A sharp Sobolev inequality on Riemannian manifolds

Yan Yan Li [a], Tonia Ricciardi [b]

[a] Department of Mathematics, Rutgers University, 110 Frelinghuysen Rd., Piscataway, NJ 08854-8019, USA

[b] Dipartimento di Matematica e Applicazioni, Università di Napoli Federico II, Via Cintia, 80126 Napoli, Italy

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Abstract   We outline our results in [11] concerning some sharp Sobolev inequalities on Riemannian manifolds. Our inequalities emphasize the role of scalar curvature in this context. ***To cite this article: Y.Y. Li, T. Ricciardi, C. R. Acad. Sci. Paris, Ser. I 335 (2002) 519–524.***
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Une inégalité de Sobolev optimale sur les variétés riemanniennes

Résumé   On présente des inégalités de Sobolev optimales sur les variétés riemanniennes. Ces inégalités contiennent un terme de courbure scalaire. Les démonstrations détaillées sont contenues dans [11]. ***Pour citer cet article : Y.Y. Li, T. Ricciardi, C. R. Acad. Sci. Paris, Ser. I 335 (2002) 519–524.***
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Soit $n \geqslant 3$. On note $2^* = 2n/(n-2)$ et

$$K^{-1} = \inf\left\{\frac{\|\nabla u\|_{L^2(\mathbb{R}^n)}}{\|u\|_{L^{2^*}(\mathbb{R}^n)}} : u \in L^{2^*}(\mathbb{R}^n) \setminus \{0\}, \ |\nabla u| \in L^2(\mathbb{R}^n)\right\}$$

la constante optimale de Sobolev. On démontre :

THÉORÈME 0.1 (Résultat principal). – *Soit (M, g) une variété riemannienne compacte, régulière, sans bord, de dimension $n \geqslant 6$. Alors il existe une constante $A > 0$, qui dépend seulement de (M, g), telle que pour toute fonction $u \in H^1(M)$ il résulte :*

$$\|u\|^2_{L^{2^*}(M,g)} \leqslant K^2 \int_M \left\{|\nabla_g u|^2 + c(n) R_g u^2\right\} \mathrm{d}v_g + A\|u\|^2_{L^{\bar{r}}(M,g)},$$

où $c(n) = (n-2)/[4(n-1)]$, $\bar{r} = 2n/(n+2)$ et où R_g dénote la courbure scalaire de g.

Le Théorème 0.1 est optimal dans le sens suivant : on ne peut pas remplacer K par un nombre plus petit, ni R_g par $R_g + f$, où $f \in C^0$ est négative à un point. En outre, si (M, g) n'est pas localement conformément plate, on ne peut pas remplacer $\bar{r}$ par un nombre plus petit.

E-mail addresses: yyli@math.rutgers.edu (Y.Y. Li); tonia.ricciardi@unina.it (T. Ricciardi).

Sur les variétés localement conformément plates, on a le résultat suivant :

THÉORÈME 0.2. – *Soit (M, g) une variété riemannienne compacte, régulière, sans bord, localement conformément plate, de dimension $n \geqslant 3$. Alors il existe une constante $A > 0$, qui dépend seulement de (M, g), telle que pour toute fonction $u \in \mathrm{H}^1(M)$ il résulte* :

$$\|u\|_{\mathrm{L}^{2^*}(M,g)}^2 \leqslant K^2 \int_M \left\{ |\nabla_g u|^2 + c(n) R_g u^2 \right\} \mathrm{d} v_g + A \|u\|_{\mathrm{L}^1(M,g)}^2.$$

Considerable work has been devoted to the analysis of sharp Sobolev-type inequalities, very often in connection with concrete problems from geometry and physics. See, e.g., Aubin [1,3], Talenti [15], Brezis and Nirenberg [6], Brezis and Lieb [5], Carlen and Loss [7], Struwe [14], Escobar [9], and references therein.

For $n \geqslant 3$ we let $2^* = 2n/(n-2)$, and we denote by K the optimal Sobolev constant determined by Aubin [1] and Talenti [15]:

$$K^{-1} = \inf\left\{ \frac{\|\nabla u\|_{\mathrm{L}^2(\mathbb{R}^n)}}{\|u\|_{\mathrm{L}^{2^*}(\mathbb{R}^n)}} : u \in \mathrm{L}^{2^*}(\mathbb{R}^n) \setminus \{0\}, \ |\nabla u| \in \mathrm{L}^2(\mathbb{R}^n) \right\}, \tag{1}$$

where $K^2 = 4/[n(n-2)\sigma_n^{2/n}]$ and where σ_n is the volume of the standard n-sphere. We denote by $\{U_{y,\lambda}; y \in \mathbb{R}^n, \ \lambda > 0\}$ the set of minimizers for (1), namely:

$$U_{y,\lambda}(x) = \lambda^{(n-2)/2} U\left(\lambda(x - y)\right),$$

$$U(x) = U_{0,1}(x) = \left(\frac{1}{1 + \bar{\lambda}^2 |x|^2} \right)^{(n-2)/2},$$

where $\bar{\lambda}^2 = [n(n-2)]^{-1} K^{-2}$.

Motivated by the Yamabe problem, Aubin [1] showed that K is the optimal Sobolev constant on any Riemannian manifold (M, g) of dimension $n \geqslant 3$. Namely, he proved that if (M, g) has constant sectional curvature, then there exists some $A > 0$ depending on (M, g) only, such that

$$\|u\|_{\mathrm{L}^{2^*}(M,g)}^2 \leqslant K^2 \|\nabla_g u\|_{\mathrm{L}^2(M,g)}^2 + A \|u\|_{\mathrm{L}^2(M,g)}^2 \quad \forall u \in \mathrm{H}^1(M), \tag{2}$$

and that on a general (M, g) a weaker version of (2) holds, where for any $\varepsilon > 0$, K is replaced by $K + \varepsilon$ and where A is allowed to depend on ε. Finally, Aubin conjectured that (2) should hold on any Riemannian manifold of dimension $n \geqslant 3$. This conjecture was proved for general manifolds by Hebey and Vaugon [10].

In general, the lower order norm on the right hand side of (2) may not be replaced by any weaker L^p-norm. Indeed, a standard check based on the family of functions $\{U_{y,\lambda}\}$ shows that if $n \geqslant 6$ and if the scalar curvature is positive somewhere, then the L^2-norm in (2) is optimal among L^p-norms. Our main result clarifies the role of scalar curvature in this context. Indeed, we have:

THEOREM 0.1 (Main result). – *Let (M, g) be a smooth compact Riemannian manifold without boundary of dimension $n \geqslant 6$. There exists a constant $A > 0$, depending on (M, g) only, such that for all $u \in \mathrm{H}^1(M)$ there holds*:

$$\|u\|_{\mathrm{L}^{2^*}(M,g)}^2 \leqslant K^2 \int_M \left\{ |\nabla_g u|^2 + c(n) R_g u^2 \right\} \mathrm{d} v_g + A \|u\|_{\mathrm{L}^{\bar{r}}(M,g)}^2, \tag{3}$$

where 2^* and K are defined above, $c(n) = (n-2)/[4(n-1)]$, $\bar{r} = 2n/(n+2)$, R_g is the scalar curvature of g.

We point out that our proof of Theorem 0.1 does not make any use of inequality (2). On the other hand, when $n \geqslant 6$, (2) is a direct consequence of (3).

Remark 1 (Sharpness). – Theorem 0.1 is *sharp*, in the sense that one can neither replace K by any smaller number, nor replace R_g by any $R_g + f$ with $f \in C^0$ negative somewhere. Moreover, if (M, g) is not locally conformally flat, one cannot replace $\bar{r}$ by any smaller number.

Remark 2. – The term $\int_M \{|\nabla u|^2 + c(n) R_g u^2 \, \mathrm{d}v_g\}$ on the right hand side of (3) is the quadratic form associated to the conformal Laplacian and it is conformally invariant.

On the other hand, using conformal invariance, we derive the following result for locally conformally flat manifolds:

THEOREM 0.2. – *Let (M, g) be a smooth compact locally conformally flat Riemannian manifold without boundary of dimension $n \geqslant 3$. There exists a constant $A > 0$, depending on (M, g) only, such that for all $u \in H^1(M)$ there holds*:

$$\|u\|^2_{L^{2^*}(M,g)} \leqslant K^2 \int_M \{|\nabla_g u|^2 + c(n) R_g u^2\} \, \mathrm{d}v_g + A \|u\|^2_{L^1(M,g)}.$$

In what follows, we denote $\| \cdot \|_q = \| \cdot \|_{L^q(M,g)}$.

Remark 3. – Denoting by Y_g the Yamabe functional for (M, g), namely:

$$Y_g(u) = \frac{\int_M \{|\nabla_g u|^2 + c(n) R_g u^2\} \, \mathrm{d}v_g}{\|u\|^2_{2^*}}, \quad u \in H^1(M) \setminus \{0\},$$

it is clear that (3) is equivalent to the following lower bound for Y_g, namely

$$Y_g(u) \geqslant K^{-2} - A \frac{\|u\|^2_{\bar{r}}}{\|u\|^2_{2^*}} \quad \forall u \in H^1(M) \setminus \{0\}. \tag{4}$$

In view of Remark 3, we can readily derive the sharpness of Theorem 0.1, as stated in Remark 1, from some well-known expansions by Aubin [2] for Y_g: denoting by $\xi_{P,\lambda}$, $P \in M$, $\lambda > 0$ the pullback of $U_{0,\lambda}$ defined above by the exponential map, i.e.:

$$\xi_{P,\lambda}(x) = \left(\frac{\lambda}{1 + \lambda^2 \operatorname{dist}^2_g(x, P)}\right)^{(n-2)/2}, \quad x \in B_\rho(P),$$

and letting $\tilde{\xi}_{P,\lambda} = \eta \xi_{P,\lambda}$, where η is a smooth cutoff function supported near P, then as $\lambda \to \infty$:

$$Y_g(\tilde{\xi}_{P,\lambda}) = \begin{cases} K^{-2} - \gamma_n |W_g(P)|^2 \lambda^{-4} + o(\lambda^{-4}) & \text{if } n \geqslant 7, \\ K^{-2} - \gamma_n |W_g(P)|^2 \lambda^{-4} \log \lambda + o(\lambda^{-4} \log \lambda) & \text{if } n = 6, \end{cases} \tag{5}$$

where $\gamma_n > 0$ is a dimensional constant, $W_g(P)$ is the Weyl tensor of g at P. Such expansions also suggest the existence of refined versions of (3) involving the Weyl tensor. Similarly, in view of the work of Schoen [13], we expect the existence of sharp Sobolev inequalities involving global geometric quantities such as "mass" for locally conformally flat manifolds and for manifolds of dimension $3 \leqslant n \leqslant 5$ with positive scalar curvature.

In the rest of this Note we outline of the proof of Theorem 0.1, and for simplicity we restrict ourselves to the case $n \geqslant 7$.

Since the "correction term" $\|\tilde{\xi}_{P,\lambda}\|_{\bar{r}}$ is of the order λ^{-4}, the expansion (5) implies that inequality (3) holds for the family $\{t\tilde{\xi}_{P,\lambda}\}$ defined above, uniformly in $t > 0$, $P \in M$, $\lambda > 0$. To obtain (3) for general $u \in H^1(M)$, we argue by contradiction, and we take a *global* approach. Negating (4), we assume that for

some sequence $\alpha \to +\infty$ there holds:

$$\inf\left\{ Y_g(u) + \alpha \frac{\|u\|_{\bar{r}}^2}{\|u\|_{2^*}^2} : u \in \mathrm{H}^1(M) \setminus \{0\} \right\} < K^{-2}. \tag{6}$$

By standard arguments, (6) implies that $\inf_{\mathrm{H}_0^1(M)\setminus\{0\}}\{Y_g + \alpha \| \cdot \|_{\bar{r}}^2 / \| \cdot \|_{2^*}^2\}$ is attained at some minimizer $u_\alpha \in \mathrm{H}^1(M)$ satisfying $u_\alpha \geqslant 0$, $\int_M u_\alpha^{2^*} \, \mathrm{d}v_g = 1$, solution to the Euler–Lagrange equation:

$$-\Delta_g u_\alpha + c(n) R_g u_\alpha + \alpha \|u_\alpha\|_{\mathrm{L}^r(M)}^{2-r} u_\alpha^{r-1} = \ell_\alpha u_\alpha^{2^*-1} \quad \text{on } M. \tag{7}$$

The proof consists in obtaining the contradiction $\alpha \leqslant C$ by blowup analysis of u_α as $\alpha \to +\infty$. We denote:

$$x_\alpha \in M: \quad u_\alpha(x_\alpha) = \max_M u_\alpha, \quad \mu_\alpha^{(n-2)/2} := u_\alpha(x_\alpha)^{-1},$$

and without loss of generality we assume that x_α converges to a point in M. It is a standard fact that u_α concentrates "in energy" at a single point. Namely:

LEMMA 0.3. – As $\alpha \to +\infty$, we have:

$$\mu_\alpha \to 0, \quad \|\nabla_g u_\alpha\|_2^2 \to K^{-2}, \quad \alpha \|u_\alpha\|_{\bar{r}}^2 \to 0,$$

$$\left\|\nabla_g(u_\alpha - \xi_{x_\alpha, \mu_\alpha^{-1}})\right\|_{\mathrm{L}^2(B_{\delta_0}(x_\alpha))} + \|u_\alpha - \xi_{x_\alpha, \mu_\alpha^{-1}}\|_{\mathrm{L}^{2^*}(B_{\delta_0}(x_\alpha))} \to 0,$$

$$\mu_\alpha^{(n-2)/2} u_\alpha\big(\exp_{x_\alpha}(\mu_\alpha \cdot)\big) \to U(\cdot) \quad \text{in } \mathrm{C}_{\mathrm{loc}}^2(\mathbb{R}^n),$$

where $\delta_0 > 0$ is small and fixed and depends only on (M, g).

The $\mathrm{C}_{\mathrm{loc}}^2(\mathbb{R}^n)$-convergence as stated in Lemma 0.3, together with a change of variables, implies the lower bound:

$$\|u_\alpha\|_{\bar{r}} \geqslant C^{-1} \mu_\alpha^2. \tag{8}$$

In order to derive a contradiction from (6) we need to estimate the rates of the asymptotic behaviors stated in Lemma 0.3. Towards this goal, the following *pointwise estimate* is a key step, which allows to neglect "boundary values":

LEMMA 0.4 (Pointwise estimate). – *The following estimate holds*:

$$u_\alpha(x) \leqslant C \mu_\alpha^{(n-2)/2} \mathrm{dist}_g^{2-n}(x, x_\alpha), \quad \forall x \in M. \tag{9}$$

Estimate (9) implies that the boundary values of u_α on $B_{\delta_\alpha}(x_\alpha)$ for some suitable $\delta_\alpha \in [\delta_0/2, \delta_0]$ are negligible in the sense of $\mathrm{L}^\infty(B_{\delta_\alpha}(x_\alpha))$ and of $\mathrm{H}^1(B_{\delta_\alpha}(x_\alpha))$.

Proof (Outline of the proof of Lemma 0.4). – We derive (9) along the line of Li and Zhu [12], adding several new ingredients. We show that setting

$$V_\alpha := \begin{cases} c(n) R_g + \alpha(\|u_\alpha\|_{\bar{r}}/u_\alpha)^{2-\bar{r}} & \text{if } u_\alpha > 0, \\ 1 & \text{if } u_\alpha = 0, \end{cases}$$

the operators $-\Delta_g + V_\alpha$ are coercive on $\mathrm{H}^1(M)$, with coercivity constant independent of α. Consequently, φ_α is well-defined by the equation

$$-\Delta_g \varphi_\alpha + V_\alpha \varphi_\alpha = \mu_\alpha^{(n-2)/2} \delta_{x_\alpha} \quad \text{on } M$$

and it satisfies

$$C^{-1} \mu_\alpha^{(n-2)/2} \mathrm{dist}_g^{2-n}(x, x_\alpha) \leqslant \varphi_\alpha(x) \leqslant C \mu_\alpha^{(n-2)/2} \mathrm{dist}_g^{2-n}(x, x_\alpha). \tag{10}$$

Setting $\widehat{g} = \varphi_\alpha^{4/(n-2)} g$, we have by conformal transformation properties

$$\begin{cases} -\Delta_{\widehat{g}} \dfrac{u_\alpha}{\varphi_\alpha} \leqslant \ell_\alpha \left(\dfrac{u_\alpha}{\varphi_\alpha} \right)^{2^*-1} & \text{on } M \setminus \{x_\alpha\}, \\[3mm] \displaystyle\int_{B_{R\mu_\alpha}(x_\alpha)} \left(\dfrac{u_\alpha}{\varphi_\alpha} \right)^{2^*} \mathrm{d}v_{\widehat{g}} \leqslant \varepsilon_0, \end{cases}$$

where ε_0 is chosen small and R is large. By Moser iterations on $M \setminus B_{R\mu_\alpha}(x_\alpha)$, we conclude $u_\alpha/\varphi_\alpha \leqslant C$ in $M \setminus B_{R\mu_\alpha}(x_\alpha)$, which suffices to conclude (9). $\square$

Another ingredient is an *energy estimate*, obtained in the spirit of Bahri and Coron [4]. We denote by $\pi_\alpha : \mathrm{H}^1(B_{\delta_\alpha}) \to \mathrm{H}^1_0(B_{\delta_\alpha})$ the standard projection operator and we let $\| \cdot \| := \| \cdot \|_{\mathrm{H}^1_0(B_{\delta_\alpha})} := \|\nabla_g \cdot \|_{\mathrm{L}^2(B_{\delta_\alpha})}$. We select $t_\alpha \in [1/2, 3/2]$, $P_\alpha \in B_{\mu_\alpha \delta_0/2}(x_\alpha)$, $\lambda_\alpha \in [1/(2\mu_\alpha), 3/(2\mu_\alpha)]$ such that

$$\begin{aligned} & \big\| \pi_\alpha(u_\alpha - t_\alpha \xi_{P_\alpha, \lambda_\alpha}) \big\| \\ & = \min\left\{ \big\| \pi_\alpha(u_\alpha - t\xi_{P,\lambda}) \big\| : t_\alpha \in \left[\frac{1}{2}, \frac{3}{2}\right], \ P_\alpha \in B_{\mu_\alpha \delta_0/2}(x_\alpha), \ \lambda_\alpha \in \left[\frac{1}{2\mu_\alpha}, \frac{3}{2\mu_\alpha}\right] \right\}. \end{aligned}$$

Setting $w_\alpha := \pi_\alpha(u_\alpha - t_\alpha \xi_{P_\alpha, \lambda_\alpha})$, our energy estimate is given by:

LEMMA 0.5 (Energy estimate). – *As $\alpha \to +\infty$, we have*:

$$\|w_\alpha\| \leqslant C\big(\mu_\alpha^2 + \big(1 + \mu_\alpha^{-2+\beta}\big)\alpha \|u_\alpha\|_{\tilde{r}}^2\big), \tag{11}$$

where $\beta = (n-6)(n-2)/[2(n+2)] > 0$ is strictly positive, *since $n \geqslant 7$.*

Lemmas 0.4 and 0.5 imply the following lower bound for the Yamabe functional evaluated at u_α:

LEMMA 0.6 (Lower bound). – *The following lower bound holds*:

$$Y_g(u_\alpha) \geqslant K^{-2} + \mathrm{o}(1)\alpha\|u_\alpha\|_{\tilde{r}}^2 - C\mu_\alpha^4. \tag{12}$$

Proof (Outline of proof). – By Taylor's expansion we have:

$$Y_g(u_\alpha) = Y_g(\tilde{\xi}_{P_\alpha, \lambda_\alpha}) + Y_g'(\tilde{\xi}_{P_\alpha, \lambda_\alpha})w_\alpha + \frac{1}{2}\big\langle Y_g''(\tilde{\xi}_{P_\alpha, \lambda_\alpha})w_\alpha, w_\alpha \big\rangle + \mathrm{o}\big(\|w_\alpha\|^2\big) + \mathrm{O}\big(\mu_\alpha^{n-2}\big), \tag{13}$$

where the error of order μ_α^{n-2} controls the "boundary part" of u_α, by Lemma 0.4. Furthermore, we have

$$\big\| Y_g'(\tilde{\xi}_{P_\alpha, \lambda_\alpha}) \big\|_{\mathrm{H}^{-1}} \leqslant C\mu_\alpha^2,$$

since $\tilde{\xi}_{P_\alpha, \lambda_\alpha}$ is a minimizer of the Sobolev quotient up to boundary errors of order μ_α^{n-2}, and since Y_g and the Sobolev quotient differ by an error of order 2. Hence, by Lemma 0.5, recalling that $\beta > 0$ (since $n \geqslant 7$), we derive that

$$\big| Y_g'(\tilde{\xi}_{P_\alpha, \lambda_\alpha})w_\alpha \big| \leqslant C\big(\mu_\alpha^4 + \mathrm{o}(1)\alpha\|u_\alpha\|_{\tilde{r}}^2\big).$$

Inserting into (13), taking into account Aubin's expansion (5), and using the coercivity property

$$\big\langle Y_g''(\tilde{\xi}_{P_\alpha, \lambda_\alpha})w_\alpha, w_\alpha \big\rangle \geqslant c_0\|w_\alpha\|^2$$

for some $c_0 > 0$ independent of α, we obtain the desired lower bound (12). $\square$

Now we are ready to conclude:

Proof of Theorem 0.1. – The contradiction assumption (6) and the lower bound (12) yield the estimate

$$\alpha \|u_\alpha\|_{\bar{r}}^2 \leqslant C\mu_\alpha^4.$$

Recalling (8), we obtain from the above that $\alpha \leqslant C$, contradiction. $\quad\square$

The case $n = 6$ is more delicate than the case $n \geqslant 7$. Indeed, it is in some sense a "limit case". We use a pointwise *lower* bound to treat the case $n = 6$, which we obtain by the maximum principle, adapting an idea in [12].

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