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# Exact inversion of a compound conical Radon transform and a novel nuclear imaging principle

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**Abstract** A new integral transform arising from a theory of imaging based on Compton scattering is introduced and the explicit expression for its inverse is established. Its properties serve as foundation to a new nuclear emission imaging principle. ***To cite this article: M.K. Nguyen, T.T. Truong, C. R. Acad. Sci. Paris, Ser. I 335 (2002) 213–217.*** © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

**Inversion exacte d'une transformation conique de Radon composée et nouveau principe d'imagerie nucléaire**

**Résumé** Une nouvelle transformation intégrale issue de la formation d'image à partir des photons diffusés par effet Compton a été établie. Sa formule d'inversion explicite a été démontrée. Ses propriétés servent de fondement à un nouveau principe d'imagerie nucléaire. ***Pour citer cet article : M.K. Nguyen, T.T. Truong, C. R. Acad. Sci. Paris, Ser. I 335 (2002) 213–217.*** © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

## *Version française abrégée*

La transformation de Radon est la base de la tomographie par émission ou par transmission. Elle permet de reconstruire en trois dimensions une fonction décrivant un objet d'étude, tel qu'un organe humain ou un objet enfoui. Dans cette Note on introduit une transformation composée de type Radon sur des surfaces coniques (au lieu de plans) qui forme le fondement d'un autre principe d'imagerie basée sur la diffusion Compton. L'équation reliant les mesures (densités surfaciques de flux de photons $g(\mathsf{D}, t = \tan\theta)$ détectées au point $\mathsf{D}$ sous l'angle de diffusion $\theta$) et l'objet recherché (densité volumique de radioactivité $f(\mathbf{V})$), définit une nouvelle tranformation intégrale $\mathcal{T}$. Cette dernière peut être considérée comme une somme pondérée de transformations de Radon conique et caractérisée par un noyau $p(\mathsf{D}, t|\mathbf{V})$. En particulier on établit la formule explicite de la transformation inverse $\mathcal{T}^{-1}$. Ce résultat théorique démontre la faisabilité

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de reconstitution d'un objet tridimensionnel à partir d'une série d'images $g(\mathsf{D}, t = \tan\theta)$ paramétrisées par l'angle de diffusion $\theta$. On peut le considérer aussi comme un nouveau principe d'imagerie nucléaire basée sur la diffusion Compton. L'obtention de la série d'images se faisant sans déplacement de l'appareil de mesure (camera gamma) comme c'est le cas de la tomographie conventionnelle, cette nouvelle modalité présente un intérêt technologique considérable.

## 1. Introduction

In his seminal work [8], J. Radon had sought the determination of functions on $\mathbb{R}^n$, $n \in \mathbb{N}$, by their integrals over manifolds of codimension one and succeeded to solve the problem when the manifolds are just hyperplanes. Over the last 75 years his work has been continued and extended to other types of manifolds, notably to spheres, paraboloids [1] or algebraic varieties [7]. In 1994, a transformation called *restricted cone surface transform* was introduced for the first time by Cree and Bones [2], in the context of image generation in Compton cameras, in which photon scattering takes place only on a plane and not in space.

In this note, we present a yet more general transform $\mathcal{T}$, which consists of a weighted sum of Conical Radon Transforms (Radon transform on surfaces of cones) and which may be called *Compound Conical Radon Transform* (*CCRT*). $\mathcal{T}$ arises from the image formation by single Compton-scattered photons (in space and not restricted to a plane) and serves as support for a new imaging principle in nuclear imaging [5,6]. Actually $\mathcal{T}$ maps the radioactivity volume density of an object $f(\mathbf{V})$, $\mathbf{V} \in \mathbb{R}^3$ inside a medium to the measured photon flux density $g(\mathsf{D}, t)$ with $\mathsf{D} \in \mathbb{R}^2$ and $t = \tan\theta$, $\theta$ being the Compton scattering angle (*see* Fig. 1).

The trajectories of photons emitted from the object and detected at site $\mathsf{D}$ all lie on a cone of apex $\mathbf{M}$, axis perpendicular to the detector and opening angle $\theta$. The transform $\mathcal{T} : f(\mathbf{V}) \mapsto g(\mathsf{D}, t)$ is given by:

$$g(\mathsf{D}, t) = K(t) \int_l^\infty \frac{\mathrm{d}\zeta_M}{\zeta_M^2} \int_0^{2\pi} \mathrm{d}\phi \int_0^\infty \frac{\mathrm{d}r}{r} f(\xi_D + r\sin\theta\cos\phi, \eta_D + r\sin\theta\sin\phi, \zeta_M + r\cos\theta), \qquad (1)$$

where $r$, $\phi$, $\zeta_M$ are integration variables as given in Fig. 1, $l$ is a cut-off and $K(t)$ is a kinematical factor due to the Compton effect and the conical geometry. From physical considerations, $f(\mathbf{V})$ and $g(\mathsf{D}, t)$

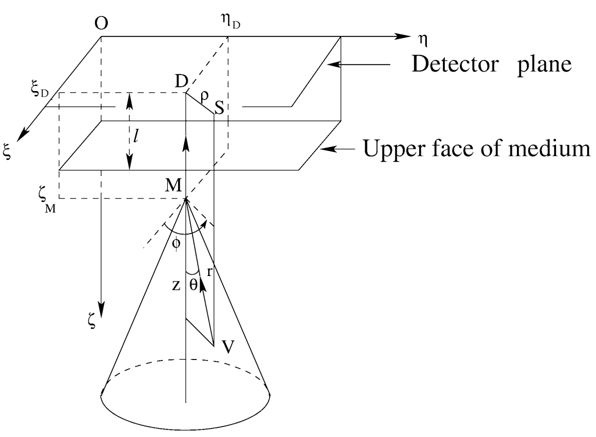


Figure 1. – Coordinate system for the calculation of $\mathcal{T}$.

*Figure 1. – Système de coordonnés pour le calcul de $\mathcal{T}$.*

have compact support in $\mathbb{R}^3$ and $\mathbb{R}^2$, respectively, are piecewise continuous and of bounded variations. For convenience we shall note $\mathbf{V} = (\mathsf{S}, \zeta_V)$, $\mathsf{S}$ being the projection of $\mathbf{V}$ onto the detector plane $O\xi\eta$.

It is known that the Radon Transform has played a crucial role in modern Computerized Tomography. The present Compound Conical Radon Transform is in fact the key to a new imaging principle based on the Compton effect. The existence of its inverse $\mathcal{T}^{-1}$ implies that a three dimensional reconstruction of $f(\mathbf{V})$ is obtainable from a series of $\theta$-indexed images. This result opens the way to a new method of object reconstruction which does not require the motion of the measuring apparatus (gamma-camera) as in standard tomographic modality. This may be of considerable technical interest for the conception of new generations of gamma-cameras [6].

## 2. Propositions

For the discussion, it is useful to consider an alternative form of Eq. (1). Let $F_2(u, v; \zeta)$ and $G_2(u, v; t)$ be the Fourier transforms of $f(\mathsf{S}, \zeta)$ and $g(\mathsf{D}, t)$, respectively. As the cone surface may lie on the forward or backward, respectively side of the direction $O\zeta$, depending on whether $0 < \theta < \pi/2$ or $\pi/2 < \theta < \pi$, the variables $t = \tan\theta$ and $z = r\cos\theta$ have always simultaneously the same sign. Thus for $z > 0$ and $t > 0$ we have:

$$G_2(u, v; \pm t) = 2\pi K(\pm t) \int_{0_+}^{\infty} z\,\mathrm{d}z\, J_0\big(2\pi z|t|\sqrt{u^2 + v^2}\big) \frac{1}{z^2} \int_{l}^{\infty} \frac{\mathrm{d}\zeta_M}{\zeta_M^2} F_2(u, v; \zeta_M \pm z), \tag{2}$$

where the choice of $\pm$ sign depends on the choice of ranges of $\theta$.

PROPOSITION 1. – *The transform $\mathcal{T}$ is defined by its kernel $p(\mathsf{D}, t|\mathbf{V})$:*

$$g(\mathsf{D}, t) = \int \mathrm{d}\mathbf{V}\, p(\mathsf{D}, t|\mathbf{V}) f(\mathbf{V}) \tag{3}$$

*with*

$$p(\mathsf{D}, t|\mathbf{V}) = \left(\frac{t}{\rho}\right)^2 \left[Y(t)K(t)\frac{Y(t(\zeta - l) - \rho)}{(t\zeta - \rho)^2} + Y(-t)K(-t)\frac{1}{(\zeta|t| + \rho)^2}\right], \tag{4}$$

*where $Y(t)$ is the Heaviside unit step distribution and $\rho = |\mathsf{DS}|$.*

To get the expression of the kernel $p(\mathsf{D}, t|\mathbf{V})$ given by Eq. (4), one sets $f(\mathbf{V})$ as a product of $\delta$-functions in the $\xi, \eta, \zeta$ directions. Integrations can be explicitly carried out in each range of $\theta$ values.

Thus $p(\mathsf{D}, t|\mathbf{V})$ is a distribution on $\mathbb{R}^3 \times \mathbb{R}^3$ with the properties:

$$p(\mathsf{D}, t|\mathbf{V}) = p(\mathsf{D}, t|\mathsf{S}, \zeta) = p\big(|\mathsf{DS}|, t, \zeta\big) = p(\rho, t, \zeta). \tag{5}$$

Let us observe a special case where $\theta \to \pi/2$, we have:

$$\lim_{|t|\to\infty} p(\mathsf{D}, t|\mathbf{V}) = K(\infty)\frac{Y(\zeta - l)}{(r\zeta)^2}. \tag{6}$$

In this case, the cone is flattened into a plane (*see* Eq. (1)) and $\mathcal{T}$ is a weighted sum of standard Radon Transforms on a set of planes parallel to the detector plane $O\xi\eta$.

The knowledge of $p(\mathsf{D}, t|\mathbf{V})$, which is the image of a point source at site $\mathbf{V}$, allows the explicit computation of $g(\mathsf{D}, t)$ for given $f(\mathbf{V})$.

PROPOSITION 2. – *The inverse transform $\mathcal{T}^{-1}$ is defined by its kernel $p^{-1}(\mathbf{V}|\mathsf{D},t)$ through the equation*:

$$f(\mathbf{V}) = \int \mathrm{d}\mathsf{D} \int \mathrm{d}t p^{-1}(\mathbf{V}|\mathsf{D},t) g(\mathsf{D},t), \tag{7}$$

*where*

$$p^{-1}(\mathbf{V}|\mathsf{D},t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{d}u\,\mathrm{d}v\big(u^2+v^2\big)\,\mathrm{e}^{2\mathrm{i}\pi[u(\xi-\xi_D)+v(\eta-\eta_D)]}$$
$$2\pi \int_{-\infty}^{\infty} z\,\mathrm{d}z\, J_0\big(2\pi|z|t\sqrt{u^2+v^2}\big) z\, H_l(\zeta+z+l)\mathcal{K}_t, \tag{8}$$

*with the definitions*:

$$H_l(\zeta) = \int_{-\infty}^{\infty} \mathrm{d}w\, \frac{\mathrm{e}^{2\mathrm{i}\pi\zeta w}}{\mathcal{J}_l(w)}, \quad \mathcal{J}_l(w) = \int_{-\infty}^{\infty} \mathrm{d}s\, \frac{Y(s)}{(s+l)^2}\,\mathrm{e}^{-2\mathrm{i}\pi s w}, \tag{9}$$

*$\mathcal{J}_l(w)$ is given by Fourier tables* [3] *and $\mathcal{K}_t$ is a discrete operator acting on $g(\mathsf{D},t)$ according to*:

$$\mathcal{K}_t g(\mathsf{D},t) = \left[Y(z)\frac{g(\mathsf{D},t)}{K(t)} + Y(-z)\frac{g(\mathsf{D},-t)}{K(-t)}\right]. \tag{10}$$

To establish this result, one should perform first the inversion of the Hankel Transform in Eq. (2), and next extract the three dimensional Fourier Transform of $F_3(u,v,w)$ of $f(\mathbf{V})$.
Then we have the following theorem:

## 3. Main theorem

*The kernels $p(\mathsf{D},t|\mathbf{V})$ and $p^{-1}(\mathbf{V}|\mathsf{D},t)$ fulfill the inversion relation*:

$$\int \mathrm{d}\mathsf{D} \int \mathrm{d}t\, p^{-1}(\mathbf{V}|\mathsf{D},t)\, p^*\big(\mathsf{D},t|\mathbf{V}'\big) = \delta\big(\mathbf{V}-\mathbf{V}'\big), \tag{11}$$

*$p^*$ being the complex conjugate of $p$.*

The proof is performed by explicit calculations, using essentially the Fourier transforms of the kernels $p(\mathsf{D},t|\mathbf{V})$ and $p^{-1}(\mathbf{V}|\mathsf{D},t)$ and in particular the Bessel function identity [4]:

$$\int_0^{\infty} s\,\mathrm{d}s\, J_0(z's)J_0(zs) = \frac{1}{z}\delta(z-z'). \tag{12}$$

## 4. Extension to the case with constant absorption

The present results, Eqs. (4), (8) and (11) concern only the emission properties of the radioactive sources in the studied object. In realistic situations, absorption of photons inside the object and by the medium should be taken into account. This problem will be addressed in future work since it is an involved one. However, if, as a first approximation one assumes a constant average absorption coefficient $\mu$, then the expression of the kernel of $\mathcal{T}$ becomes

$$p_\mu(\mathsf{D},t|\mathbf{V}) = \mathrm{e}^{-\mu[(\zeta-l)+\rho\tan\theta/2]} p(\mathsf{D},t|\mathbf{V}). \tag{13}$$

and the kernel $p_\mu^{-1}$ keeps the same form as given by Eq. (8) but with the following modified quantities:

$$H_{\mu,l}(\zeta) = \int_{-\infty}^{\infty} \mathrm{d}w \, \frac{\mathrm{e}^{2\mathrm{i}\pi\zeta w}}{\mathcal{J}_{\mu,l}(w)}, \quad \mathcal{J}_{\mu,l}(w) = \int_{-\infty}^{\infty} \mathrm{d}s \, \frac{Y(s)\,\mathrm{e}^{-\mu s}}{(s+l)^2} \, \mathrm{e}^{-2\mathrm{i}\pi s w}. \tag{14}$$

The main theorem remains also valid.

## 5. Concluding remarks

The new integral transform $\mathcal{T}$ presented in this note belongs to a large class of linear integral transforms which is widely used in the field of nuclear imaging, where numerous potential applications are possible (medical imaging, non-destructive control, material testing etc.). This new transform provides the foundation for a novel nuclear emission imaging principle based on Compton scattering whereby the source distribution is reconstructed without the motion of the detector, as it is the case in standard tomographic procedure.

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