int_x^y \widetilde{G}_{p-1}(t'+1-y)\,\mathrm{d}t\,\mathrm{d}t'\right. \\
& \left. + \int_1^x \widetilde{G}_{p-1}(t+1-x)\int_x^y \widetilde{G}_{p-1}(t'+1-y)t'\,\mathrm{d}t\,\mathrm{d}t'\right).
\end{aligned}$$

2.2. Asymptotic expansion of $(T_{N,|1-\chi|^{2p}f_1})^{-1}$

Let $A(\mathbb{T})$ be the algebra of functions defined by:

$$A(\mathbb{T}) = \left\{\alpha \in \mathrm{L}^1(\mathbb{T}) : \sum_{k\in\mathbb{Z}}\left|\widehat{\alpha(k)}\right| < \infty\right\}.$$

We can now state our main result:

THEOREM 6 (main theorem). – *If* $f_1 \in \mathrm{C}^1(\mathbb{T})$ *and* $f_1' \in A(\mathbb{T})$ *we have*:

$$\left(T_{N,|1-\chi|^{2p}f_1}\right)^{-1}_{[Nx]+1,\,[Ny]+1} = \frac{2}{\sigma^2}N^{2p-1}G_p(x,y) + \mathrm{o}\left(N^{2p-1}\right),$$

where $2/\sigma^2 = |\sum_{u=0}^{N+1}\beta_u|^2$, and $\beta_u = (T_{N,f_1})^{-1}_{u+1,1}$.

3. Applications

Applications to probability theory and statistics are important, we plan to develop some consequences in a forthcoming paper. Here we give only few consequences to trace and determinant computations.

3.1. Trace asymptotic expansion

THEOREM 7 (trace theorem). – *With the main theorem's hypothesis we can write*:

$$\mathrm{Tr}\left(T_{N,|1-\chi|^{2p}f_1}\right)^{-1} = N^{2p}\frac{2}{\sigma^2}\int_0^1 G_p(x,x)\,\mathrm{d}x + \mathrm{o}\left(N^{2p}\right).$$

3.2. Determinant asymptotic expansion

The Hartwig–Fisher conjecture (see [3,2,1]) which has been confirmed for integers exponents by Widom [8] gives the first order of the asymptotic expansion of Toeplitz matrix determinant with symbol:

$$f(\chi) = h(\chi)\prod_{r=1}^R\left(\chi^{\beta_r}/\chi_r^{\beta_r}\right)\exp(-\mathrm{i}\beta_r\pi)|\chi_r-\chi|^{2\alpha_r},$$

where h is a regular function on $\mathbb{T}$ such $\mathrm{Ind}(h) = 0$ and $\Re\alpha_r > -1/2$.

If we note $D_N(f)$ this determinant the conjecture is

$$D_N(f) = G[h]N^{\Gamma}E + \mathrm{o}\left(G[h]N^{\Gamma}E\right),$$

where $G[h]$ is a constant depending only of h and E another constant depending of $\alpha, \beta,$ and h, with the Barnes function [7]. Moreover:

$$\Gamma = \sum_{r=1}^{R} \alpha_r^2 - \beta_r^2.$$

By using the fundamental remark $D_{N-1}/D_N = (T_{N,f})_{1,1}^{-1}$ and the equality

$$\left(T_{N,|1-\chi|^2/|P_{N+1}|^2}\right)_{1,1}^{-1} = \left|P_{N+1}(0)\right|^2 \left(1 - 1/\left(N + 2 + A(P_{N+1})\right)\right),$$

where $A(P_{N+1}) = -2\Re(\overline{P'_{N+1}}(1) P_{N+1}(1))/|P_{N+1}(1)|^2$, we obtain an infinite product which gives us, for smooth function classes, an asymptotic expansion of larger order than Hartwig–Fisher. Putting: $P(\chi) = \sum_{u=0}^{m} \beta_u \chi^u$, P without zero in: $\{z/|z| \leqslant 1\}$ we can precise some examples.

- If $f_1(\chi) = (|1-\chi|^2/|P|^2)$ we obtain that D_N is the product of a constant by: $\left(\prod_{j=m}^{N} |\beta_0|^2 (1 - 1/(j + 2 - A(P)))\right)^{-1}$ and, lastly:

$$D_N(f) = |\beta_0|^{2N} E\left(N + 4 - A(P) + \mathrm{o}(1)\right),$$

  where E is the constant given by the Hartwig–Fisher conjecture for f_1.
- Let $f_2(\chi)$ be $|1-\chi|^4/|P|^2$ and put $P_{2,N+1}$ be the predictor polynomial of order $N+1$ of $|1-\chi|^2/|P|^2$. Then the determinant is the product of a constant by: $\prod_{j=0}^{N} |\beta_0|^2 (1 - 1/(j + C))(1 - 1/(j + 2 - A(P_{2,j})))$.

  If $A(P_{2,N+1}) = -\frac{2}{3}N + \beta + \mathrm{o}(1)$ is an asymptotic expansion of $A(P_{2,N+1})$ we have:

$$D_N(f) = |\beta_0|^{2N} E\left(N^4 + \left(9\beta + 17 - 2A(P)\right)N^3 + \mathrm{o}\left(N^3\right)\right),$$

  where E is the constant given by the Hartwig–Fisher conjecture for f_2.
- If $f_3 = (|1-\chi|^4/|1 + a\chi|^2)$, $a \in \mathbb{R}$, $|a| < 1$, then

$$D_N(f) = E\left(N^4 + \left(\frac{27a - 31}{1 + a}\right)N^3 + \mathrm{o}\left(N^3\right)\right),$$

  where E is the constant given by the Hartwig–Fisher conjecture for f_3.

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References

[1] A. Böttcher, B. Silbermann, The asymptotic behavior of Toeplitz determinants for generating functions with zeros of integral orders, Math. Nachr. 102 (1981) 79–105.
[2] T. Ehrhardt, B. Silbermann, Toeplitz determinants with One Fisher–Hartwig singularity, J. Funct. Anal. 148 (1997) 229–256.
[3] M.E. Fisher, R.E. Hartwig, Toeplitz determinants: Some applications, theorems, and conjectures, Adv. Chem. Phys. 15 (1968) 333–353.
[4] H.J. Landau, Maximum entropy and the moment problem, Bull. Amer. Math. Soc. (N.S.) 1 (1) (1986) 47–77.
[5] P. Rambour, J.-M. Rinkel, A. Seghier, Inverse asymptotique de la matrice de Toeplitz et noyau de Green, C. R. Acad. Sci. Paris 331 (2000) 857–860.
[6] F.L. Spitzer, C.J. Stone, A class of Toeplitz forms and their applications to probability theory, Illinois J. Math. 4 (1960) 253–277.
[7] E.T. Whittaker, G.N. Watson, A Course of Modern Analysis, 4th ed., Cambridge University Press, London, 1952.
[8] H. Widom, Toeplitz determinants with singular generating functions, Amer. J. Math. 95 (1973) 333–383.