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The Neumann problem for free boundaries in two dimensions

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Abstract We extend to the case with surface tension a theorem of H. Lewy concerning the a priori regularity of traveling waves solutions of the free surface problem of water waves. ***To cite this article: A.-M. Matei, C. R. Acad. Sci. Paris, Ser. I 335 (2002) 597–602.***

Le problème de Neumann pour les frontières libres en deux dimensions

Résumé Nous généralisons au cas avec tension superficielle un résultat de H. Lewy concernant la régularité a priori des solutions en ondes progressives pour le problème d'écoulements à surface libre. ***Pour citer cet article : A.-M. Matei, C. R. Acad. Sci. Paris, Ser. I 335 (2002) 597–602.***

Version française abrégée

Soit Ω un domaine dans $\mathbb{R}^2$ dans lequel on considère l'équation de Laplace $\Delta\phi = 0$ avec une condition de Neumann sur le bord de Ω. Un sous-ensemble Γ du bord de Ω est une « frontière libre » si une solution de l'équation de Laplace satisfait une condition supplémentaire sur Γ. Ce type de problème apparaît dans de nombreux contextes physiques, plus particulièrement dans les problèmes de frontière libre en hydrodynamique. Il est admis que pour des nombreuses conditions supplémentaires, le sous-ensemble Γ ne peut pas être arbitraire. Il doit au moins posséder une certaine régularité a priori. Ce phénomène est mis en évidence dans un théorème de H. Lewy [6] concernant la régularité a priori des solutions en ondes progressives pour le problème d'écoulements à surface libre. Son résultat porte sur le cas où la tension superficielle est nulle.

La principale contribution de ce travail est d'étendre son résultat au problème de frontière libre avec tension superficielle. Soit ϕ le potentiel des vitesses pour un fluide parfait stationaire en deux dimensions d'espace et supposons que la frontière libre d'une région du fluide soit localement donnée par

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$\Gamma = \{(x_1, \eta(x_1))\}$. Alors les fonctions $\phi(x_1, x_2)$ et $\eta(x_1)$ satisfont les équations de Bernoulli :

$$\begin{cases} \Delta\phi = 0 & \text{dans } \Omega, \\ \partial_N \phi = 0 & \text{sur } \{x_2 = \eta(x_1)\}, \\ \frac{1}{2}|\nabla\phi|^2 + g\eta - \frac{c^2}{2} + \sigma H(\Gamma) = 0 & \text{sur } \{x_2 = \eta(x_1)\}, \end{cases}$$

où N est la normale unitaire extérieure à Γ, c est la constante de Bernoulli, et $H(\Gamma)$ est la courbure moyenne de Γ.

THÉORÈME 1 (Lewy [6]). – *Supposons que $\sigma = 0$ et que $\eta \in C^1$ au voisinage d'un point $p \in \Gamma$, alors η et ϕ sont C^ω dans un voisinage de p.*

Le théorème est précis car il y a des solutions telles que $\eta \in \mathrm{Lip}(\mathbb{R}) \setminus C^1(\mathbb{R})$. En effet, il s'agit des ondes extrêmes de Stokes qui ont des crêtes Lipschitz d'angle $2\pi/3$ aux points $p = (x_1, x_2)$ où $\eta(x_1) = c^2/2g$ [2].

Ces dernières années il y a eu un intérêt croissant pour les problèmes de frontière libre dans lesquels l'effet de la tension superficielle est non négligeable (c'est-à-dire $\sigma > 0$ dans les équations de Bernoulli). L'objet de ce travail est de prouver un résultat de régularité a priori similaire dans ce cas. En fait on démontre un résultat général pour les solutions du problème :

$$\begin{cases} \Delta u = 0 & \text{dans } \Omega, \\ \partial_N u = 0 & \text{sur } \Gamma, \\ G(x_1, x_2, \nabla u) + \sigma H(\Gamma) = 0 & \text{sur } \Gamma, \end{cases}$$

où la fonction G est $C^{k,\alpha}$, $k \geqslant 1$ (respectivement C^ω) dans les variables (x_1, x_2, q).

THÉORÈME 2. – *Supposons que $\eta \in C^2$, $u \in C^{2,\mu}(\Omega \cup \Gamma)$ $(\mu > 0)$, localement dans un voisinage d'un point $p \in \Gamma$ et que $G(\cdot, \cdot, 0) + \sigma H(\Gamma) \neq 0$ au point p. Alors η et u sont $C^{k+2,\alpha}$ (respectivement C^ω) dans un voisinage (possiblement plus petit) de p.*

Le problème hydrodynamique d'écoulements à surface libre est donné par

$$G(x_1, x_2, p) = \frac{1}{\sigma}\left(\frac{1}{2}p^2 + g\eta - \frac{c^2}{2}\right)$$

qui correspond au cas analytique du théorème.

Des méthodes pour résoudre des problèmes similaires ont été développées dans d'autres contextes, plus particulièrement dans les travaux de Kinderlehrer, Nirenberg et Spruck [3–5]. Ces auteurs ont considéré principalement des équations similaires, mais avec première condition sur le bord la condition de Dirichlet où la condition $\partial_n \phi = 0$ (pour un champ de vecteurs constant $X = \partial_n$). Leur travaux ont été étendus par Caffarelli dans une série de papiers portant sur les problèmes de frontière libre en magnéto-hydrodynamique et du confinement de plasma.

Pour prouver le Théorème 2, on transforme d'abord le problème de Neumann pour u en un problème de Dirichlet pour une conjuguée harmonique de u. Dans cette nouvelle formulation, on applique la méthode introduite dans [3]. On utilise la condition de Dirichlet pour faire un changement de variables qui transforme (localement) notre problème en un problème non-linéaire elliptique dans un demi-plan. On peut appliquer alors des résulats de régularité classiques [1,7] pour conclure.

Notre travail est généralisé en toute dimension dans un travail en collaboration avec Craig, en cours de préparation. A présent le problème général de Neumann sans tension superficielle reste ouvert.

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1. Introduction

Let Ω be a domain in $\mathbb{R}^2$ in which we are to solve Laplace's equation. In this work we will consider the problem with Neumann boundary conditions imposed on the boundary of Ω. A subset Γ of the boundary is a "free boundary" if a solution to Laplace's equation satisfies an additional boundary condition on Γ. Such problems arise in many physical contexts, most classically in free boundary problems in hydrodynamics. It is a fact that for many cases of additional boundary conditions, the subset Γ cannot be arbitrary, but must at a minimum possess certain higher a priori regularity properties. This phenomenon is exhibited in a theorem of Lewy [6] concerning the a priori regularity of traveling waves solutions of the free surface problem of water waves. His result concerns the case of zero surface tension.

The principal contribution of the present paper is to extend his results to the free surface problem with surface tension present. Let ϕ be the velocity potential for an ideal fluid in two dimensions which is stationary, and suppose that the free surface of a fluid region is given (locally) by $\Gamma = \{(x_1, \eta(x_1)\}$. Then the functions $\phi(x_1, x_2)$ and $\eta(x_1)$ satisfy the classical equations of Bernoulli

$$\begin{cases} \Delta\phi = 0 & \text{in } \Omega, \\ \partial_N \phi = 0 & \text{on } \{x_2 = \eta(x_1)\}, \\ \frac{1}{2}|\nabla\phi|^2 + g\eta - \frac{c^2}{2} + \sigma H(\Gamma) = 0 & \text{on } \{x_2 = \eta(x_1)\}, \end{cases} \tag{1}$$

where N is the unit exterior normal to Γ, c is the Bernoulli constant, and $H(\Gamma)$ is the mean curvature of Γ.

THEOREM 1 (Lewy [6]). – *Suppose that $\sigma = 0$ and that $\eta \in C^1$ near a point $p \in \Gamma$, then in fact η and ϕ are real analytic in a neighborhood of p.*

The theorem is sharp, since there are solutions of (1) for which $\eta \in \mathrm{Lip}(\mathbb{R}) \setminus C^1(\mathbb{R})$. Indeed these are the famous Stokes waves of extremal forms, known to possess Lipschitz crests of open angle $2\pi/3$ at points $p = (x_1, x_2)$ at which $\eta(x_1) = c^2/2g$ [2].

Over the last decade there has been an increased interest in free surface problems where the effects of surface tension are not negligible, meaning $\sigma > 0$ in Eq. (1). The object of this paper is to prove a similar a priori regularity result in this case. In fact we are able to prove quite a general result in the presence of surface tension addressing solutions of an equation of the form

$$\begin{cases} \Delta u = 0 & \text{in } \Omega, \\ \partial_N u = 0 & \text{on } \Gamma, \\ G(x_1, x_2, \nabla u) + \sigma H(\Gamma) = 0 & \text{on } \Gamma, \end{cases}$$

where the function $G(x_1, x_2, q)$ is $C^{k,\alpha}$, $k \geqslant 1$ (respectively C^ω) in the variables (x_1, x_2, q).

THEOREM 2. – *Suppose that $\eta \in C^2$, $u \in C^{2,\mu}(\Omega \cup \Gamma)$ $(\mu > 0)$, locally in a neighborhood of a point $p \in \Gamma$ and that $G(\cdot,\cdot,0) + \sigma H(\Gamma) \neq 0$ at p. Then in fact η and u are $C^{k+2,\alpha}$ (respectively C^ω) in a (possibly smaller) neighborhood of p.*

The hydrodynamic problem of free surface water waves is given by $G(x_1, x_2, p) = \frac{1}{\sigma}(\frac{1}{2}p^2 + g\eta - \frac{c^2}{2})$, which invokes the real analytic case of the theorem. This result does not depend upon the nondegeneracency of $\partial_N u$. In fact it holds independently of this.

It remains a question in higher dimensions whether the Neumann problem for free surfaces admits the a priori regularity results of the kind that appear in Theorems 1 and 2.

Techniques for addressing related free boundary problems have appeared in many contexts, in particular in the foundational work of Kinderlehrer, Nirenberg and Spruck [3–5]. These authors have principally addressed the Dirichlet problem for equations similar to (1), or to problems involving as a first boundary condition the boundary condition $\partial_n \phi = 0$ for a given fixed vector field $X = \partial_n$. This work has been further developed in a series of papers of Caffarelli which have been directed in particular at free boundary problems arising in magneto-hydrodynamics and the plasma confinement problem.

The Neumann problem arises in particular in the context of hydrodynamics, but as well in a geometric form in the question of free boundaries at the contact line of minimal surfaces with hypersurfaces, of regularity $C^{k,\alpha}$, C^{ω} respectively. In fact the original 1952 paper of Lewy [6] represents a specialization of his results for minimal surfaces to the hydrodynamic equations.

Our own work is extended to Neumann free boundary problems in n-dimensions with surface tension, in fourthcoming work with Craig. At present, the general Neumann problem without surface tension remains open.

2. Main result

Let Ω be a domain in $\mathbb{R}^2$ and u a harmonic function in Ω. Let Γ be a connected open subset of $\partial\Omega$; Γ is not arbitrary but u satisfies an overdetermined boundary value problem on Γ:

$$\begin{cases} \Delta u = 0 & \text{in } \Omega, \\ \partial_N u = 0 & \text{on } \Gamma, \\ G(x_1, x_2, \nabla u) + \sigma H(\Gamma) = 0 & \text{on } \Gamma, \end{cases} \tag{2}$$

where the function $G(x_1, x_2, q)$ is $C^{k,\alpha}$, $k \geqslant 1$ (respectively C^{ω}) in the variables (x_1, x_2, q).

THEOREM 2. – *Suppose that $\eta \in C^2$, $u \in C^{2,\mu}(\Omega \cup \Gamma)$ $(\mu > 0)$, locally in a neighborhood of a point $p \in \Gamma$ and that $G(\cdot,\cdot,0) + \sigma H(\Gamma) \neq 0$ at p. Then in fact η and u are $C^{k+2,\alpha}$ (respectively C^{ω}) in a (possibly smaller) neighborhood of p.*

Proof of Theorem 2. – Let v be a conjugate harmonic function of u. Since $u \in C^{2,\mu}(\Omega \cup \Gamma)$ it follows that $v \in C^{2,\mu}(\Omega \cup \Gamma)$ near p. Modulo subtraction of a constant, v is a solution of the Dirichlet problem

$$\begin{cases} \Delta v = 0 & \text{in } \Omega, \\ v = 0 & \text{on } \Gamma, \\ \widetilde{G}(x_1, x_2, \nabla v) + \sigma H(\Gamma) = 0 & \text{on } \Gamma, \end{cases} \tag{3}$$

where $\widetilde{G}(x_1, x_2, \nabla v) = G(x_1, x_2, J\nabla v)$, $J = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$.

We can apply now the methods introduced in [3]. We use the Dirichlet boundary condition to make a change of variables so as to transform $\Omega \cup \Gamma$ locally in a half space and problem (3) into an elliptic problem with nonlinear boundary conditions. We show this boundary condition to be coercive. This permits the application of known elliptic regularity theory.

Consider a coordinate system such that $p = 0$ and the x_2-axis is normal to Γ at p and pointing into Ω. We denote by v_1, v_2 the partial derivatives of v with respect to x_1, respectively x_2. Then $v_1(0) = 0$ and the assumption on G implies that $\nabla v(0) \neq 0$, i.e., $v_2(0) \neq 0$. We may assume for definiteness that $v_2(0) = 1$.

Since Γ is a level set for v it follows that in fact Γ (and η) is $C^{2,\mu}$ in a neighborhood of 0 and we can extend v to a full neighborhood of 0 in $\mathbb{R}^2$ as a one-to-one $C^{2,\mu}$ function. In a neighborhood of the origin we consider the partial hodograph transformation [3]

$$x = (x_1, x_2) \mapsto y = (x_1, v).$$

Since $v_2(0) \neq 0$, this transformation is one to one and maps locally $\Omega \mapsto U \subset \{y_2 \geqslant 0\}$ and $\Gamma \mapsto S \subset \{y_2 = 0\}$, thus in effect transforming the original problem (3) to a problem in new dependent and independent variables, posed on a domain such that the image of the free boundary Γ is a subset of the fixed boundary $\{y_2 = 0\}$.

The associated hodograph transform function $x_2 = \psi(y)$ is of class $C^{2,\mu}$ locally in $U \cup S$. The x-derivatives of v can be expressed in terms of the y-derivatives of ψ by the formulae

$$\frac{\partial}{\partial x_1} = \frac{\partial}{\partial y_1} + v_1 \frac{\partial}{\partial y_2}, \qquad \frac{\partial}{\partial x_2} = v_2 \frac{\partial}{\partial y_2},$$

and we have

$$v_1 = -\frac{\psi_1}{\psi_2}, \qquad v_2 = \frac{1}{\psi_2},$$

where ψ_1, ψ_2 denote the derivatives of ψ w.r.t. y_1, y_2. Differentiating again we get

$$v_{11} = -\left(\frac{\psi_1}{\psi_2}\right)_1 + \frac{\psi_1}{\psi_2}\left(\frac{\psi_1}{\psi_2}\right)_2, \qquad v_{22} = \frac{1}{\psi_2}\left(\frac{1}{\psi_2}\right)_2.$$

Note that since Γ can be parametrized by $x_2 = \psi(x_1, 0)$ the mean curvature of Γ can be written in terms of y_1, y_2 and ψ as

$$H = \frac{\partial}{\partial y_1}\left(\frac{\psi_1}{\sqrt{1+\psi_1^2}}\right) = \frac{\psi_{11}}{(1+\psi_1^2)^{3/2}}.$$

Furthermore, the second order linear equation for v is transformed to a second order nonlinear equation for ψ:

$$\begin{cases} -\left(\dfrac{\psi_1}{\psi_2}\right)_1 + \dfrac{\psi_1}{\psi_2}\left(\dfrac{\psi_1}{\psi_2}\right)_2 + \dfrac{1}{\psi_2}\left(\dfrac{1}{\psi_2}\right)_2 = 0 & \text{in } U, \\ \widetilde{G}\left(y_1, \psi(y), \left(-\dfrac{\psi_1}{\psi_2}, \dfrac{1}{\psi_2}\right)\right) + \sigma \dfrac{\psi_{11}}{(1+\psi_1^2)^{3/2}} = 0 & \text{on } S. \end{cases} \tag{4}$$

LEMMA 3. – *Problem* (4) *is elliptic and coercive.*

Proof of Lemma 3. – Since we work locally and ellipticity and coercivity are open conditions, it suffices to verify them only at the origin. We will use the notation and the system of weights introduced in [1]. We associate the weight 2 to the variable ψ, and the weight 0 to both the equation and the boundary condition. We consider the linear homogenous problem obtained by linearizing (4) at the origin and then taking only the terms of order $2+0=2$ in both the equation and the boundary condition. If $\Psi = \delta\psi$ we obtain:

$$\begin{cases} -\dfrac{1}{\psi_2(0)}\Psi_{11} + \dfrac{\psi_1(0)\psi_2(0) - (\psi_1^2(0)+1)}{\psi_2^3(0)}\Psi_{22} - \dfrac{\psi_1(0)}{\psi_2^2(0)}\Psi_{12} = 0 & \text{in } U, \\ \dfrac{1}{(1+\psi_1^2(0))^{3/2}}\Psi_{11} = 0 & \text{on } S. \end{cases} \tag{5}$$

The quasilinear problem (4) is elliptic and coercive if the linear homogenous problem (5) is elliptic and has no nontrivial exponential solution of the form $\Psi = e^{i\lambda y_1}\phi(y_2)$, $\lambda \neq 0$, which decays as $y_2 \to \infty$. Note that since $v_1(0) = 0$ and $v_2(0) = 1$ we have $\psi_1(0) = 0$, $\psi_2(0) = 1$ and problem (5) becomes

$$\begin{cases} \Psi_{11} + \Psi_{22} = 0 & \text{in } U, \\ \Psi_{11} = 0 & \text{on } S. \end{cases} \tag{6}$$

Ellipticity is now obvious. To check coercivity we replace $\Psi = \mathrm{e}^{\mathrm{i}\lambda y_1}\phi(y_2)$ in (6) and obtain

$$\begin{cases} -\lambda^2\phi + \phi'' = 0 \quad \text{in } y_2 \geqslant 0, \\ \phi(0) = 0. \end{cases}$$

The general solution for the equation is $\phi(y_2) = A\,\mathrm{e}^{|\lambda| y_2} + B\,\mathrm{e}^{-|\lambda| y_2}$. Since we want only solutions which decay for large y_2, we must have $A = 0$. The condition $\phi(0) = 0$ implies $B = 0$, which implies directly that $\phi = 0$ and $\Psi = 0$. This ends the proof of the lemma. □

We can apply now regularity theory for nonlinear elliptic boundary value problems. If G is $\mathrm{C}^{k,\alpha}$ ($k \geqslant 1$), so is $\widetilde{G}$, and Theorem 11.1 of [1] implies that ψ is $\mathrm{C}^{k+2,\min(\alpha,\mu)}$. In particular, ψ is $\mathrm{C}^{2,\alpha}$. Applying again the same theorem we obtain that ψ is $\mathrm{C}^{k+2,\alpha}$ in $y_1 \geqslant 0$ near the origin. Hence v and u are locally $\mathrm{C}^{k+2,\alpha}$ near p. Since $\eta(x_1) = \psi(y_1, 0)$ it follows that η is also $\mathrm{C}^{k+2,\alpha}$ in a neighborhood of p.

Finally, if G is C^ω, so is $\widetilde{G}$, and using the results in Chapter 6.7 of [7] we obtain that ψ is C^ω in $y_1 \geqslant 0$ near the origin. Hence η, u, v are locally C^ω. This ends the proof of the theorem. □

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