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L^1 contraction property for a Boltzmann equation with Pauli statistics

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Abstract We establish a L^1 contraction property for solutions to the Boltzmann equation when collisions are taken into account through the Pauli operator. The Pauli operator is a non-linear integral operator, that we consider here in full generality, without assuming relations such as the detailed balance principle. It takes into account the Pauli exclusion principle and appears especially to describe the flow of electrons and holes in some semi-conductor devices. ***To cite this article: A. Mellet, B. Perthame, C. R. Acad. Sci. Paris, Ser. I 335 (2002) 337–340.***

Propriété de contraction L^1 pour l'équation de Boltzmann avec la statistique de Pauli

Résumé Nous établissons une propriété de contraction L^1 pour les solutions de l'équation de Boltzmann lorsque les collisions sont décrites par l'opérateur de Pauli. L'opérateur de Pauli est un opérateur intégral non-linéaire, qui prend en compte le principe d'exclusion de Pauli et qui est utilisé pour décrire les flots d'électrons et de trous dans certains dispositifs semiconducteurs. On le considère ici sans hypothèse suplémentaire, telle que la relation d'équilibre en détail. ***Pour citer cet article : A. Mellet, B. Perthame, C. R. Acad. Sci. Paris, Ser. I 335 (2002) 337–340.***

1. The Pauli collision operator in semiconductor cristal

When modelling the evolution of a cloud of particles in a semiconductor device, one usually considers its distribution function $f(t,x,k)$, where t denotes the time variable, x denotes the space variable, lying in a subset Ω of $\mathbb{R}^3$, and k denotes the wave vector, lying in the first Brillouin zone B, a torus in $\mathbb{R}^3$.

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Collisions that particles may undergo when crossing the device can be described through integral operators, such as the so-called *Pauli operator*:

$$Q(f)(k) = \int_{k' \in B} \sigma(k,k')\bigl(1 - f(k)\bigr) f(k') - \sigma(k',k)\bigl(1 - f(k')\bigr) f(k)\,\mathrm{d}k', \tag{1}$$

where the cross section $\sigma(k,k')$ is non-negative, and may depend on x.

Such an operator models the collisions of particles against the semiconductor or against impurities; and the terms $1 - f(k)$ and $1 - f(k')$ take into account the Pauli exclusion principle. As a consequence, we shall deal with functions satisfying $0 \leqslant f \leqslant 1$, an invariant region for the corresponding evolution equation (*see* (4) below). We refer to [6] and references therein for more informations about the physical meaning of this operator.

When no further hypotheses are assumed concerning the cross-section $\sigma(k,k')$, very little is known about the properties of such an operator, and especially the existence of an entropy (H-theorem) is not known. Nevertheless, in [4] and [5], the following result is established:

PROPOSITION 1.1. – *Assume that there exist σ_1 and σ_2 such that*

$$0 < \sigma_1 \leqslant \sigma(k,k') \leqslant \sigma_2, \quad \forall k, k' \in B \times B.$$

Then, for all $\rho \in [0,1]$, there exists a unique $F(\rho,k) \in \mathrm{L}^\infty(B)$ which verifies

$$\begin{cases} 0 \leqslant F(\rho,k) \leqslant 1, & \displaystyle\int_B F(\rho,k)\,\mathrm{d}k = \rho, \\ Q\bigl(F(\rho)\bigr) = 0. & \end{cases}$$

Notice that the linear case, and the diffusion limit especially, was first studied by Degond, Goudon and Poupaud [2] in the case where the detailed balance principle (DBP in short) is not fulfilled.

Indeed, one can also assume that σ satisfies the following DBP:

$$\sigma(k,k')M(k) = \sigma(k',k)M(k'),$$

where $M(k) = \exp(-\mathcal{E}(k))$, with $\mathcal{E}$ a (usually convex) function describing energy distribution, is called the Maxwellian distribution. Then, it is possible to explicit these equilibrium states: They are known as the Fermi–Dirac distributions. Moreover, in this situation, Golse and Poupaud, [3], established the following entropy inequality, for increasing functions χ,

$$-\int_B Q(f)\chi(f)\,\mathrm{d}k \geqslant \alpha \bigl\| f - F(\rho) \bigr\|^2_{\mathrm{L}^2(B)}.$$

2. Contraction property

For general cross-section σ, the above relation does not hold true and an entropy principle is still lacking.

In this Note, we provide a related structure which is natural from the point of view of hyperbolic equations, see Serre [8], Dafermos [1]. It provides an alternative property: the L^1 contraction principle. We prove the following proposition:

PROPOSITION 2.1. – *Assume $\sigma(\cdot,\cdot) \geqslant 0$ and let f and $g \in \mathrm{L}^\infty(B)$ satisfy*

$$0 \leqslant f(k) \leqslant 1, \qquad 0 \leqslant g(k) \leqslant 1, \tag{2}$$

then we have, with $\operatorname{sgn}_+(s) = +1$ *if* $s > 0$, *and* $\operatorname{sgn}_+(s) = 0$ *otherwise,*

$$\int_B \big(Q(f) - Q(g)\big) \operatorname{sgn}_+(f-g)\,\mathrm{d}k \leqslant 0.$$

Note that when $g(k) = F(\rho, k)$, Proposition 2.1 yields:

$$\int_B Q(f) \operatorname{sgn}_+\big(f - F(\rho)\big)\,\mathrm{d}k \leqslant 0, \quad \forall \rho \in [0,1]. \tag{3}$$

As an application, we consider the following Boltzmann equation:

$$\partial_t f + v(k)\cdot\nabla_x f = Q(f), \quad x \in \Omega,\ k \in B,\ t \geqslant 0, \tag{4}$$

which is commonly used to describe the evolution of the distribution function f. Then the following local L^1-contraction property follows easily from Proposition 2.1:

PROPOSITION 2.2. – *Let* $f, g \in \mathrm{L}^\infty((0,+\infty)\times\Omega\times B)$ *be two solutions of* (4) *with initial data satisfying* (2). *Then we have*:

$$\partial_t |f-g| + v(k)\cdot\nabla_x |f-g| = m(t,x,k),$$

with

$$\int_B m(t,x,k)\,\mathrm{d}k \leqslant 0.$$

This structure, combined to (3) ressembles a kinetic formulation, see Perthame [7]. The study of hyperbolic limits (to the entropy solution) follows by a standard uniqueness argument for measure valued of kinetic solutions in the limit. Indeed the choice of $g = F(\bar\rho)$, $\bar\rho \in \mathbb{R}$, as mentioned above, provides all the entropy inequalities. In the full space, or when Ω is bounded and under appropriate boundary conditions (e.g., if $f|_{\partial\Omega} = g|_{\partial\Omega}$ or for specular reflections), we immediately deduce the following L^1 contraction properties:

$$\big\|f(T) - g(T)\big\|_{\mathrm{L}^1(\Omega\times B)} \leqslant \big\|f^0 - g^0\big\|_{\mathrm{L}^1(\Omega\times B)}, \quad \forall T \geqslant 0,$$
$$\big\|\partial_t f(T)\big\|_{\mathrm{L}^1(\Omega\times B)} \leqslant \big\|v(k)\cdot\nabla_x f^0 - Q\big(f^0\big)\big\|_{\mathrm{L}^1(\Omega\times B)}, \quad \forall T \geqslant 0.$$

In the full space and using the translational invariance (when σ is independent of x) we deduce a Total Variation property

$$\big\|\nabla_x f(T)\big\|_{\mathrm{L}^1(\mathbb{R}^3\times B)} \leqslant \big\|\nabla_x f^0\big\|_{\mathrm{L}^1(\mathbb{R}^3\times B)}, \quad \forall T \geqslant 0.$$

Of course the hyperbolic limit, see [3], follows from these properties using classical methods [1,7,8].

3. Proof of Proposition 2.1

We only prove Proposition 2.1, the other results being classical consequences (*see* [7] for instance).

In the following, we use the classical notations $f = f(k)$, $f' = f(k')$, ... then, we write:

$$\begin{aligned}
&\int_B \big(Q(f) - Q(g)\big)\operatorname{sgn}_+(f-g)\,\mathrm{d}k \\
&\quad = \int_B\int_B \sigma(k,k')\Big(\big(1-f(k)\big)f(k') - \big(1-g(k)\big)g(k')\Big)\operatorname{sgn}_+(f-g)(k)\,\mathrm{d}k\,\mathrm{d}k' \\
&\qquad - \int_B\int_B \sigma(k',k)\Big(\big(1-f(k')\big)f(k) - \big(1-g(k')\big)g(k)\Big)\operatorname{sgn}_+(f-g)(k)\,\mathrm{d}k\,\mathrm{d}k'.
\end{aligned}$$

Exchanging the role of variable k and k' in the last term, we get

$$\int_B \big(Q(f)-Q(g)\big)\,\mathrm{sgn}_+(f-g)\,\mathrm{d}k$$
$$= \int_B\int_B \sigma(k,k')\Big(\big(1-f(k)\big)f(k')-\big(1-g(k)\big)g(k')\Big)\Big(\mathrm{sgn}_+(f-g)(k)-\mathrm{sgn}_+(f-g)(k')\Big)\,\mathrm{d}k\,\mathrm{d}k'.$$

We now note that the term $\mathrm{sgn}_+(f-g)(k)-\mathrm{sgn}_+(f-g)(k')$ is equal to

$$\begin{array}{rl} 1 & \text{when } f(k)\geqslant g(k), \text{ and } f(k')\leqslant g(k'),\\ -1 & \text{when } f(k)\leqslant g(k), \text{ and } f(k')\geqslant g(k'),\\ 0 & \text{otherwise.}\end{array}$$

It is now easy to check, recalling the property (2), that $(1-f(k))f(k')-(1-g(k))g(k')$ is non-positive in the first case, and non-negative in the second one. Therefore the sign property follows directly.

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