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The Neumann problem in the half-space

Chérif Amrouche

Laboratoire de mathématiques appliquées, Université de Pau et des Pays de l'Adour, 64000 Pau, France

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Abstract We study here the nonhomogeneous Neumann problem in the half-space $\mathbb{R}^N_+$ with $N \geqslant 2$. We give in L^p theory, with $1 < p < \infty$, a basic existence and regularity results in weighted Sobolev spaces. ***To cite this article: C. Amrouche, C. R. Acad. Sci. Paris, Ser. I 335 (2002) 151–156.*** © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

Le problème de Neumannn dans le demi-espace

Résumé Nous étudions ici le problème non homogène de Neumann dans le demi-espace $\mathbb{R}^N_+$ avec $N \geqslant 2$. Nous donnons des résultats fondamentaux d'existence et de régularité en théorie L^p, avec $1 < p < \infty$, dans des espaces de Sobolev avec poids. ***Pour citer cet article : C. Amrouche, C. R. Acad. Sci. Paris, Ser. I 335 (2002) 151–156.*** © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

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L'objet de cette Note est de résoudre le problème de Neumann (P_N) dans le demi-espace. Comme pour le problème de Poisson ou celui de Laplace dans un domaine extérieur, les espaces de Sobolev avec poids (1.1) et (2.2) fournissent un cadre approprié pour la recherche de solutions. La principale différence est due à la nature de la frontière et l'une des difficultés est d'obtenir l'espace de traces qui convient (*voir* Lemme 2.1 qui généralise ainsi le résultat de Hanouzet, (*cf.* [3])). En particulier dans certains cas, qu'on qualifiera de critiques, il est nécessaire d'introduire un facteur logarithmique supplémentaire dans le poids (ce qui est le cas, par exemple de la dimension 2 lorsque l'exposant p vaut 2). Dans un précédent travail, des résultats similaires ont été établis pour le problème de Dirichlet (*cf.* [1]) étendant ainsi ceux de Boulmezaoud (*cf.* [2]) valables seulement dans le cadre hilbertien $p = 2$ et $N \geqslant 3$. Le Corollaire 3.3 et le Théorème 3.7 sont les principaux résultats de ce travail.

1. Introduction

The purpose of this paper is to solve the Neumann problem $(\mathcal{P}_N)$:

$$-\Delta u = f \quad \text{in } \mathbb{R}^N_+,$$
$$\frac{\partial u}{\partial x_N} = g \quad \text{on } \mathbb{R}^{N-1}.$$

The approach is based on the use of a special class of weighted Sobolev spaces for describing the behavior at infinity. Many authors have studied the Laplace equation in the whole space $\mathbb{R}^N$ or in an exterior domain. The main difference is due to the nature of the boundary and one of the difficulties is to obtain the appropriate trace spaces. However, the half-space has a useful symmetric property.

E-mail address: Cherif.Amrouche@univ-pau.fr (C. Amrouche).

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Problem $(\mathcal{P}_N)$ has been investigated in weighted Sobolev spaces by several authors, but only in the Hilbert cases ($p=2$) and without the critical cases corresponding to logarithmic factor (*cf.* [2]).

Let Ω be an open set of $\mathbb{R}^N$, $N \geqslant 2$, $\varrho = (1+|x|^2)^{1/2}$ and $\lg \varrho = \ln(2+|x|^2)$. For any nonnegative integer m, real numbers $p>1$, α and β and setting $k = m - N/p - \alpha$ if $N/p+\alpha \in \{1,\dots,m\}$, and $k=-1$ otherwise, we define the following space:

$$\begin{aligned}\mathrm{W}^{m,p}_{\alpha,\beta}(\Omega) = \big\{ & u \in \mathcal{D}'(\Omega);\ 0 \leqslant |\lambda| \leqslant k,\ \varrho^{\alpha-m+|\lambda|}(\lg \varrho)^{\beta-1} D^\lambda u \in \mathrm{L}^p(\Omega), \\ & k+1 \leqslant |\lambda| \leqslant m,\ \varrho^{\alpha-m+|\lambda|}(\lg\varrho)^{\beta} D^\lambda u \in \mathrm{L}^p(\Omega)\big\}.\end{aligned} \tag{1.1}$$

In the case $\beta = 0$, we simply denote the space by $\mathrm{W}^{m,p}_{\alpha}(\Omega)$. Note that $\mathrm{W}^{m,p}_{\alpha,\beta}(\Omega)$ is a reflexive Banach space equipped with its natural norm:

$$\|u\|_{\mathrm{W}^{m,p}_{\alpha,\beta}(\Omega)} = \Bigg[\sum_{0\leqslant|\lambda|\leqslant k} \big\|\varrho^{\alpha-m+|\lambda|}(\lg\varrho)^{\beta-1}D^\lambda u\big\|^p_{\mathrm{L}^p(\Omega)} + \sum_{k+1\leqslant|\lambda|\leqslant m} \big\|\varrho^{\alpha-m+|\lambda|}(\lg\varrho)^{\beta}D^\lambda u\big\|^p_{\mathrm{L}^p(\Omega)}\Bigg]^{1/p}.$$

We also define the semi-norm:

$$|u|_{\mathrm{W}^{m,p}_{\alpha,\beta}(\Omega)} = \Bigg(\sum_{|\lambda|=m} \big\|\varrho^{\alpha}(\lg\varrho)^{\beta}D^\lambda u\big\|^p_{\mathrm{L}^p(\Omega)}\Bigg)^{1/p},$$

and for any integer q, we denote by $\mathcal{P}_q$ the space of polynomials in N variables of the degree smaller than or equal to q, with the convention that $\mathcal{P}_q$ is reduced to $\{0\}$ when q is negative. The weights in definition (1.1) are chosen so that the corresponding space satisfies two properties: $\mathcal{D}(\overline{\mathbb{R}^N_+})$ is dense in $\mathrm{W}^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+)$ and the following Poincaré-type inequality: if

$$\frac{N}{p} + \alpha \notin \{1,\dots,m\} \quad \text{or} \quad (\beta-1)p \neq -1. \tag{1.2}$$

then the semi-norm $|\cdot|_{\mathrm{W}^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+)}$ defines on $\mathrm{W}^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+)/P_{q'}$ a norm which is equivalent to the quotient norm, with $q' = \inf(q, m-1)$, where q is the highest degree of the polynomials contained in $\mathrm{W}^{m,p}_{\alpha}(\mathbb{R}^N_+)$.

Now, we define the space $\mathring{\mathrm{W}}^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+) = \overline{\mathcal{D}(\mathbb{R}^N_+)}^{\|\cdot\|_{\mathrm{W}^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+)}}$ and the dual of $\mathring{\mathrm{W}}^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+)$ is denoted by $\mathrm{W}^{-m,p'}_{-\alpha,-\beta}(\mathbb{R}^N_+)$, where p' is the conjugate of p. Under the assumptions (1.2), the semi-norm $|\cdot|_{\mathrm{W}^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+)}$ is a norm on $\mathring{\mathrm{W}}^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+)$ such that is equivalent to the full norm $\|\cdot\|_{\mathrm{W}^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+)}$.

We recall now some properties of the weighted Sobolev spaces $\mathrm{W}^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+)$. We have the algebraic and topological imbeddings:

$$\mathrm{W}^{m,p}_{\alpha,\beta}\big(\mathbb{R}^N_+\big) \subset \mathrm{W}^{m-1,p}_{\alpha-1,\beta}\big(\mathbb{R}^N_+\big) \subset \cdots \subset \mathrm{W}^{0,p}_{\alpha-m,\beta}\big(\mathbb{R}^N_+\big) \quad \text{if } \frac{N}{p}+\alpha \notin \{1,\dots,m\}. \tag{1.3}$$

When $N/p + \alpha = j \in \{1,\dots,m\}$, then we have:

$$\mathrm{W}^{m,p}_{\alpha,\beta}\big(\mathbb{R}^N_+\big) \subset \cdots \subset \mathrm{W}^{m-j+1,p}_{\alpha-j+1,\beta}\big(\mathbb{R}^N_+\big) \subset \mathrm{W}^{m-j,p}_{\alpha-j,\beta-1}\big(\mathbb{R}^N_+\big) \subset \cdots \subset \mathrm{W}^{0,p}_{\alpha-m,\beta-1}\big(\mathbb{R}^N_+\big). \tag{1.4}$$

Note that in the first case and if moreover $\frac{N}{p}+\alpha-\gamma \notin \{1,\dots,m\}$, the mapping $u \to \varrho^\gamma u$ is an isomorphism from $\mathrm{W}^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+)$ on $\mathrm{W}^{m,p}_{\alpha-\gamma,\beta}(\mathbb{R}^N_+)$ for any integer m. In both cases and for any multi-index $\lambda \in \mathbb{N}^N$, the mapping $u \in \mathrm{W}^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+) \to D^\lambda u \in \mathrm{W}^{m-|\lambda|,p}_{\alpha,\beta}(\mathbb{R}^N_+)$ is continuous.

Finally, it can be readily checked that the highest degree q of the polynomials contained in $\mathrm{W}^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+)$ is given by

$$q = \begin{cases} m - \left(\dfrac{N}{p}+\alpha\right) - 1 & \text{if } \begin{cases} \frac{N}{p}+\alpha \in \{1,\dots,m\} \text{ and } (\beta-1)p \geqslant -1 \text{ or} \\ \frac{N}{p}+\alpha \in \{j \in Z;\ j \leqslant 0\} \text{ and } \beta p \geqslant -1, \end{cases} \\ \left[m - \left(\dfrac{N}{p}+\alpha\right)\right] & \text{otherwise,} \end{cases}$$

where $[s]$ denote the integer part of s.
In the sequel, for any integer $q \geqslant 0$, we shall use the following polynomial spaces:
– $\mathcal{P}_q$ (respectively $\mathcal{P}_q^\Delta$) is the space of polynomials (respectively harmonic polynomials) of degree $\leqslant q$,
– $\mathcal{P}'_q$ is the subspace of polynomials in $\mathcal{P}_q$ depending only on the $N-1$ first variables, $x' = (x_1, \ldots, x_{N-1})$,
– $\mathcal{A}_q^\Delta$ (respectively $\mathcal{N}_q^\Delta$) the subspace of polynomials $\mathcal{P}_q^\Delta$ satisfying the condition $p(x', 0) = 0$ (respectively $\frac{\partial p}{\partial x_N}(x', 0) = 0$) or equivalently odd with respects to x_N (respectively even with respect to x_N), with the convention that $\mathcal{P}_q, \mathcal{P}_q^\Delta, \mathcal{P}'_q, \ldots$ are reduced to $\{0\}$ when q is negative.

2. The spaces of traces

In order to define the traces of functions of $\mathrm{W}_{\alpha,\beta}^{m,p}(\mathbb{R}_+^N)$, we introduce for any $\sigma \in]0,1[$ the space:

$$\mathrm{W}_0^{\sigma,p}(\mathbb{R}^N) = \left\{ u \in \mathcal{D}'(\mathbb{R}^N);\ w^{-\sigma} u \in \mathrm{L}^p(\mathbb{R}^N),\ \int_0^{+\infty} t^{-1-\sigma p}\,\mathrm{d}t \int_{\mathbb{R}^N} \left| u(x+te_i) - u(x) \right|^p \mathrm{d}x < \infty \right\}, \tag{2.1}$$

where $w = \varrho$ if $N/p \neq \sigma$ and $w = \varrho(\lg \varrho)^{1/\sigma}$ if $N/p = \sigma$, and $e_1, \ldots, e_N$ is a canonical basic of $\mathbb{R}^N$. It is a reflexive Banach space equipped with its natural norm:

$$\|u\|_{\mathrm{W}_0^{\sigma,p}(\mathbb{R}^N)} = \left(\left\| \frac{u}{w^\sigma} \right\|_{\mathrm{L}^p(\mathbb{R}^N)}^p + \sum_{i=1}^N \int_0^\infty t^{-1-\sigma p}\,\mathrm{d}t \int_{\mathbb{R}^N} \left| u(x+te_i) - u(x) \right|^p \mathrm{d}x \right)^{1/p}$$

which is equivalent to the norm

$$\left(\left\| \frac{u}{w^\sigma} \right\|_{\mathrm{L}^p(\mathbb{R}^N)}^p + \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x-y|^{N+\sigma p}}\,\mathrm{d}x\,\mathrm{d}y \right)^{1/p}.$$

Similary, for each real number α, we define the space:

$$\mathrm{W}_\alpha^{\sigma,p}(\mathbb{R}^N) = \left\{ u \in \mathcal{D}'(\mathbb{R}^N);\ w^{\alpha-\sigma} u \in \mathrm{L}^p(\mathbb{R}^N),\ \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|\varrho^\alpha(x)u(x) - \varrho^\alpha(y)u(y)|^p}{|x-y|^{N+\sigma p}}\,\mathrm{d}x\,\mathrm{d}y < +\infty \right\},$$

where $w = \varrho$ if $N/p + \alpha \neq \sigma$ and $w = \varrho(\lg \varrho)^{1/(\sigma-\alpha)}$ if $N/p + \alpha \neq \sigma$.

Remark 2.1. – If $\sigma = 1$, the definition (2.1) coincides with the definition (1.1).

For any $s \in \mathbb{R}^+$, we set

$$\begin{aligned} \mathrm{W}_\alpha^{s,p}(\mathbb{R}^N) = \big\{ & u \in \mathcal{D}'(\mathbb{R}^N);\ 0 \leqslant |\lambda| \leqslant k,\ \varrho^{\alpha-s+|\lambda|}(\lg \varrho)^{-1} D^\lambda u \in \mathrm{L}^p(\mathbb{R}^N), \\ & k+1 \leqslant |\lambda| \leqslant [s]-1,\ \varrho^{\alpha-s+|\lambda|} D^\lambda u \in \mathrm{L}^p(\mathbb{R}^N);\ D^{[s]} u \in \mathrm{W}_\alpha^{\sigma,p}(\mathbb{R}^N) \big\}, \end{aligned} \tag{2.2}$$

where $k = s - N/p - \alpha$ if $N/p + \alpha \in \{\sigma, \ldots, \sigma + [s]\}$, with $\sigma = s - [s]$ and $k = -1$ otherwise. It is a reflexive Banach space equipped with the norm:

$$\begin{aligned} \|u\|_{\mathrm{W}_\alpha^{s,p}(\mathbb{R}^N)} = \Bigg[& \sum_{0 \leqslant |\lambda| \leqslant k} \left\| \varrho^{\alpha-s+|\lambda|} (\lg \varrho)^{-1} D^\lambda u \right\|_{\mathrm{L}^p(\Omega)}^p \\ & + \sum_{k+1 \leqslant |\lambda| \leqslant [s]-1} \left\| \varrho^{\alpha-s+|\lambda|} D^\lambda u \right\|_{\mathrm{L}^p(\Omega)}^p \Bigg]^{1/p} + \sum_{|\lambda| = [s]} \left\| D^\lambda u \right\|_{\mathrm{W}_\alpha^{\sigma,p}(\mathbb{R}^N)}. \end{aligned}$$

Remark 2.2. – (i) If $N/p + \alpha \notin \{\sigma, \ldots, \sigma + [s]\}$ and $N/p \notin \{\sigma, \ldots, \sigma + [s]\}$ then

$$u \in \mathrm{W}_\alpha^{s,p}(\mathbb{R}^N) \Longleftrightarrow \varrho^\alpha u \in \mathrm{W}_0^{s,p}(\mathbb{R}^N) \Longleftrightarrow u \in \mathrm{W}_{\alpha+[s]-s}^{[s],p}(\mathbb{R}^N) \text{ and } \forall |\lambda| = [s],\ \varrho^\alpha D^\lambda u \in \mathrm{W}_0^{s-[s],p}(\mathbb{R}^N).$$

(ii) We can similarly define, for any real number β, the space:

$$\mathrm{W}^{s,p}_{\alpha,\beta}(\mathbb{R}^N) = \{v \in \mathcal{D}'(\mathbb{R}^N);\ (\lg\varrho)^{\beta} v \in \mathrm{W}^{s,p}_{\alpha}(\mathbb{R}^N)\}. \tag{2.3}$$

We can prove some properties of the weighted Sobolev spaces $\mathrm{W}^{s,p}_{\alpha,\beta}(\mathbb{R}^N)$. We have the algebraic and topological imbeddings in the case where $N/p + \alpha \notin \{\sigma, \ldots, \sigma + [s]\}$:

$$\mathrm{W}^{s,p}_{\alpha,\beta}(\mathbb{R}^N) \subset \mathrm{W}^{s-1,p}_{\alpha-1,\beta}(\mathbb{R}^N) \subset \cdots \subset \mathrm{W}^{\sigma,p}_{\alpha-[s],\beta}(\mathbb{R}^N), \tag{2.4}$$

$$\mathrm{W}^{s,p}_{\alpha,\beta}(\mathbb{R}^N) \subset \mathrm{W}^{[s],p}_{\alpha+[s]-s,\beta}(\mathbb{R}^N) \subset \cdots \subset \mathrm{W}^{0,p}_{\alpha-s,\beta}(\mathbb{R}^N). \tag{2.5}$$

When $N/p + \alpha = j \in \{\sigma, \ldots, \sigma + [s]\}$, then we have:

$$\mathrm{W}^{s,p}_{\alpha,\beta} \subset \cdots \subset \mathrm{W}^{s-j+1,p}_{\alpha-j+1,\beta} \subset \mathrm{W}^{s-j,p}_{\alpha-j,\beta-1} \subset \cdots \subset \mathrm{W}^{\sigma,p}_{\alpha-[s],\beta-1}, \tag{2.6}$$

$$\mathrm{W}^{s,p}_{\alpha,\beta} \subset \mathrm{W}^{[s],p}_{\alpha+[s]-s,\beta} \subset \cdots \subset \mathrm{W}^{[s]-j+1,p}_{\alpha-\sigma-j+1,\beta} \subset \mathrm{W}^{[s]-j,p}_{\alpha-\sigma-j,\beta-1} \subset \cdots \subset \mathrm{W}^{0,p}_{\alpha-s,\beta-1}. \tag{2.7}$$

If u is a function on $\mathbb{R}^N_+$, we denote its traces on $\Gamma = \mathbb{R}^{N-1}$ by:

$$x' \in \mathbb{R}^{N-1}, \quad \gamma_0 u(x') = u(x', 0), \ldots, \gamma_j u(x') = \frac{\partial^j u}{\partial x_N^j}(x', 0).$$

As in [3], we can prove the following trace lemma:

LEMMA 2.1. – *For any integer $m \geqslant 1$ and real number α, the mapping*

$$\gamma : u \in \mathcal{D}(\overline{\mathbb{R}^N_+}) \to (\gamma_0 u, \ldots, \gamma_{m-1} u) \in \prod_{j=0}^{m-1} \mathcal{D}(\mathbb{R}^{N-1})$$

can be extended by continuity to a linear and continuous mapping still denoted by γ from $\mathrm{W}^{m,p}_{\alpha}(\mathbb{R}^N_+)$ to $\prod_{j=0}^{m-1} \mathrm{W}^{m-j-1/p,p}_{\alpha}(\mathbb{R}^{N-1})$. Moreover γ is onto and $\operatorname{Ker}\gamma = \mathring{\mathrm{W}}^{m,p}_{\alpha}(\mathbb{R}^N_+)$.

3. The Laplace equation with Neumann condition

The aim of this section is to study the problem $(\mathcal{P}_N)$.

THEOREM 3.1. – *Let l be an integer and assume that*

$$\frac{N}{p'} \notin \{1, \ldots, l\} \quad \text{if } l \geqslant 1 \quad \text{and} \quad \frac{N}{p} \notin \{1, \ldots, -l\} \quad \text{if } l \leqslant -1. \tag{3.1}$$

For any f in $\mathrm{W}^{0,p}_{l}(\mathbb{R}^N_+)$ satisfying the compatibility condition

$$\forall \varphi \in \mathcal{N}^{\Delta}_{[l-N/p']}, \quad \langle f, \varphi \rangle_{\mathrm{W}^{0,p}_{l}(\mathbb{R}^N_+) \times \mathrm{W}^{0,p'}_{-l}(\mathbb{R}^N_+)} = 0, \tag{3.2}$$

problem $(\mathcal{P}_N)$, with $g = 0$, has a unique solution $u \in \mathrm{W}^{2,p}_{l}(\mathbb{R}^N_+)/\mathcal{N}^{\Delta}_{[2-l-N/p]}$.

Proof. – The uniqueness is obvious.

(i) *First step*: $l \leqslant 1$. We set $\tilde{f} = f$ in $\mathbb{R}^N_+$ and $\tilde{f} = 0$ elsewhere. Then $\tilde{f}$ belongs to $\mathrm{W}^{0,p}_{l}(\mathbb{R}^N)$ and for any $\varphi \in \mathcal{N}^{\Delta}_{[l-N/p']} = \mathcal{P}^{\Delta}_{[l-N/p']}$, we have:

$$\langle \tilde{f}, \varphi \rangle_{\mathrm{W}^{0,p}_{l}(\mathbb{R}^N) \times \mathrm{W}^{0,p'}_{-l}(\mathbb{R}^N)} = \int_{\mathbb{R}^N} \tilde{f}\varphi \,\mathrm{d}x = \int_{\mathbb{R}^N_+} f\varphi \,\mathrm{d}x = \langle f, \varphi \rangle_{\mathrm{W}^{0,p}_{l}(\mathbb{R}^N_+) \times \mathrm{W}^{0,p'}_{-l}(\mathbb{R}^N_+)} = 0.$$

Therefore, there exists a function $w \in \mathrm{W}^{2,p}_{l}(\mathbb{R}^N)$ such that $\Delta w = \tilde{f}$ in $\mathbb{R}^N$. The function u defined by

$$u(x', x_N) = \frac{1}{2}\left[w(x', x_N) + w(x', -x_N)\right] \quad \text{with } x_N > 0,$$

belongs to $\mathrm{W}^{2,p}_{l}(\mathbb{R}^N_+)$ and satisfies $-\Delta u = f$ in $\mathbb{R}^N_+$ and $\frac{\partial u}{\partial x_N} = 0$ on Γ.

(ii) *Second step*: $l \geqslant 2$. The function defined by $\tilde{f}(x', x_N) = f(x', x_N)$ if $x_N > 0$ and $\tilde{f}(x', x_N) = f(x', -x_N)$ if $x_N < 0$ belongs to $\mathrm{W}_l^{0,p}(\mathbb{R}^N)$ and verifies:

$$\forall \varphi \in \mathcal{P}_{[l-N/p']}^{\Delta}, \quad \langle \tilde{f}, \varphi \rangle_{\mathrm{W}_l^{0,p}(\mathbb{R}^N) \times \mathrm{W}_{-l}^{0,p'}(\mathbb{R}^N)} = 0.$$

Thus, there exists a unique function $w \in \mathrm{W}_l^{2,p}(\mathbb{R}^N)$ such that $-\Delta w = \tilde{f}$ in $\mathbb{R}^N$. However, note that the function v defined by $v(x', x_N) = w(x', -x_N)$ belongs to $\mathrm{W}_l^{2,p}(\mathbb{R}^N)$ and verifies also this equation. The uniqueness of solutions yields $v = w$ and $\partial w / \partial x_N = 0$ on Γ. Now, the function $u = w_{|\mathbb{R}_+^N}$ is solution of our problem.

Remark 3.2. – When $N/p' = l$, if f belongs to $\mathrm{W}_l^{0,p}(\mathbb{R}_+^N) \cap \mathrm{W}_0^{-l,p}(\mathbb{R}_+^N)$ and satisfies the compatibility condition $\langle f, 1 \rangle_{\mathrm{W}_0^{-l,p} \times \mathrm{W}_0^{l,p'}} = 0$, then the previous theorem holds (the unicity is up to a constant if $N = 2$).

COROLLARY 3.3. – *Let l be an integer and $g \in \mathrm{W}_l^{1-1/p,p}(\Gamma)$. We suppose that one on both conditions* (i) *or* (ii) *holds*:

(i) *We assume that the hypothesis* (3.1) *holds and $f \in \mathrm{W}_l^{0,p}(\mathbb{R}_+^N)$ and satisfies the compatibility condition*:

$$\forall \varphi \in \mathcal{N}_{[l-N/p']}^{\Delta}, \quad \langle f, \varphi \rangle_{\mathrm{W}_l^{0,p}(\mathbb{R}_+^N) \times \mathrm{W}_{-l}^{0,p'}(\mathbb{R}_+^N)} = \langle g, \varphi \rangle_{\mathrm{W}_l^{1-1/p,p}(\Gamma) \times \mathrm{W}_{-l}^{-1/p',p'}(\Gamma)}. \tag{3.3}$$

(ii) *We assume that $N/p' = l$ and f in $\mathrm{W}_l^{0,p}(\mathbb{R}_+^N) \cap \mathrm{W}_0^{-l,p}(\mathbb{R}_+^N)$ satisfying the compatibility condition*

$$\langle f, 1 \rangle_{\mathrm{W}_0^{-l,p} \times \mathrm{W}_0^{l,p'}} = \langle g, 1 \rangle_{\mathrm{W}_l^{1-1/p,p}(\Gamma) \times \mathrm{W}_{-l}^{-1/p',p'}(\Gamma)}. \tag{3.4}$$

Then, problem $(\mathcal{P}_N)$ has a unique solution $u \in \mathrm{W}_l^{2,p}(\mathbb{R}_+^N)/\mathcal{N}_{[2-l-N/p]}^{\Delta}$.

Proof. – It is a consequence of Lemma 2.1, Theorem 3.1 and Remark 3.2.

THEOREM 3.4. – *Let l be an integer satisfying the hypothesis* (3.1) *and $g \in \mathrm{W}_{-l}^{-1-1/p',p'}(\Gamma)$ satisfying the compatibility condition*:

$$\forall \varphi \in \mathcal{N}_{[2-l-N/p]}^{\Delta}, \quad \langle g, \varphi \rangle_{\mathrm{W}_{-l}^{-1-1/p',p'}(\Gamma) \times \mathrm{W}_l^{2-1/p,p}(\Gamma)} = 0. \tag{3.5}$$

Then, problem $(\mathcal{P}_N)$, with $f = 0$, has a unique solution $v \in \mathrm{W}_{-l}^{0,p'}(\mathbb{R}_+^N)/\mathcal{N}_{[l-N/p']}^{\Delta}$.

Proof. – (i) *First step*. We define the space

$$T(\mathbb{R}_+^N) = \{ v \in \mathrm{W}_{-l}^{0,p'}(\mathbb{R}_+^N);\ \Delta v \in \mathrm{W}_{-l+2}^{0,p'}(\mathbb{R}_+^N) \},$$

and we can prove that if $v \in T(\mathbb{R}_+^N)$, then $\partial v / \partial x_N \in \mathrm{W}_{-l}^{-1-1/p',p'}(\Gamma)$. Then, let us remark that the problem $(\mathcal{P}_N)$ with $f = 0$ is equivalent to find $v \in \mathrm{W}_{-l}^{0,p'}(\mathbb{R}_+^N)/\mathcal{N}_{[l-N/p']}^{\Delta}$ such that for any $\varphi \in W_l^{2,p}(\mathbb{R}_+^N)$ with $\partial \varphi / \partial x_N = 0$ on Γ,

$$\langle v, \Delta \varphi \rangle_{\mathrm{W}_{-l}^{0,p'}(\mathbb{R}_+^N) \times \mathrm{W}_l^{0,p}(\mathbb{R}_+^N)} = -\langle g, \varphi \rangle_{\mathrm{W}_{-l}^{-1-1/p',p'}(\Gamma) \times \mathrm{W}_l^{2-1/p,p}(\Gamma)}.$$

Now, according to Theorem 3.1, for any $h \in \mathrm{W}_l^{0,p}(\mathbb{R}_+^N)$ satisfying the compatibility condition (3.2), there exists a unique $\varphi \in \mathrm{W}_l^{2,p}(\mathbb{R}_+^N)/\mathcal{N}_{[2-l-N/p]}^{\Delta}$ such that $-\Delta \varphi = h$ in $\mathbb{R}_+^N$ and $\partial \varphi / \partial x_N = 0$ on Γ, with the estimate

$$\|\varphi\|_{\mathrm{W}_l^{2,p}(\mathbb{R}_+^N)/\mathcal{N}^{\Delta}{}_{[2-l-N/p]}} \leqslant C \|h\|_{\mathrm{W}_l^{0,p}(\mathbb{R}_+^N)}.$$

Using (3.5), we have:

$$\forall \lambda \in \mathcal{N}^{\Delta}{}_{[2-l-N/p]}, \quad |\langle g, \varphi \rangle| = |\langle g, \varphi + \lambda \rangle| \leqslant C \|g\|_{\mathrm{W}_{-l}^{-1-1/p',p'}(\Gamma)} \|h\|_{\mathrm{W}_l^{0,p}(\mathbb{R}_+^N)}.$$

In other words, thanks to Riesz' representation theorem, the mapping $h \mapsto < g, \varphi >$ defines a unique $v \in [\mathrm{W}_l^{0,p}(\mathbb{R}_+^N) \perp \mathcal{N}_{[l-N/p']}^{\Delta}]' = \mathrm{W}_{-l}^{0,p'}(\mathbb{R}_+^N)/\mathcal{N}_{[l-N/p']}^{\Delta}$ solution of $(\mathcal{P}_N)$, with $f = 0$.

Remark 3.5. – When $N/p' = l$, if $g \in W_{-l}^{-1-1/p',p'}(\Gamma)$ and satisfies the compatibility condition:

$$\forall \varphi \in \mathcal{N}_{[2-N]}^{\Delta}, \quad \langle g, \varphi \rangle_{W_{l}^{-1-1/p',p'}(\Gamma) \times W_{-l}^{2-1/p,p}(\Gamma)} = 0,$$

then the previous theorem holds with $v \in W_{-l,-1}^{0,p'}(\mathbb{R}_+^N)$ (the unicity up to additive constant).

COROLLARY 3.6. – *Let l be an integer satisfying the hypothesis* (3.1) *and $g \in W_{-l}^{-1/p',p'}(\Gamma)$ satisfying the compatibility condition*

$$\forall \varphi \in \mathcal{N}_{[1-l-N/p]}^{\Delta}, \quad \langle g, \varphi \rangle_{W_{-l}^{-1/p',p'}(\Gamma) \times W_{l}^{1-1/p,p}(\Gamma)} = 0.$$

Then, the problem $(\mathcal{P}_N)$, with $f = 0$, has a unique solution $v \in W_{-l}^{1,p'}(\mathbb{R}_+^N)/\mathcal{N}^{\Delta}{}_{[l+1-N/p']}$.

THEOREM 3.7. – *Let l be an integer satisfying the hypothesis* (3.1). *For any f in $W_{l}^{0,p}(\mathbb{R}_+^N)$ and $g \in W_{l-1}^{-1/p,p}(\Gamma)$ satisfying the compatibility condition*

$$\forall \varphi \in \mathcal{N}_{[l-N/p']}^{\Delta}, \quad \langle f, \varphi \rangle_{W_{l}^{0,p}(\mathbb{R}_+^N) \times W_{-l}^{0,p'}(\mathbb{R}_+^N)} = \langle g, \varphi \rangle_{W_{l-1}^{-1/p,p}(\Gamma) \times W_{-l+1}^{1-1/p',p'}(\Gamma)}$$

problem $(\mathcal{P}_N)$ has a unique solution $u \in W_{l-1}^{1,p}(\mathbb{R}_+^N)/\mathcal{N}^{\Delta}{}_{[2-l-N/p]}$.

Proof. – It is enough to prove the existence of a solution.

(i) *First step*: we suppose that $l < N/p'$. Hence the set $\mathcal{N}_{[l-N/p']}^{\Delta}$ is empty. Thanks to the imbedding $W_{l}^{1-1/p,p}(\Gamma) \subset W_{l-1}^{-1/p,p}(\Gamma)$ and to Corollary 3.6 , there exists $v \in W_{l-1}^{1,p}(\mathbb{R}_+^N)$ such that $-\Delta v = 0$ in $\mathbb{R}_+^N$ and $\partial v/\partial x_N = g$ on Γ. We deduce by Theorem 3.1 the existence of unique $v \in W_{l}^{2,p}(\mathbb{R}_+^N)$ satisfying $-\Delta v = f$ in $\mathbb{R}_+^N$ and $\partial v/\partial x_N = 0$ on Γ. The function $u = v + w$ is an answer to the question.

(ii) *Second step*: we suppose that $l > N/p'$. We denote by n the dimension of the subspace $N = \mathcal{N}_{[l-N/p']}^{\Delta}$ of $W_{-l+2}^{2,p'}(\mathbb{R}_+^N)$ included in $W_{-l}^{0,p'}(\mathbb{R}_+^N)$, $e_1, \dots, e_n$ a basis of N and $e_1^*, \dots, e_n^*$ the dual basis of the dual space N'. The restriction to Γ of e_j is denoted by h_j, i.e., $\gamma_0(e_j) = h_j$. With a suitable numbering of the family $h_1, \dots, h_d$, named in the same way, the elements $h_1, \dots, h_d$ form a basis of $\gamma_0(N)$ and $h_j = 0$ for $j = d+1, \dots, n$. We note that for each j, h_j belongs to $W_{-l+1}^{1-1/p',p'}(\Gamma) \subset W_{-l+1/p'}^{0,p'}(\Gamma) \subset W_{-l}^{-1/p',p'}(\Gamma)$. Let $\{h_1^*, \dots, h_d^*\}$ the dual basis of $\{h_1, \dots, h_d\}$. Now, let us consider the functions defined by

$$F = \sum_{i=1}^{i=n} e_i^* \langle f, e_i \rangle_{W_{l}^{0,p}(\mathbb{R}_+^N) \times W_{-l}^{0,p'}(\mathbb{R}_+^N)} \quad \text{and} \quad G = \sum_{i=1}^{i=d} h_i^* \langle g, h_i \rangle_{W_{l-1}^{-1/p,p}(\Gamma) \times W_{-l+1}^{1-1/p',p'}(\Gamma)}.$$

They satisfy: $\langle F, e_k \rangle = \langle f, e_k \rangle = \langle g, e_k \rangle$ if $k = 1, \dots, n$, $\langle F, e_k \rangle = \langle G, e_k \rangle = \langle g, e_k \rangle$ if $k = 1, \dots, d$ and $\langle F, e_k \rangle = \langle G, e_k \rangle = 0$ if $k = d+1, \dots, n$. So, by using Theorem 3.1, there exists $w \in W_{l}^{2,p}(\mathbb{R}_+^N)$ satisfying

$$-\Delta w = f - F \quad \text{in } \mathbb{R}_+^N \quad \text{and} \quad \frac{\partial w}{\partial x_N} = 0 \quad \text{on } \Gamma.$$

Thanks to Corollary 3.6, there exists $v \in W_{l-1}^{1,p}(\mathbb{R}_+^N)$ satisfying

$$-\Delta v = 0 \quad \text{in } \mathbb{R}_+^N \quad \text{and} \quad \frac{\partial v}{\partial x_N} = g - G \quad \text{on } \Gamma.$$

As F belongs to $W_{l}^{0,p}(\mathbb{R}_+^N)$ and G to $W_{l}^{1-1/p,p}(\Gamma)$, the conditions of Corollary 3.3 are verify. Then, there exists a function $z \in W_{l}^{2,p}(\mathbb{R}_+^N)$ such that $-\Delta z = F$ in $\mathbb{R}_+^N$ and $\partial z/\partial x_N = G$ on Γ. Finally, the function $u = v + w + z$ is an answer to our question.

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